Invariant Tensors and Wheeled PROP

joint work with Visu Makam

5/20/2021

$k$ field, $\text{char } k = 0$, e.g., $k = \mathbb{R}$ or $\mathbb{C}$

$V$ $n$-dim $k$-vector space

$V \otimes^d = V \otimes V \otimes \ldots \otimes V$
\[ \mathcal{V}_q^p = (V^*)^p \otimes V^q \]

diagrams: \( T \in \mathcal{V}_4^3 \):

we can construct new tensors from old ones:

tensor product: if \( T_1 \in \mathcal{V}_{q_1}^{p_1} \), \( T_2 \in \mathcal{V}_{q_2}^{p_2} \) then \( T_1 \otimes T_2 \in \mathcal{V}_{q_1+q_2}^{p_1+p_2} \)

e.g., if \( T_1 \in \mathcal{V}_2^1 \), \( T_2 \in \mathcal{V}_2^1 \) then \( T_1 \otimes T_2 \in \mathcal{V}_3^3 \):

\[ \Rightarrow \]

\[ T_1 \quad T_2 \]

\[ \Rightarrow \]

\[ T_1 \otimes T_2 \]

contraction: \( \delta^i_i : \mathcal{V}_0^p \rightarrow \mathcal{V}_{0-1}^{p-1} \)
contract $i$-th copy $V_i^*$ with $j$-th copy $V$ in $V_q^p$ e.g., $T \in U_q^3$,

$\partial_2^3(T)$:

There are also some special tensors:

$1 \in U_0^0 = K$

$id \in U_1^1 = V^* \otimes V \cong \text{End} V$

(note: $id \circ id = id$)

one can also permute the inputs and outputs
This can be achieved using \( \otimes \) and contractions, e.g.

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
T
\end{array}
\end{array}
\end{align*}
\begin{array}{c}
\begin{array}{c}
\otimes
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
T
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\otimes
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{id}
\end{array}
\end{array}
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\partial^2_3 (T \otimes \text{id})
\end{array}
\end{array}
\end{align*}

More examples:

If \( A, B \in \text{End} V \otimes V' \), then \( AB \) corresponds to

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
AB
\end{array}
\end{array}
\end{align*}
\begin{array}{c}
\begin{array}{c}
\text{id}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\partial^1_2 (A \otimes B)
\end{array}
\end{array}
\end{align*}

Trace of \( A \) is

\[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{id}
\end{array}
\end{array}
\end{align*}
In particular, \[ \epsilon = \epsilon \circ \text{id} = \text{Tr}(\text{id}) = n = \dim V \]

**GL(V)** acts on \( V_q^p \)

\[ 1 \in V_0^q = K \text{ and } \text{id} \in \mathcal{V}_1 = \text{End} V \text{ are invariant operations} \quad \epsilon, \quad \circ \text{ are GL(V)-equivariant (i.e., operations commute with GL(V)-action)} \]

If \( G \subseteq \text{GL}(V) \) subgroup, then \( (V_q^p)^G = \{ T \in V_q^p \mid g \cdot T = T \text{ for all } g \in G \} \)

**First Fundamental Theorem of Invariant Theory** (for \( \text{GL}(V) \)):

- if \( p \neq q \) then \( (V_q^p)^{\text{GL}(V)} = 0 \)
- if \( p = q \) then \( V_q \) spanned by all permutations of \( q \)
e.g. $(\mathbb{R}^3)^{GL(V)}$ spanned by:

\begin{align*}
&111, x1, xh, 1x, xh, xh \in S_3 \\
&\text{(se}_{p} \text{ form basis of } (\mathbb{R}^p)^{GL(V)} \text{ if } p \leq n)}
\end{align*}

\[ V = \bigoplus_{p,q} V^p_q \] is a bigraded assoc. algebra with multiplication $\otimes$ and unit $1 \in K = V^0_0$. Together with id \in V^1_0 and contractions it has the structure of a wheeled prop.

