Asymptotics of Ranks of Tensors

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Definition

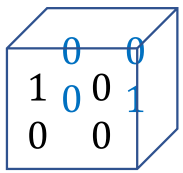
- pick field F I usually pick the complex numbers
- pick an order k I usually pick k=2 (matrix case) or k=3
- pick number d not so relevant as long as large enough

- A **tensor t** is an element in $F^d \otimes F^d \otimes \ldots \otimes \mathrm{F}^d$ (k factors)

- k=2 matrix case

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

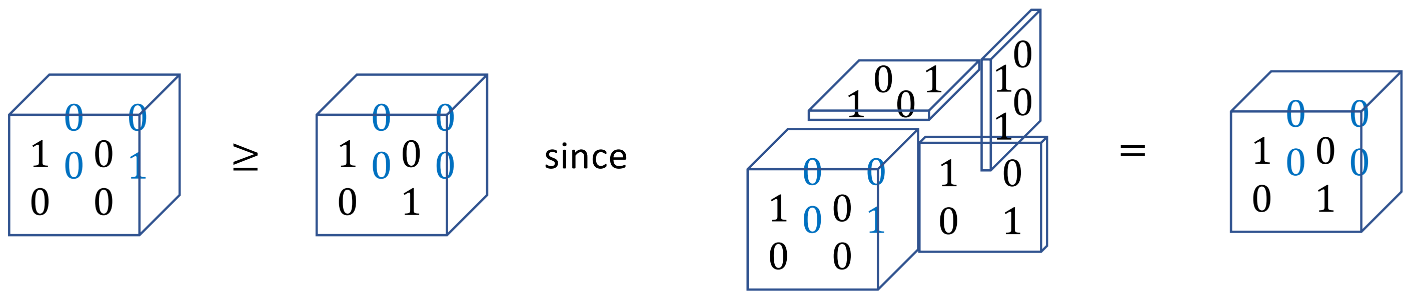
k=3



Restriction

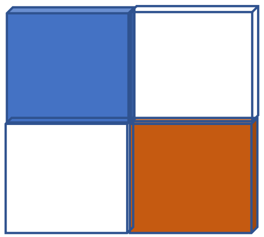
- Pick **t** in $F^d \otimes F^d \otimes \ldots \otimes \mathrm{F}^d$ and **t'** in $F^{d'} \otimes F^{d'} \otimes \ldots \otimes \mathrm{F}^{d'}$ (k factors)
- We say that **t restricts to t' ($\boldsymbol{t} \geq \boldsymbol{t}'$)**
  if there are matrices: $a_1, a_2, \ldots, a_k$ s.th. $(a_1 \otimes \mathrm{a}_2 \ldots \otimes \mathrm{a}_\mathrm{k})t = t'$

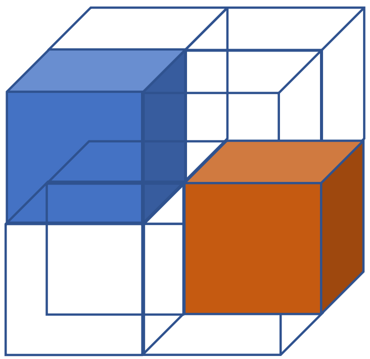
$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \geq \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \text{ since } \underbrace{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}}_{a_1}\underbrace{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}_{t}\underbrace{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}_{a_2} = \underbrace{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}}_{t'}$$



Direct sum

- $t \oplus t'$ in $(F^d \oplus F^{d'}) \otimes (F^d \oplus F^{d'}) \otimes \ldots \otimes (F^d \oplus F^{d'})$

