Inner and outer approximations of tropical polytopes and their applications in tropical tensors

Yang Qi

INRIA and CMAP École Polytechnique

(joint with Marianne Akian, Stéphane Gaubert, and Omar Saadi)

IPAM Workshop III: Mathematical Foundations and Algorithms for Tensor Computations

May 5, 2021
The Tropical Semiring

- The tropical semiring is \((\mathbb{R}_{\text{max}}, \oplus, \odot)\), with:
  \[ \mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\} \]

- The neutral element for \(\oplus\) is "0" = \(-\infty\), and for \(\odot\) is "1" = 0.

- The operations are defined for matrices:
  \[
  (A \oplus B)_{ij} = \max(A_{ij}, B_{ij})
  \]
  \[
  (A \odot B)_{ij} = \max_k (A_{ik} + B_{kj})
  \]

- A scalar acts on a vector by "\(\lambda x\)" = \(\lambda \odot x = \lambda + x\) entrywise.

- Eigenvalues and eigenvectors of a matrix \(A\):
  \[
  A \odot v = \lambda + v
  \]
The Tropical Semiring

- The tropical semiring is \((\mathbb{R}_{\text{max}}, \oplus, \otimes)\), with:
  \[
  \mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\},
  \]
  \[
  x \oplus y = \max(x, y);
  \]
- The neutral element for \(\oplus\) is "0" = \(-\infty\), and for \(\otimes\) is "1" = 0.
- The operations are defined for matrices:
  \[
  (A \oplus B)_{ij} = \max(A_{ij}, B_{ij}),
  \]
  \[
  (A \otimes B)_{ij} = \max_{k}(A_{ik} + B_{kj}),
  \]
- A scalar acts on a vector by "\(\lambda x\)" = \(\lambda \otimes x\) = \(\lambda + x\) entrywise.
- Eigenvalues and eigenvectors of a matrix \(A\):
  \[
  A \otimes v = \lambda \oplus v,
  \]
The Tropical Semiring

- The tropical semiring is \((\mathbb{R}_{\text{max}}, \oplus, \odot)\), with:
  \[
  \mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}
  \]
  \[
  x \oplus y = \max(x, y); \quad 3 \oplus 4 = 4,
  \]
The tropical semiring is \((\mathbb{R}_{\text{max}}, \oplus, \odot)\), with:

- \(\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}\)
- \(x \oplus y = \max(x, y)\); \(3 \oplus 4 = 4\),
- \(x \odot y = x + y\);
The tropical semiring is \((\mathbb{R}_{\text{max}}, \oplus, \otimes)\), with:

\[
\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}
\]

\[
x \oplus y = \max(x, y); \quad 3 \oplus 4 = 4,
\]

\[
x \otimes y = x + y; \quad 3 \otimes 4 = 7.
\]
The Tropical Semiring

- The tropical semiring is \((\mathbb{R}_{\max}, \oplus, \odot)\), with:
  \[
  \mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}
  \]
  \[
  x \oplus y = \max(x, y); \quad 3 \oplus 4 = 4,
  \]
  \[
  x \odot y = x + y; \quad 3 \odot 4 = 7.
  \]
- The neutral element for \(\oplus\) is “0” = \(-\infty\), and for \(\odot\) is “1” = 0.
The Tropical Semiring

- The tropical semiring is \((\mathbb{R}_{\text{max}}, \oplus, \odot)\), with:
  \[ \mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\} \]
  \[ x \oplus y = \max(x, y); \quad 3 \oplus 4 = 4, \]
  \[ x \odot y = x + y; \quad 3 \odot 4 = 7. \]
- The neutral element for \(\oplus\) is “0” = \(-\infty\), and for \(\odot\) is “1” = 0
- The operations are defined for matrices:
  \[ (A \oplus B)_{ij} = \max(A_{ij}, B_{ij}) \]
  \[ (A \odot B)_{ij} = \max_k(A_{ik} + B_{kj}) \]
The tropical semiring is \((\mathbb{R}_{\max}, \oplus, \odot)\), with:
\[
\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}
\]
\[
x \oplus y = \max(x, y); \quad 3 \oplus 4 = 4,
\]
\[
x \odot y = x + y; \quad 3 \odot 4 = 7.
\]
- The neutral element for \(\oplus\) is “0” = \(-\infty\), and for \(\odot\) is “1” = 0
- The operations are defined for matrices:
\[
(A \oplus B)_{ij} = \max(A_{ij}, B_{ij})
\]
\[
(A \odot B)_{ij} = \max_k(A_{ik} + B_{kj})
\]
- A scalar acts on vector by “\(\lambda x\)” = \(\lambda \odot x = \lambda + x\) entrywise.
The Tropical Semiring

- The tropical semiring is \((\mathbb{R}_{\text{max}}, \oplus, \odot)\), with:
  \[
  \mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}
  \]
  \[
  x \oplus y = \max(x, y); \quad 3 \oplus 4 = 4,
  \]
  \[
  x \odot y = x + y; \quad 3 \odot 4 = 7.
  \]
- The neutral element for \(\oplus\) is “0” = −\(\infty\), and for \(\odot\) is “1” = 0
- The operations are defined for matrices:
  \[
  (A \oplus B)_{ij} = \max(A_{ij}, B_{ij})
  \]
  \[
  (A \odot B)_{ij} = \max_k (A_{ik} + B_{kj})
  \]
- A scalar acts on vector by “\(\lambda x\)” = \(\lambda \odot x = \lambda + x\) entrywise.
- Eigenvalues and eigenvectors of a matrix \(A\):
  \[
  A \odot v = \lambda + v
  \]
Applications

Tropical tensors arise naturally:
- Optimization
- Game theory
- Approximation theory
- ...

