On the dimension of Tensor Network Varieties

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Efficient Tensor Representations for Learning and Computational Complexity

Overview

- Motivations
- Definition and Properties
- Dimension
- Further Questions

Motivations

From Quantum Physics

- Tensor space has high dimension: $\dim(V^{\otimes d}) = \dim(V_i)^d$. Quickly intractable. Requires too large memory to reprensent a tensor.
- Given a quantum many-body wave function, specifying its coefficients in a given local basis does not give any intuition about the structure of the *entanglement* between its constituents:

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$$

$$T = \sum_{i,j,k=1}^{d} t_{i,j,k} e_i \otimes e_j \otimes e_k$$

with $\{e_l\}$ orthonormal and $t_{i,j,k} \in \mathbb{R}_{>0}$

Motivations

A Tensor Network has this information directly available in its description in terms of a network of quantum correlations.

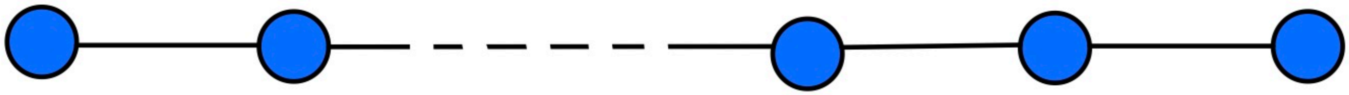
Matrix product $AB = C$: $\sum_{j=1}^{m} a_{i,j} b_{j,k} = (c_{i,k})_{i=1,\dots,n_1, k=1,\dots,n_2}$.

The network of correlations makes explicit the effective lattice geometry in which the state actually lives

A TN is a set of tensors where some, or all, indices are contracted according to some pattern.

Motivations

Matrix product states



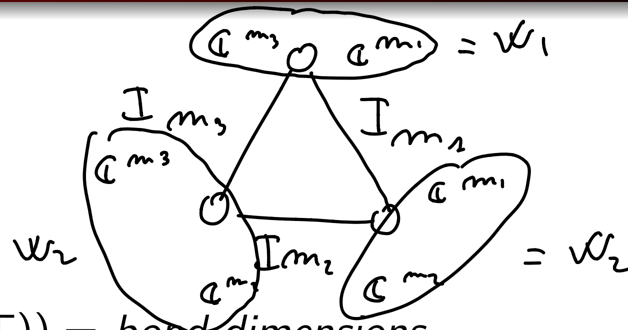
Reduced number of parameters

$$dm^2 \dim(V) << \dim(V)^d$$

MPS are accurate representations of physical states with limited bond length m.

Highlight entangled structure of state. The corresponding spaces of tensors are only locally entangled because interactions (entanglement) in the physical world appear to just happen locally.

Definition - Graph Tensor



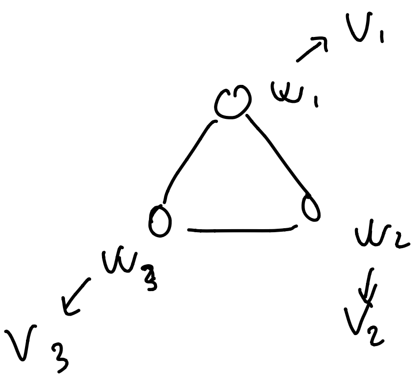
- Fix a graph $\Gamma(\mathrm{v}(\Gamma), \mathrm{e}(\Gamma))$
- Fix the weights $\mathrm{m} = (m_e, e \in \mathrm{e}(\Gamma)) =$ *bond dimensions*
- $d := \sharp v(\Gamma)$
- Consider $I_{m_e} \in \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e}$ at e
- Tensor them: $\bigotimes_{e \in e(\Gamma)} I_{m_e}$
- It naturally lives in $\bigotimes_{e \in e(\Gamma)} \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e}$ but we think it as an element of $\bigotimes_{v \in v(\Gamma)} (\bigotimes_{e \ni v} \mathbb{C}^{m_e}) := \bigotimes_{v \in v(\Gamma)} W_v$ obtained by grouping together the spaces incident at the same vertex:

$$T(\Gamma, \mathrm{m}) := \bigotimes_{e \in e(\Gamma)} I_{m_e} \in \bigotimes_{v \in v(\Gamma)} W_e$$

Definition - TNS

$TNS^{\Gamma}_{\mathrm{m},\mathrm{n}} \subset V_1 \otimes \cdots \otimes V_d$ associated to the tensor network $(\Gamma, \mathrm{m}, \mathrm{n})$