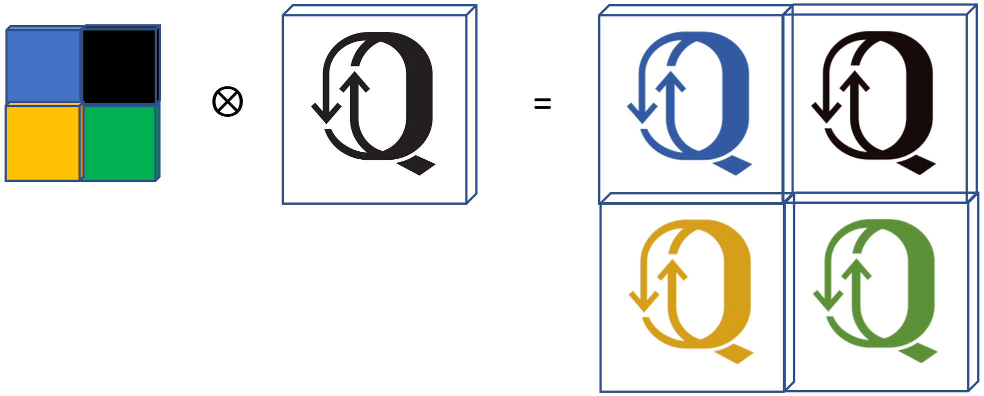


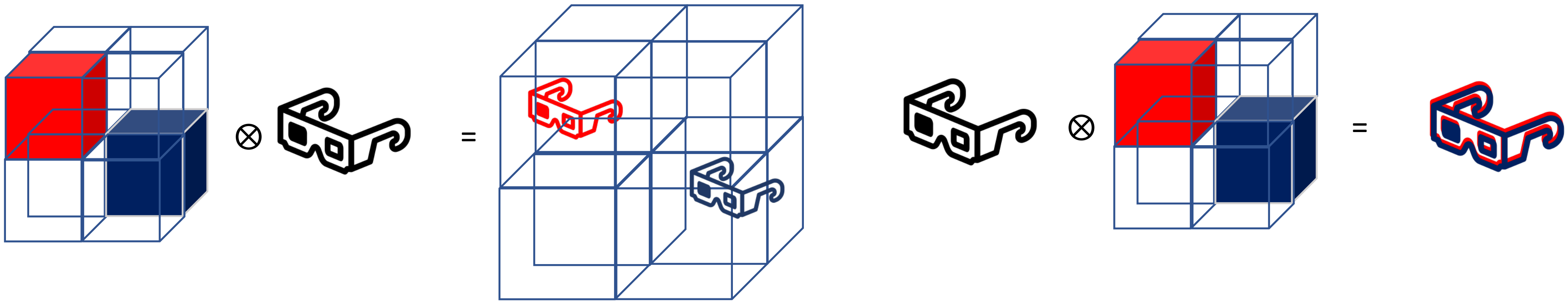


<d>:=unit tensor size d=diagonal size d

Tensor product

- $t \otimes t'$ in $(F^d \otimes F^{d'}) \otimes (F^d \otimes F^{d'}) \otimes \ldots \otimes (F^d \otimes F^{d'})$

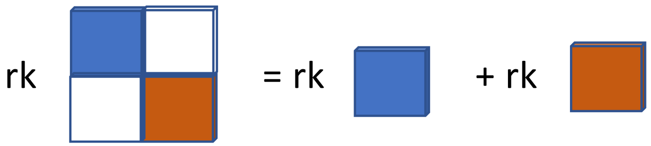




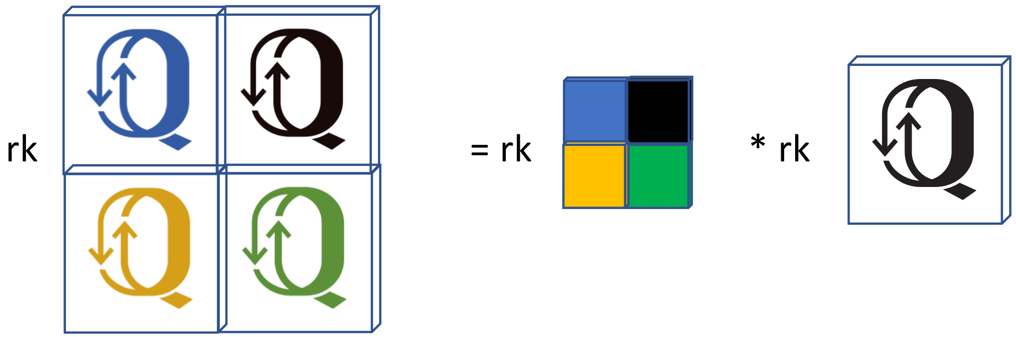
Matrix rank

- monotone, i.e.
  non-increasing under restriction $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \geq \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, since $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

- additive
  under direct sum



- multiplicative
  under tensor product



- normalized

rk $\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}$ =d

Rank zoo

tensor rank

$$R(t) \coloneqq \min\{r\colon \langle r\rangle \geq t\}$$
$$= \min\left\{r\colon\ t = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i\right\}$$

