Wednesday, May 19, 2021 5:57 PM

Invariant Tensors and Wheeled PROP

Product and Permutation category

joint work with Visu Makam

5/20/2021

K field, char $K=0$, e.g., $K=\mathbb{R}$ or $\mathbb{C}$

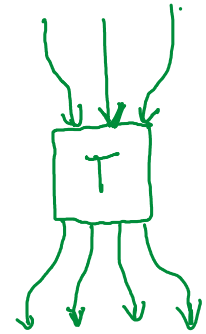
V n-dim K-vector space

$$V^{\otimes d} = \underbrace{V \otimes V \otimes \cdots \otimes V}$$

d

$$\mathcal{V}^p_q = (V^*)^{\otimes p} \otimes V^{\otimes q}$$

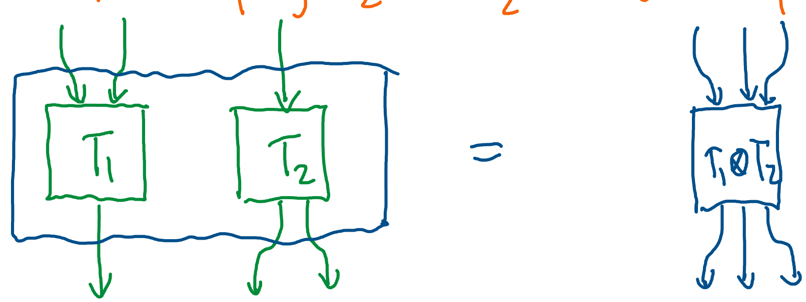
diagrams: $T \in \mathcal{V}^3_4$:



we can construct new tensors from old ones:

tensor product: if $T_1 \in \mathcal{V}^{p_1}_{q_1}$, $T_2 \in \mathcal{V}^{p_2}_{q_2}$ then $T_1 \otimes T_2 \in \mathcal{V}^{p_1+p_2}_{q_1+q_2}$

e.g., if $T_1 \in \mathcal{V}^2_1$, $T_2 \in \mathcal{V}^1_2$ then $T_1 \otimes T_2 \in \mathcal{V}^3_3$:

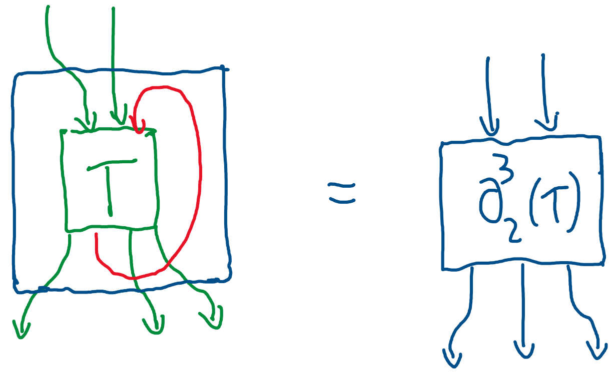


contraction: $\partial^i_j : \mathcal{V}^p_q \to \mathcal{V}^{p-1}_{q-1}$

contract i-th copy V^* with j-th copy V in $\mathcal{V}^p_q$

e.g., $T \in \mathcal{V}^3_4$,

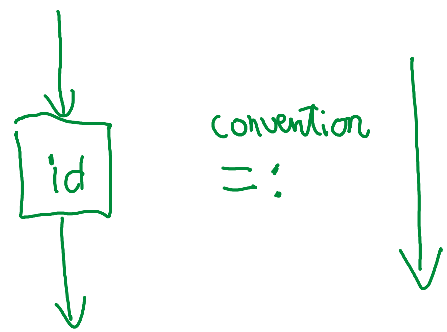
$\partial^3_2(T)$:



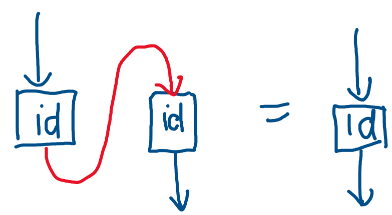
There are also some special tensors:

$1 \in \mathcal{V}^0_0 = K$

$\mathrm{id} \in \mathcal{V}^1_1 = V^* \otimes V \cong \mathrm{End}\, V$



(note:



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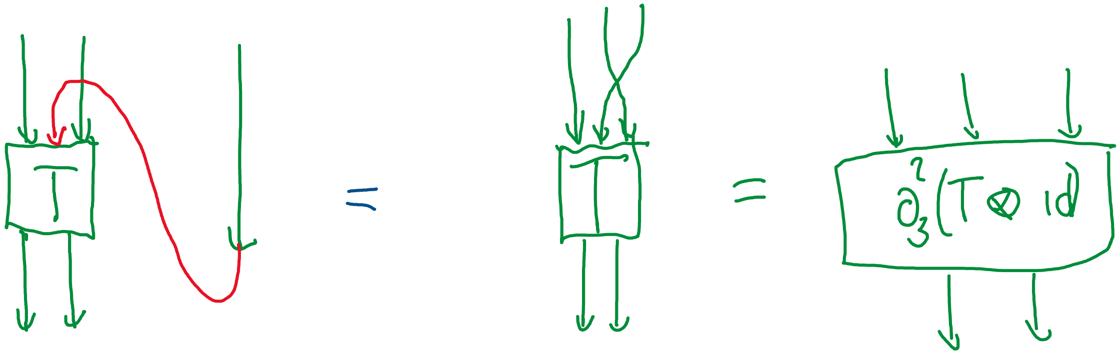
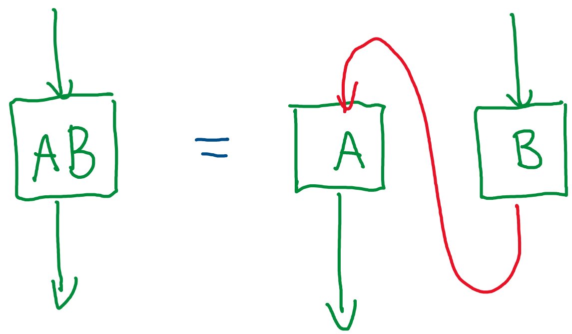
one can also permute the inputs and outputs this can be achieved using $\otimes$ and contractions, e.g.



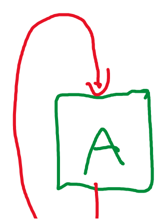
more examples:

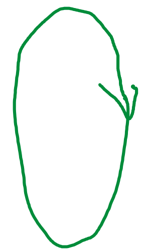
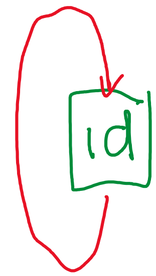
if $A, B \in \mathrm{End}\, V = V^1_1$, then AB corresponds to



$= \partial^1_2(A\otimes B)$

Trace of A is



In particular,  $\asymp$ 

$= \mathrm{Tr}(\mathrm{id}) = n = \dim V$

$GL(V)$ acts on $\mathcal{V}^p_q$

$1 \in \mathcal{V}^0_0 = K$ and $\mathrm{id} \in \mathcal{V}^1_1 = \mathrm{End}\,V$ are invariant

operations $\otimes$, ∂^i_j are $GL(V)$-equivariant

(i.e., operations commute with $GL(V)$-action)

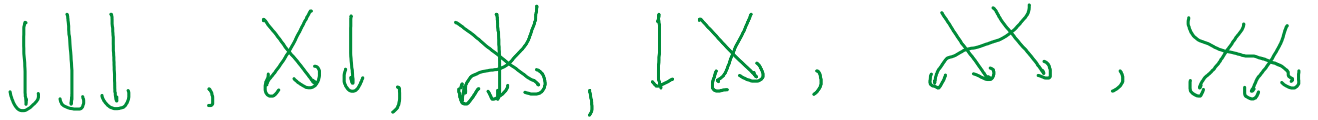
if $G \subseteq GL(V)$ subgroup, then $(\mathcal{V}^p_q)^G = \{ T \in \mathcal{V}^p_q \mid g \cdot T = T \text{ for all } g \in G \}$

First Fundamental Theorem of Invariant Theory (for $GL(V)$):