**Def.** A wheeled PROP over $K$ is a bigraded associative algebra $R = \bigoplus_{p,q} R^p_q$ with unit (1 \in R^0_0) together with an element id \in R^1_0 and contraction maps

\[ e_j : R^p_q \rightarrow R^p_{q-1} \quad (1 \leq i \leq p, 1 \leq j \leq q) \] satisfying certain axioms.

https://northeastern-my.sharepoint.com/personal/hderksen_northeastern_edu/_layouts/15/Doc.aspx?sourcedoc={71cc6cb7-f584-4ad4-8df4-8e98bb9ff4e0}&action=edit&wd=target%28New Section 1.0... 6/13
if \( G \leq \text{GL}(V) \) subgroup, then \( \mathcal{U}^G = \bigoplus_{p,q} \mathcal{U}_{p,q}^G \) is also a wheeled PROP (sub-wheeled PROP)

**Theorem (Schrijver):** suppose \( K = \mathbb{C} \), \( V \) Hilbert space, let \( \ast : \mathcal{U}_q^p \to \mathcal{U}_q^p \) be involution (from \( V = V^* \)) if \( W \leq \mathcal{U} \) is a sub-wheeled PROP closed under \( \ast \) then \( W = \mathcal{U}^G \) for some compact subgroup \( G \leq \text{GL}(V) \).

**Thm (D.-Makam):** char \( K = 0 \). if \( W \leq \mathcal{U} \) is a subwheeled PROP and the restriction of the pairing \( \mathcal{U}_q^p \times \mathcal{U}_q^p \to K \) to \( W_q^p \times W_q^p \to K \) is nondegenerate, then \( W = \mathcal{U}^G \) for some closed reductive subgroup \( G \leq \text{GL}(V) \).
We can define homomorphisms, ideals, prime/maximal ideals etc. for wheeled PROP.
Wheeled PROPs form a category.

There is an initial object $\mathbb{Z}$ in category of wheeled PROPs.
For every wheeled PROP $R$ there is a unique homomorphism of wheeled PROPs $\mathbb{Z} \rightarrow R$.
( the role of $\mathbb{Z}$ for wheeled PROPs is similar to role of $\mathbb{Z}$ for rings with 1)

$\mathbb{Z} = \bigoplus_{p>0} \mathbb{Z}_p \quad (\mathbb{Z}_q = 0 \text{ if } p \neq q)$

$\mathbb{Z}_0 = K[t]$ , where $t = \ominus$ "formal dim"

$\mathbb{Z}_p \cong K[t]S_p$争取 algebra of symmetric group $S_p$. 
D-Makam: complete classification of all ideals / prime ideals / max ideals

for example, the kernel of $\mathbb{Z} \to V$ is ideal generated by:

$$\bigoplus_{n \in \mathbb{N}} \mathbb{Z}^n$$

where

$$\begin{array}{c}
\begin{array}{c}
6 \\
6 \\
\vdots
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\vdots
\end{array}
\end{array}$$

one may think of this as 2nd fundamental
Theorem of invariant theory.

Other maximal ideals:

- Ideal generated by $O + P$, $\sum_{6 \in S_{m+1}}$.

- $O - a$ where $a \in K \setminus \mathbb{Z}$

Thm (D. - Mahan): Suppose $P$ is a wheeled PROP and the kernel of the (unique) homomorphism $\mathbb{Z} \rightarrow P$ contains $O - n$, $\sum_{6 \in S_{m+1}} \text{sgn}(6)$.
Then $P$ is isomorphic to subalgebra of $A \otimes V$ where $A$ is commutative, associative $K$-algebra.

Generalizations/Extensions:

- One may start vector space $V$ with nondegenerate bilinear form. Then $V \cong V^*$, "inputs = outputs" diagrams with undirected edges instead of arrows.

- May have more than 1 vector space to start with, $V_1, V_2, \ldots, V_d$. Use different colors to arrows (edges) corresponding to different vector spaces.

  e.g. $R, G, B$ $\mathbb{R}$-vector spaces with inner products $T \in R \otimes G \otimes B$ 3-way tensor.
Tokman-Guyak-Najemian-D:

\[
\left( \int \langle T, \varphi \psi \varphi \psi \rangle^2 \, d\mu \right)^{1/2} \text{ converges to spectral norm of } T
\]

\[
\int \langle T, \varphi \psi \varphi \psi \rangle^4 = \]

\[
\|\psi\| = \|\varphi\| = \|\varphi\|
\]

\[
= \]

\[
\begin{align*}
&\begin{array}{c}
\text{T} \\
\text{T}
\end{array}
+ 2
\begin{array}{c}
\text{T} \\
\text{T}
\end{array}
+ 2
\end{align*}
\]
\[
+2 \quad \begin{array}{c}
\text{Diagram 1}
\end{array} \quad \begin{array}{c}
\text{Diagram 2}
\end{array} \quad + \quad 2
\]

up to constant.