Your favorite animal

smoothable rank

border rank

cactus rank

max flattening rank

Waring rank
(symmetric rank)

min flattening rank

slice rank

analytic rank

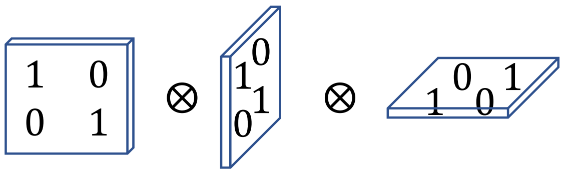
geometric rank

partition rank

$$\mathrm{Q}(t) \coloneqq max\{r\colon t \geq \langle r\rangle\}$$

subrank

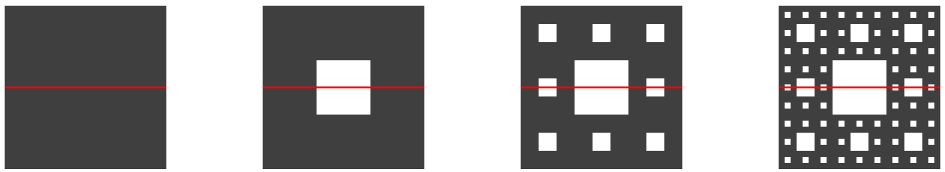
- What do they have in common?
  - monotone, normalized, but not multiplicative under tensor product



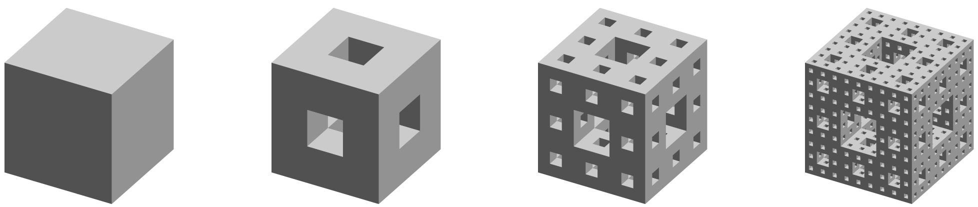
= 2x2 matrix multiplication tensor (tensor rank=7<8, Strassen ‘69)

Iterating the tensor product

- $1 \quad \to \quad t \quad \to \quad t^{\otimes 2} \quad \to \quad t^{\otimes 3} \quad \to \cdots$
- Sierpinsky carpet



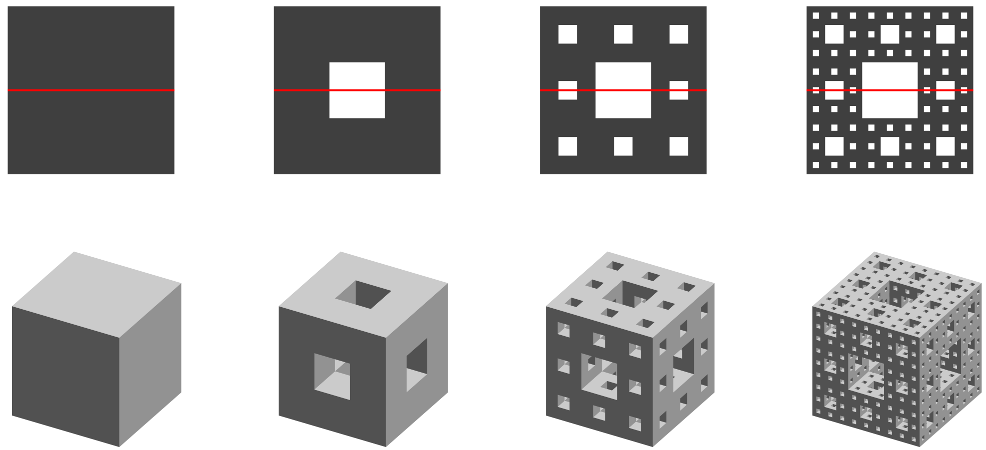
- Menger sponge



taken from Gelss and Schütte, https://arxiv.org/abs/1812.00814

Regularize/Amortize/Asymptotize

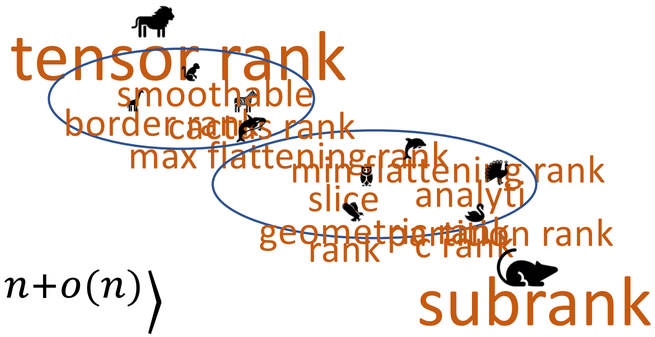
- $f: tensors \to F$
- $\tilde{f}(t) := \lim_{n\to\infty} f\left(t^{\otimes n}\right)^{\frac{1}{n}}$
- $\tilde{f}$ measures growth under tensor product
- f multiplicative→ $\tilde{f} = f$
  - ex: matrix case ☺
  - converse not true