- if $p \neq q$ then $(\mathcal{V}^p_q)^{GL(V)} = 0$
- if $p = q$ then $\mathcal{V}^p$ spanned by all permutations $\sigma \in S$

... spanned by all permutations of S_p.

e.g. $(\mathcal{V}_3^3)^{GL(V)}$ spanned by:



$\in S_3$

($\sigma \in S_p$ form basis of $(\mathcal{V}_p^p)^{GL(V)}$ if $p \leq n$)

$\mathcal{V} = \bigoplus_{p,q} \mathcal{V}_q^p$ is a bigraded assoc. algebra with multiplication $\otimes$ and unit $1 \in K = \mathcal{V}_0^0$. Together with $\mathrm{id} \in \mathcal{V}_1^1$ and contractions it has the structure of a wheeled prop.

Def. A wheeled PROP over K is a bigraded associative algebra $\mathcal{R} = \bigoplus_{p,q=0}^{\infty} \mathcal{R}_q^p$ with unit ($1 \in \mathcal{R}_0^0$) together with an element $\mathrm{id} \in \mathcal{R}_1^1$ and contraction maps $\partial_j^i : \mathcal{R}_q^p \to \mathcal{R}_{q-1}^{p-1}$ $(1 \leq i \leq p,\ 1 \leq j \leq q)$ satisfying certain axioms.

(wheeled PROPs were defined by Markl, Merkulov, Shadrin '09)

if $G \subseteq GL(V)$ subgroup, then $\mathcal{V}^G = \bigoplus_{p,q=0} (\mathcal{V}_q^p)^G$ is also a wheeled PROP (sub-wheeled PROP)

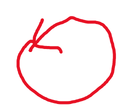
Theorem (Schrijver): suppose $K = \mathbb{C}$, V Hilbert space, let $* : \mathcal{V}_q^p \to \mathcal{V}_p^q$ be involution (from $V \cong V^*$) if $\mathcal{W} \subseteq \mathcal{V}$ is a sub-wheeled PROP closed under $*$ then $\mathcal{W} = \mathcal{V}^G$ for some compact subgroup $G \subseteq U(V)$.
unitary gp.

Thm (D.-Makam): char $K = 0$. if $\mathcal{W} \subseteq \mathcal{V}$ is a subwheeled PROP and the restriction of the pairing $\mathcal{V}_q^p \times \mathcal{V}_p^q \to K$ to $\mathcal{W}_q^p \times \mathcal{W}_p^q \to K$ is nondegenerate, then $\mathcal{W} = \mathcal{V}^G$ for some closed reductive subgroup $G \subseteq GL(V)$.

We can define homomorphisms, ideals, prime/maximal ideals etc. for wheeled PROP.
wheeled PROPS form a category

There is an initial object $\mathcal{Z}$ in category of wheeled PROPs.
For every wheeled PROP R there is a unique homomorphism of wheeled PROPs. $\mathcal{Z} \longrightarrow R$.
(the role of $\mathcal{Z}$ for wheeled PROPS is similar to role of $\mathbb{Z}$ for rings with 1)

$$\mathcal{Z} = \bigoplus_{p=0}^{\infty} \mathcal{Z}_p^p \qquad (\mathcal{Z}_q^p = 0 \text{ if } p \neq q)$$

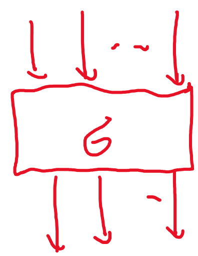
$\mathcal{Z}_0^0 = K[t]$, where $t =$  "formal dim"

$\mathcal{Z}_p^p \cong K[t]S_p$ group algebra of symmetric group S_p

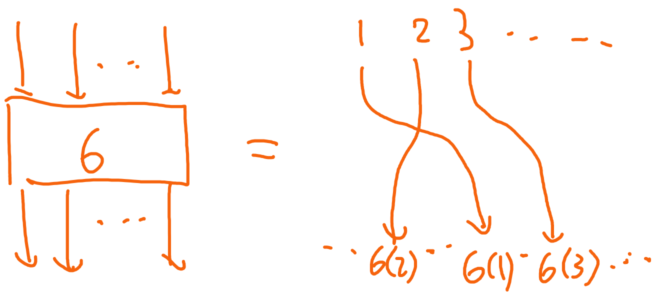
over $K[t]$.