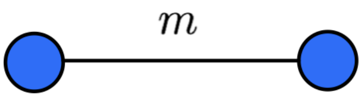
$$\begin{array}{rcl} \Phi : Hom(W_1, V_1) \times \cdots \times Hom(W_d, V_d) & \to & V_1 \otimes \cdots \otimes V_d \\ (X_1, \ldots, X_d) & \mapsto & (X_1 \otimes \cdots \otimes X_d)(T(\Gamma, \mathrm{m})) \end{array}$$



$$Im(\Phi) = TNS^{\Gamma,0}_{\mathrm{m},\mathrm{n}}$$

$$TNS^{\Gamma}_{\mathrm{m},\mathrm{n}} = \overline{Im(\Phi)} \subset V_1 \otimes \cdots \otimes V_d$$

Example Matrix multiplication



$$T(\Gamma, m) = I_m \in \mathbb{C}^m \otimes \mathbb{C}^m = W_1 \otimes W_2 \text{ Fix } V_1, V_2$$

$$\Phi : Hom(W_1, V_1) \times Hom(W_2, V_2) \to V_1 \otimes V_2$$

$$\Phi(X_1, X_2) = (X_1, X_2) \cdot I_m = (X_1, X_2) \cdot \sum_{i=1}^{m} e_i \otimes e_i = \sum_{i=1}^{m} X_1 e_i \otimes X_2 e_i =$$

$$= \sum_{i=1}^{m} X_1 e_i (X_2 e_i)^T = X_1 I_m X_2^T = X_1 X_2^T$$

In this case $TNS^{\Gamma}_{\mathrm{m,n}} = \{M \in V_1 \otimes V_2 : rank(M) \leq m\} = TNS^{\Gamma,0}_{\mathrm{m,n}}$

Why graph tensor is better

The multilinear multiplication is nothing but evaluation. Evaluating the graph tensor $T(\Gamma, m)$ is easier than evaluating other tensors.

- Given $T \in V_1 \otimes \cdots \otimes V_d$ and a graph Γ
- start with small m and evaluate $T(\Gamma, m)$: hope to find linear maps $X_1, \ldots, X_d$ s.t.

$$(X_1 \otimes \cdots \otimes X_d)(T(\Gamma, m)) = T$$

Properties

- One can assume that all $m_e > 1$, otherwise remove the edge from the graph.
- Monotonicity:

$$\text{If } m' \leq m \text{ (entry-wise) then } TNS^{\Gamma}_{m',n} \subseteq TNS^{\Gamma}_{m,n}$$

- Universality: If Γ is connected then

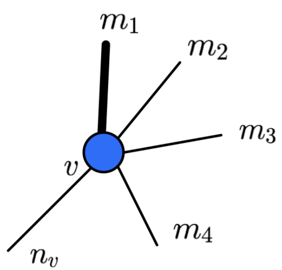
$$TNS^{\Gamma}_{m,n} = V_1 \otimes \cdots \otimes V_d$$

if m_e is large enough for every $e \in e(\Gamma)$.

Reductions

- We may assume all bond dimensions associated to the edges incident a fixed vertex are *balanced*: Fix a vertex v and $e_1, \ldots, e_k \in v$; If

$$m_{e_k} > n_v \cdot m_{e_1} \cdots m_{e_{k-1}}, \qquad m_{e_k} \text{ is } \textit{overabundant}$$



then

$$TNS_{\mathrm{m,n}} = TNS_{\overline{\mathrm{m}},\mathrm{n}}$$

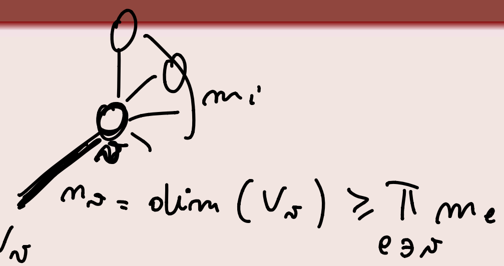
where $\overline{m}_e = m_e$ if $e \neq e_k$ and $\overline{m}_{e_k} = n_v \cdot m_1 \cdots m_{e_{k-1}}$.

Reductions

Definition (Landsberg-Qi-Ye '12)

A vertex $v \in \mathrm{v}$ is called

- *subcritical* if $\prod_{e \ni v} m_e \geq n_v$;
- *supercritical* if $\prod_{e \ni v} m_e \leq n_v$;
- *critical* if v is both subcritical and supercritical.



