Minrank: On the complexity of orbit closures

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and “Variety Membership Testing, Algebraic Natural Proofs, and Geometric Complexity Theory”
with Bläser, Lysikov, Pandey, Schreyer
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1. Tensor rank and border rank
2. Minrank
3. Quadrics and NP-hardness of Minrank
4. Orbit closures
5. Stabilizers and Multiplicities
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A matrix $A$ has rank $\leq r$ iff all $(r + 1) \times (r + 1)$ minors of $A$ vanish.

Given a sequence of matrices $A_1, A_2, A_3, \ldots$ in $\mathbb{C}^{n \times n}$ with $\lim_{i \to \infty} A_i = A$ and $\forall i : \text{rk}(A_i) \leq r$. Then by the continuity of the minors we have $\text{rk}(A) \leq r$.

**Definition (tensor rank)**

A tensor $t \in U \otimes V \otimes W$ has rank 1 (also called decomposable) iff there exist vectors $u \in U, v \in V, w \in W$ with $t = u \otimes v \otimes w$.

The tensor rank $R(t)$ is the smallest $r$ such that $t$ can be written as a sum of $r$ rank 1 tensors.

$$(e_1 + \frac{1}{i}e_2)^{\otimes 3} = e_1^{\otimes 3} + \frac{1}{i}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) + \frac{1}{i^2}(\ldots) + \frac{1}{i^3}(\ldots)$$

Hence $\lim_{i \to \infty} i\left((e_1 + \frac{1}{i}e_2)^{\otimes 3} - e_1^{\otimes 3}\right) = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$

$e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$ is a rank 3 tensor that is a limit of rank 2 tensors.

**Definition (border rank)**

The border rank $R(t)$ is the smallest $r$ such that $t$ can be approximated arbitrarily closely by tensors of rank $\leq r$.

Example: $R(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) = 2$

Given $t$ and $r$, it is NP-hard to decide whether or not $R(t) \leq r$ [Håstad 1990].

**Open Question**

Given $t$ and $r$, how hard is it to decide $R(t) \leq r$?
Minrank

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The Minrank problem

Given a tensor $t \in U \otimes V \otimes W$, the **minrank** of $t$ is defined as the smallest rank $r$ of any matrix that can be obtained as a nontrivial linear combination of the $\dim U$ many slice matrices of format $\dim V \times \dim W$. In other words,

$$\text{minrank}(t) = \min_{0 \neq \varphi \in U^*} \{\text{rk}(\varphi t)\}.$$ 

Example: Let $U = V = W = \mathbb{C}^2$. Let $t$ be the $2 \times 2 \times 2$ tensor with slices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We have $\text{rk}(A_1 + iA_2) = 1$, so $\text{minrank}(t) = 1$. (Remark: over the reals the minrank is 2.)

- We can define **border minrank** analogously to border rank:
  The border minrank of $t$, denoted $\overline{\text{minrank}}(t)$, is the smallest $r$ such that $t$ can be approximated arbitrarily closely by tensors of minrank $\leq r$.
- We will see now that $\overline{\text{minrank}}(t) = \text{minrank}(t)$.
Recall: \( \text{minrank}(t) = \min_{0 \neq \varphi \in U^*} \{ \text{rk}(\varphi t) \} \).

**Proposition: Minrank equals border minrank**

For all \( t \) we have \( \text{minrank}(t) = \text{minrank}(t) \).

**Proof:** We mimic [Sawin-Tao, 2016] on slice rank: We write the minrank-variety as a projection along a projective variety.

Define the variety

\[
P \mathcal{X}_{U \otimes V \otimes W, r} = \{ ([t], [\varphi]) \in \mathbb{P}(U \otimes V \otimes W) \times \mathbb{P}U^* \mid \text{rk}(\varphi t) \leq r \} \subset \mathbb{P}(U \otimes V \otimes W) \times \mathbb{P}U^*.
\]

Let \( \pi : \mathbb{P}(U \otimes V \otimes W) \times \mathbb{P}U^* \to \mathbb{P}(U \otimes V \otimes W) \) be the projection onto the first component of the product.

\[
\pi P \mathcal{X}_{U \otimes V \otimes W, r} = \{ [t] \in \mathbb{P}(U \otimes V \otimes W) \mid \exists \varphi \neq 0 : \text{rk}(\varphi t) \leq r \}.
\]

As an image of a projective variety, it is a closed subvariety of \( \mathbb{P}(U \otimes V \otimes W) \) (see e. g. Shafarevich). The affine cone over this subvariety is therefore also closed. This affine cone is exactly the set of tensors of minrank \( \leq r \).
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**MinRank1**

Given a tensor $t$ with entries from $\mathbb{Q}$, the MinRank1 problem is to decide if $\text{minrank}(t) \leq 1$.

**Quad**

Given a set of quadratic forms with coefficients from $\mathbb{Q}$, the Quad problem is to decide if they have a common solution.

**Theorem**

Quad is NP-hard, even if all coefficients are from $\{-1, 0, 1\}$.