D-Makam: complete classification of all ideals / prime ideals / max ideals

for example, the kernel of $Z \to V$ is ideal generated by:

$\bigcirc - n$, $\sum_{\sigma \in S_{n+1}} \operatorname{sgn} \sigma$ 

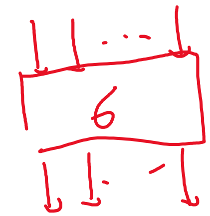
where



one may think of this as 2nd fundamental

Theorem of invariant theory.

other maximal ideals:

- ideal generated by $\circlearrowleft + p$, $\sum_{\sigma \in S_{p+1}}$ 

  "$(-p)$-dimensional vector space?"

- $\circlearrowleft - \alpha$ where $\alpha \in K \setminus \mathbb{Z}$

Thm (D.-Makam): suppose $\mathcal{P}$ is a wheeled PROP and the kernel of the (unique) homomorphism $\mathcal{Z} \to \mathcal{P}$ contains

$\circlearrowleft - n$, $\sum_{\sigma \in S_{n+1}} \operatorname{sgn} \sigma$ 

Then Γ is isomorphic to subalgebra of $A \otimes V$ where A is commutative, associative K-algebra.

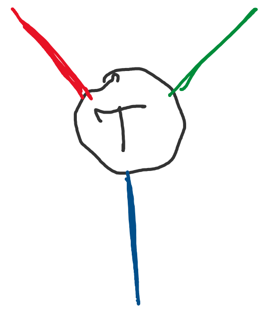
generalizations / extensions:

- One may start vector space V with nondegenerate bilinear form. Then $V \cong V^*$, "inputs = outputs" diagrams with with undirected edges instead of arrows

- May have more than 1 vector space to start with. $V_1, V_2, \ldots, V_d$. Use different colors to arrows (edges) corresponding to different vector spaces.

e.g. R, G, B $\mathbb{R}$-vector spaces with inner products

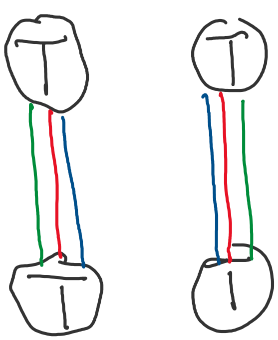
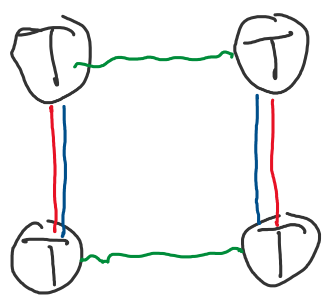
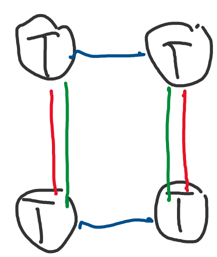
$T \in R \otimes G \otimes B$ 3-way tensor

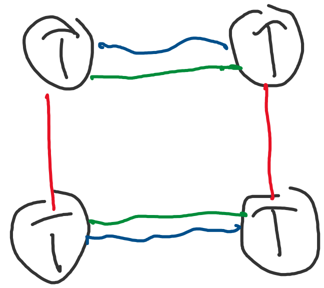
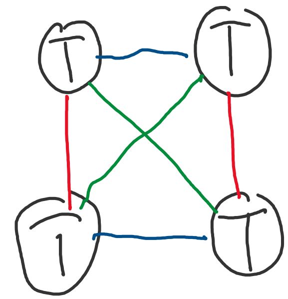


Tokcan-Gyak-Najarian-D.:

$$\left(\int_{\|u\|=\|v\|=\|w\|=1} \langle T, u \otimes v \otimes w \rangle^{2d} \, d\mu \right)^{\frac{1}{d}}$$ converges to spectral norm of T

$$\int_{\|u\|=\|v\|=\|w\|=1} \langle T, u \otimes v \otimes w \rangle^{4} =$$

 $+ 2$  $+ 2$ 

+2  + 2 

up to constant.