Folkore

- Theorem: Let f monotone, normalized and $\tilde{f}$ exists. Then

$$\tilde{R} \geq \tilde{f} \geq \tilde{Q}$$



- Proof:
  - By definition
$$\langle \tilde{R}(t)^{n+o(n)} \rangle \geq t^{\otimes n} \geq \langle \tilde{Q}(t)^{n+o(n)} \rangle$$
  - Apply f. Due to monotonicity
$$f(\langle \tilde{R}(t)^{n+o(n)} \rangle) \geq f(t^{\otimes n}) \geq f(\langle \tilde{Q}(t)^{n+o(n)} \rangle)$$
  - Due to normalization
$$\tilde{R}(t)^{n+o(n)} \geq f(t^{\otimes n}) \geq \tilde{Q}(t)^{n+o(n)}$$

qed

Are there multiplicative functions?

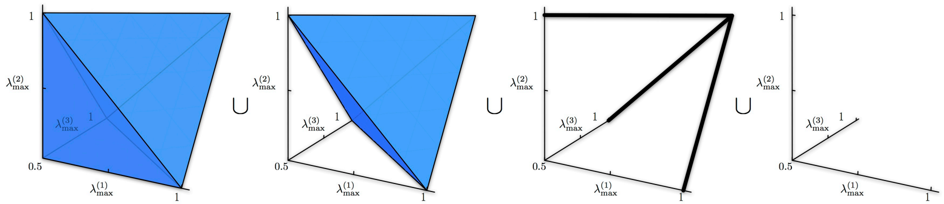
- Theorem (Strassen):
  - There are normalized, multiplicative functions F, s.th.

$$\tilde{R} = \max F \text{ and } \tilde{Q} = \min F$$

  - Furthermore
    - the F's are additive
    - they determine asymptotic restriction

$$\text{If } F(t) \geq F(t')\forall F, then\ t\ \geq t'$$

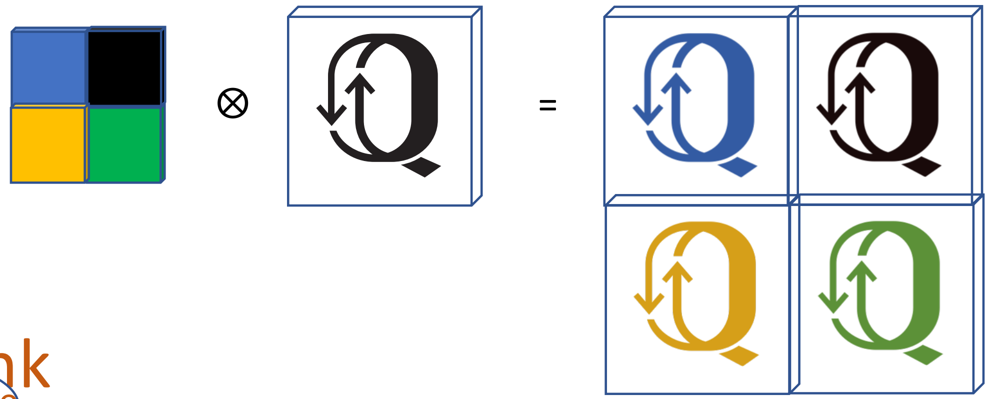
Some results



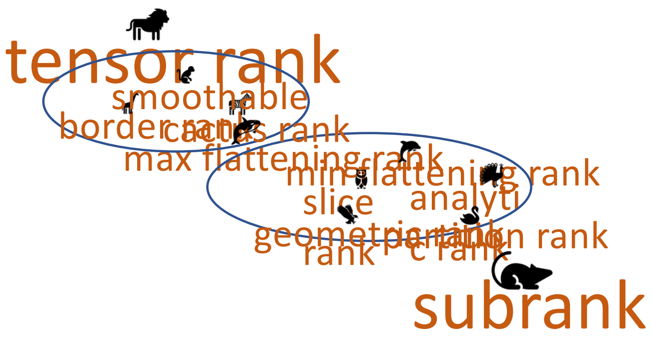
- Quantum functionals
  - Strassen F’s (STOC’18)
- Weighted slice rank
  - = Legendre of Quantum functionals
  - https://arxiv.org/abs/2012.14412
- Asymptotic symmetric subrank (of symmetric tensors)
  - = symptotic subrank
  - barrier for cap-set like problems
  - https://arxiv.org/abs/2104.01130

Summary

- QMATH – hiring!



- Zoo



- Asymptotic ranks (cool results ☺)