**Proof:**

We reduce from graph 3-colorability. Let $G = (V, E)$ be a graph.

- Two variables for each vertex: $x_v$ and $y_v$. One global additional variable $z$.
- Three equations for each vertex: $x_v y_v = 0$, $x_v^2 - x_v z = 0$, $y_v^2 - y_v z = 0$
- One equation for each edge $(v, w)$: $x_v^2 + y_v^2 + x_w^2 + y_w^2 - x_v y_w - x_w y_w - z^2 = 0$

If $z = 0$, there is only the trivial solution. Assume $z \neq 0$, rescale $z = 1$.

When $z = 1$, the vertex equations give three possibilities: $(x_v, y_v) \in \{(0, 0), (0, 1), (1, 0)\}$.

The edge equation on these inputs is satisfied iff $(x_v, y_v) \neq (x_w, y_w)$.

Hence, for $z = 1$ we get a 1:1 correspondence between solutions and 3-colorings. □
MinRank1

Given a tensor $t$ with entries from $\mathbb{Q}$, the MinRank1 problem is to decide if $\minrank(t) \leq 1$.

Quad (NP-hard)

Given a set of quadratic forms with coefficients from $\mathbb{Q}$, the Quad problem is to decide if they have a common solution.

Theorem

MinRank1 and Quad are polynomial-time equivalent. Hence MinRank1 is NP-hard.

Proof: One direction is clear, because MinRank1 is decided by the vanishing of $2 \times 2$ minors. Other direction:

Given $k$ quadrics on $\mathbb{C}^n$. Each quadric $q(x) = \sum_{i,j} a_{ij} x_i x_j$ corresponds to the linear map $Q(x) = \sum_{i,j} a_{ij} x_{i,j}$ on $\text{Sym}^2 \mathbb{C}^n$. We have $q(x) = 0$ iff $Q(x \otimes x) = 0$.

$k$ linear forms give a linear map $L : \text{Sym}^2 \mathbb{C}^n \rightarrow \mathbb{C}^k$ with $x \otimes x \in \ker(L)$ iff $x$ is a common zero of the quadrics.

Compute $\ker(L) = \langle A_1, \ldots, A_m \rangle$ with linearly independent symmetric matrices $A_i$.

Set $t = \sum_{i=1}^m e_i \otimes A_i$.

If $x$ is a nontrivial zero of the quadrics, then $x \otimes x \in \ker L$, hence $x \otimes x = \sum_{i=1}^m y_i A_i$, so $\minrank(t) \leq 1$.

If $\minrank(t) = 1$, then $\exists y : \sum_{i=1}^m y_i A_i$ is symmetric and rank 1, so $\sum_{i=1}^m y_i A_i = x \otimes x$, hence $x$ is a common 0.

If $\minrank(t) = 0$ is impossible, because the $A_i$ are linearly independent. \qed
NP-hard:

**MinRank1**

Given a tensor $t$ with entries from $\mathbb{Q}$, the MinRank1 problem is to decide if $\text{minrank}(t) \leq 1$.

This makes determining lower bounds coNP-hard:

**MinRank>1**

Given a tensor $t$ with entries from $\mathbb{Q}$, the MinRank>1 problem is to decide if $\text{minrank}(t) > 1$.

If $\text{NP} \neq \text{coNP}$, then MinRank>1 is not in $\text{NP}$, i.e., there are no polytime-verifiable minrank lower bound proofs.
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• $U = \mathbb{C}^k$, $V = W = \mathbb{C}^n$.
• Embed $\mathbb{C}^n \hookrightarrow \mathbb{C}^s$ with $s = n(k - 1) + r$

$$t_{k,n,r} := e_1 \otimes \left( \sum_{j=1}^{r} e_{1j} \otimes e_{1j} \right) + \sum_{i=2}^{k} e_i \otimes \left( \sum_{j=1}^{n} e_{ij} \otimes e_{ij} \right) \in \mathbb{C}^k \otimes \mathbb{C}^s \otimes \mathbb{C}^s$$

Block diagonal, with each block on its own slice.

Group action of $G := \text{GL}(U) \times \text{GL}(V) \times \text{GL}(W)$ on $U \otimes V \otimes W$: $(g_1, g_2, g_3)(u \otimes v \otimes w) := (g_1u) \otimes (g_2v) \otimes (g_3w)$.

**Proposition**

For $t \in U \otimes V \otimes W$ we have $\text{minrank}(t) \leq r$ iff $t \in (\text{GL}_k \times \text{GL}_s \times \text{GL}_s)t_{k,n,r}$.

(note the similarity of this formulation to the null cone problem)

**Corollary**

For tensors of order $\geq 3$ and for the standard action of $\text{GL} \times \text{GL} \times \cdots \times \text{GL}$ it is NP-hard to decide if a tensor lies in the orbit closure of another tensor.
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[Mulmuley-Sohoni 2001, 2008] introduce the method of \textbf{multiplicity obstructions} that can sometimes be used to prove that a point does not lie in an orbit closure.