Theorem (BDG)

If the vertex v is supercritical let $N = \dim W_d = \prod_{e \ni d} m_e$ and $\mathrm{n}' = (n'_v : v \in \mathrm{v}(\Gamma))$ be the d-tuple of local dimensions s.t. $n'_v = n_v$ if $v \neq d$ and $n'_d = N$. Then

$$\dim TNS^{\Gamma}_{\mathrm{m},\mathrm{n}} = N(n_d - N) + \dim(TNS^{\Gamma}_{\mathrm{m},\mathrm{n}'})$$

$\in V_{v_1} \otimes \cdots \otimes V_{v_n}$

Studying the orbit of $T(\Gamma, m)$ does not say anything about tensors in $TNS_\Gamma(m,n) \setminus TNS^0_\Gamma(m,n)$.

Theorem (Landsberg-Qi-Ye '12)

- *If* Γ *doesn't have cycles, then* $TNS^0_\Gamma(m,n) = TNS_\Gamma(m,n)$
- *otherwise* $TNS_\Gamma(m,n) \setminus TNS^0_\Gamma(m,n) \neq \emptyset$

Dimension

If $f : X \to Y$ map between varieties, then

$$\dim(\overline{Im(f)}) = \dim X - \dim f^{-1}(y)$$

for y generic in $Im(f)$.

We study the fibers of

$$\Phi : \mathrm{Hom}(W_1, V_1) \times \cdots \times \mathrm{Hom}(W_d, V_d) \to V_1 \otimes \cdots \otimes V_d$$

$$(X_1, \ldots, X_d) \mapsto (X_1 \otimes \cdots \otimes X_d)(T(\Gamma, \mathrm{m}))$$

Obviously in the fiber

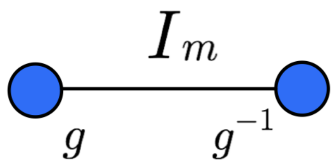
Ex: Matrix case

$$\Phi : Hom(\mathbb{C}^m, V_1) \times Hom(\mathbb{C}^m, V_2) \to V_1 \otimes V_2$$

with $\Phi(X_1, X_2) = X_1 I_m X_2^t$.

$\Phi(X_1, X_2) = \Phi(X_1 g, X_2 (g^{-1})^t)$ for every $g \in GL_m$.

The fiber containing (X_1, X_2) contains the entire GL_m-orbit.



The fiber containing $(X_v : v \in \mathrm{v}(\Gamma))$ contains its entire $\mathcal{G}_{\Gamma,\mathrm{m}}$-orbit, where

$$\mathcal{G}_{\Gamma,\mathrm{m}} = \times_{e \in \mathrm{e}(\Gamma)} GL_{m_e} \qquad \textit{gauge subgroup} \text{ of } \Gamma.$$

The role of this group in the theory of tensor network was known and it is expected that it entirely controls the value of dim TNS. In fact, it is **expected** that in "most" cases the exact value of the dimension is

$$\min\{\underbrace{\sum_v (n_v \times \prod_{e \ni v} m_e) - d + 1}_{\dim \times_v \mathbb{P}(Hom(W_v, V_v))} - \underbrace{\sum_e (m_e^2 - 1)}_{\dim \mathcal{G}_{\Gamma,m}}, \prod_v n_v\}$$

This computation does not take care of two facts:

- the possible existence of the stabilizer under the action of the gauge subgroup of a generic d-tuple of linear maps,
- there may be something else in the fiber.

Main theorem

Theorem (BDG'21)

$$\dim(TNS^{\Gamma}_{m,n}) \leq$$

$$\min\{\underbrace{\sum_{v}(n_v \times \prod_{e \ni v} m_e) - d + 1}_{\dim \times_v \mathbb{P}(Hom(W_v, V_v))} - (\underbrace{\sum_{e}(m_e^2 - 1)}_{\dim \mathcal{G}_{\Gamma,m}} - \underbrace{\dim Stab_{\mathcal{G}_{\Gamma,m}}(X)}_{??}), \prod_{v} n_v\}$$

Luckily...

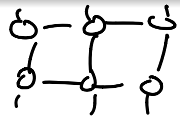
Theorem (Derksen-Makam-Walter'20)

$\dim(Stab_{\mathcal{G}_{\Gamma,\mathrm{m}}}(X)) = 0$ *in "most" cases*

(the action of $\mathcal{G}_{\Gamma,m}$ *on* $\times_v Hom(W_v, V_v)$ *is generically stable, i.e. there exists an element* v *in the parameter space s.t.* $Stab_G(v)$ *is a finite group).*

Two important ones:

- Γ *is a cycle, called matrix product states;* 
- Γ *is a grid, called projected entangled pair states.* PEPS



Theorem (Haegeman-Mariën-Osborne-Verstraete '14)