[B¨urgisser-I 2011, 2013] use this method to prove modest lower bounds for the border rank of the matrix multiplication tensor.

This method can also be used for lower bounds on minrank.

\[
t_{k,n,r} := e_1 \otimes \left( \sum_{j=1}^{r} e_{1j} \otimes e_{1j} \right) + \sum_{i=2}^{k} e_i \otimes \left( \sum_{j=1}^{n} e_{ij} \otimes e_{ij} \right) \in \mathbb{C}^k \otimes \mathbb{C}^s \otimes \mathbb{C}^s
\]

\[
G := \text{GL}_k \times \text{GL}_s \times \text{GL}_s
\]

The stabilizer \( \text{stab}_{k,n,r} \) of \( t_{k,n,r} \) is isomorphic to \((\text{GL}_r \times \mathbb{C}^\times) \times (\text{GL}_n \times \mathbb{C}^\times)^{k-1} \rtimes \mathcal{G}_{k-1} \).

\[
G_{t_{k,n,r}} \simeq G/\text{stab}_{k,n,r}.
\]
Let \( G := \text{GL}_k \times \text{GL}_s \times \text{GL}_s \) and \( t_{k,n,r} := e_1 \otimes (\sum_{j=1}^r e_{1j} \otimes e_{1j}) + \sum_{i=2}^k e_i \otimes (\sum_{j=1}^n e_{ij} \otimes e_{ij}) \in \mathbb{C}^k \otimes \mathbb{C}^s \otimes \mathbb{C}^s \).

Let \( I(\overline{G t_{k,n,r}})_\delta \) denote the homogeneous degree \( \delta \) part of the vanishing ideal. It is a \( G \)-representation.

Let \( \text{mult}(\lambda,\mu,\nu) I(\overline{G t_{k,n,r}})_\delta \) denote the multiplicity of the irred. \( G \)-representation of type \((\lambda,\mu,\nu)\) in \( I(\overline{G t_{k,n,r}})_\delta \).

We mimic Mulmuley-Sohoni’s representation theoretic lower bounds method:

\[
\text{mult}(\lambda,\mu,\nu) I(\overline{G t_{k,n,r}})_\delta = k(\lambda,\mu,\nu) - \text{mult}(\lambda,\mu,\nu) \mathbb{C}[G t_{k,n,r}]_\delta \leq \text{mult}(\lambda,\mu,\nu) \mathbb{C}[G t_{k,n,r}]_\delta
\]

where \( k(\lambda,\mu,\nu) \) is the Kronecker coefficient.

So to find representations in \( I(\overline{G t_{k,n,r}})_\delta \) it suffices to find \((\lambda,\mu,\nu)\) with \( \text{mult}(\lambda,\mu,\nu) \mathbb{C}[G t_{k,n,r}]_\delta < k(\lambda,\mu,\nu) \).

\[
\text{mult}(\lambda,\mu,\nu) \mathbb{C}[G t_{k,n,r}] = \text{mult}(\lambda,\mu,\nu) \mathbb{C}[G/\text{stab}_{k,n,r}] = \text{mult}(\lambda,\mu,\nu) \mathbb{C}[G]^{\text{stab}_{k,n,r}} =
\]

\[
\sum_{\xi \leq \lambda, \mu^1 \vdash \lambda - |\xi| \leq \mu^2, \ldots, \mu^k \vdash m} \sum_{\delta^1, \ldots, \delta^{k-1} \vdash \ell(\xi)} C^{\mu}_{\mu^1, \ldots, \mu^k} C^{\nu}_{\mu^1, \ldots, \mu^k} C^{\xi}_{\delta^1, \ldots, \delta^{k-1}} \prod_{i=1}^{|\xi|} a_{\delta_i}(\kappa_i, i),
\]

where the \( \kappa_i \) denotes the number of times \( i \) occurs in \((|\mu^2|, \ldots, |\mu^k|)\).

\[
\text{mult}(\lambda,\mu,\nu) \mathbb{C}[G t_{2,3,1}]_6 = 0 < 1 = k(\lambda,\mu,\nu) \quad \text{for} \quad (\lambda,\mu,\nu) \in \big\{ (3,3); (2,2,2), (3,3) \big\}, \big\{ ((3,3), (3,3), (2,2,2)), (3,3), (2,2,2), (4,1,1) \big\}, \big\{ (3,3), (4,1,1), (2,2,2) \big\}
\]

The only type we miss is \((3,3), (3,2,1), (3,2,1)\), where both multiplicities are 2.
Summary

- Deciding tensor rank $\leq r$ is NP-hard.

- Open question: Can border rank $\leq r$ be decided in polynomial time?

- Minrank = border minrank, and deciding minrank $\leq r$ is NP-hard. (also true for slice rank)

- Corollary: Detecting if a tensor is in the orbit closure of another tensor is NP-hard.

- If $\text{NP} \neq \text{coNP}$, then minrank lower bounds are not in NP.

- But still, representation theoretic multiplicities based on symmetries give equations for minrank. How far can this approach go in other settings such as geometric complexity theory?

Thank you for your attention! Questions/comments?