Matrix product states with open boundary conditions $(m_0 = m_d = 1)$

$$\dim TNS^{\Gamma}_{\mathrm{m},\mathrm{n}} = \min \left\{ \sum_{i=1}^{d} n_i m_{i-1} m_i - \sum_{j=1}^{d-1} m_j^2, \prod_{1}^{d} n_i \right\}$$

Main theorem

Theorem (BDG'21)

If $(\Gamma, \mathrm{m}, \mathrm{n})$ *is a subcritical tensor network with no overabundant bond dimension, then*

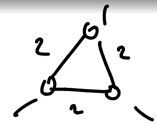
$$\dim(TNS^{\Gamma}_{m,n}) \leq$$

$$\min\{\underbrace{\sum_{v}(n_v \times \prod_{e \ni v} m_e) - d + 1}_{\dim \times_v \mathbb{P}(Hom(W_v, V_v))} - (\underbrace{\sum_{e}(m_e^2 - 1)}_{\dim \mathcal{G}_{\Gamma,m}} - \underbrace{\dim Stab_{\mathcal{G}_{\Gamma,m}}(X)}_{??}), \prod_{v} n_v\}$$

If $(\Gamma, \mathrm{m}, \mathrm{n})$ *is a supercritical case the the bound is sharp and* $\dim Stab_{\mathcal{G}_{\Gamma,m}}(X)) = 0$

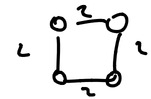
$$\dim(TNS^{\Gamma}_{m,n}) = \min\{\sum_{v}(n_v \times \prod_{e \ni v} m_e) - d + 1 - \sum_{e}(m_e^2 - 1), \prod_{v} n_v\}$$

$m = (2, 2, 2)$



	n	lower bound	upper bound
	$(2,2,2)$	8	8
	$(2,2,3)$	12	12
	$(2,2,4)$	16	16
	$(2,3,3)$	18	18
*	$(2,3,4)$	22	24
*	$(2,4,4)$	26	29
	$(3,3,3)$	25	25
	$(3,3,4)$	29	29
	$(3,4,4)$	31	31
	$(4,4,4)$	37	37

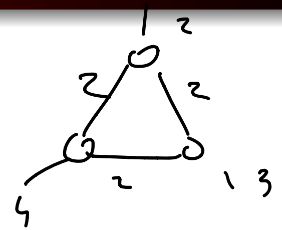
$m = (2, 2, 2, 2)$



	n	lower bound	upper bound
*	$(2,2,2,2)$	15	16
*	$(2,2,2,3)$	20	21
*	$(2,2,2,4)$	24	25
	$(2,2,3,3)$	25	25
	$(2,2,3,4)$	29	29
	$(2,2,4,4)$	33	33
*	$(2,3,2,3)$	24	25
*	$(2,3,2,4)$	28	29
	$(2,3,3,3)$	29	29
	$(2,3,3,4)$	33	33
	$(2,3,4,3)$	33	33
	$(2,3,4,4)$	37	37
*	$(2,4,2,4)$	32	33
	$(2,4,3,4)$	37	37
	$(2,4,4,4)$	41	41
	$(3,3,3,3)$	33	33
	$(3,3,3,4)$	37	37

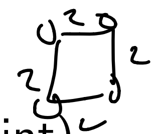
$m = (2,2,2),\ n = (2,3,4)$

- $T(\Gamma, \mathrm{m}) \in \mathbb{C}^{2\times 2} \otimes \mathbb{C}^{2\times 2} \otimes \mathbb{C}^{2\times 2}$
- $TNS_\Gamma(\mathrm{m},\mathrm{n}) \subseteq \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4)$.



Let $T \in \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$. Consider the flattening

$$T_1 : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^4.$$



Then $L_T = \mathbb{P}(Im(T_1))$ is a line in $\mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^4)$ (or a single point).

Theorem (BDG'21)

$T \in TNS_\Gamma(\mathrm{m},\mathrm{n})$ if and only if

- *either $rank(L_T) = 1$*
- *or L_T intersects $\{A : rank(A) \leq 2\}$ in at least two points (counted with multiplicity).*

In particular $\dim TNS_\Gamma(\mathrm{m},\mathrm{n}) \leq (=) 24 - 2 = 22 < 24$.

Further Questions

- Classify all sub-critical cases where the upper bound is not reached: they have some interesting peculiar geometric properties.
- Which is "the best" $TNS^{\Gamma}_{m,n}$ a given T belongs to?
    - Γ can be reasonably chosen from the context. One may work on decreasing m. How to choose m s.t. a given $T \in TNS^{\Gamma,0}_{m,n}$?
    - Very well established procedures to find a "good enough" approximation of T on a given $TNS^{\Gamma}_{m,n}$.