A baby example:
- Given a direct-distances matrix $A = (a_{ij})$, find the longest distances.
- Consider $A^2 = A \odot A$, i.e., $(A^2)_{ij} = \max_k (a_{ik} + a_{kj})$, which gives the longest path between $i$ and $j$ of length 2.
- $A + A^2 + \cdots$ gives longest paths of all lengths.
Tropical tensors arise naturally:

- Optimization
- Game theory
- Approximation theory
- ...

A baby example:

- Given a direct-distances matrix $A = (a_{ij})$, find the longest distances.
- Consider $A^2 = A \odot A$, i.e.,

$$
(A^2)_{ij} = \max_k (a_{ik} + a_{kj}),
$$

which gives the longest path between $i$ and $j$ of length 2.
- $A + A^2 + \cdots$ gives longest paths of all lengths.
A decision maker and \( n \) local companies
Invitation to tender markets

- A decision maker and $n$ local companies
- For invitation $j \in [q]$, company $i \in [n]$ asks for the price $p_{ij}$ (public)
Invitation to tender markets

- A decision maker and \( n \) local companies
- For invitation \( j \in [q] \), company \( i \in [n] \) asks for the price \( p_{ij} \) (public)
- Secret evaluation \( 0 < f_i \leq 1 \) (technical quality)
Invitation to tender markets

- A decision maker and $n$ local companies
- For invitation $j \in [q]$, company $i \in [n]$ asks for the price $p_{ij}$ (public)
- Secret evaluation $0 < f_i \leq 1$ (technical quality)
- The decision maker minimizes his expected cost:

$$\min_{i \in [n]} p_{ij} f_i^{-1}$$
Equilibrium in invitation to tender markets

The prices constitute an equilibrium:

$$\min_{i \in [q]} p_{ij} f_i^{-1}$$ is achieved twice at least
The prices constitute an equilibrium:

\[
\min_{i \in [q]} p_{ij} f_i^{-1} \text{ is achieved twice at least}
\]

Indeed, if \( p_{ij} f_i^{-1} < p_{kj} f_k^{-1}, \forall k \neq i \), then company \( i \) may raise its price and still wins the offer.
The prices constitute an equilibrium:

$$\min_{i \in [q]} p_{ij} f_i^{-1}$$

is achieved twice at least

Indeed, if $p_{ij} f_i^{-1} < p_{kj} f_k^{-1}, \forall k \neq i$, then company $i$ may raise its price and still wins the offer.

Let $V_{ij} = -\log(p_{ij})$ and $a_i = \log(f_i)$, the equilibrium is:

$$\max_{i \in [n]} (V_{ij} + a_i)$$

is achieved twice at least
The prices constitute an equilibrium:

\[
\min_{i \in [q]} p_{ij} f_i^{-1} \text{ is achieved twice at least.}
\]

Indeed, if \( p_{ij} f_i^{-1} < p_{kj} f_k^{-1}, \forall k \neq i \), then company \( i \) may raise its price and still wins the offer.

Let \( V_{ij} = -\log(p_{ij}) \) and \( a_i = \log(f_i) \), the equilibrium is:

\[
\max_{i \in [n]} (V_{ij} + a_i) \text{ is achieved twice at least.}
\]

will see later: \( \forall j \in [q], \ V_j \text{ lies in some hyperplane } H_a \)
Given tropical data, we would like to do

- Tropical linear regression:
  Given $m$ points, find a hyperplane which best fits these data.
- Tropical tensor approximation:
  Given a tropical tensor, find a best rank-one approximation.
Tropical cones

- $C \subseteq (\mathbb{R}_{\text{max}})^n$ is a tropical (convex) cone if $\forall x, \ y \in C, \forall \lambda \in \mathbb{R}_{\text{max}}$:
  \[
  \lambda + x \in C \text{ and } \max(x, y) \in C.
  \]


- $C \subseteq (\mathbb{R}_{\max})^n$ is a **tropical (convex) cone** if $\forall x, y \in C, \forall \lambda \in \mathbb{R}_{\max}$:
  $$\lambda + x \in C \text{ and } \max(x, y) \in C.$$

- For a set $\mathcal{V} \subseteq (\mathbb{R}_{\max})^n$, the tropical cone generated by $\mathcal{V}$:
  $$\text{Span}(\mathcal{V}) := \left\{ \sup_{v \in \mathcal{V}} (\lambda_v + v) \mid \lambda_v \in \mathbb{R}_{\max} \right\}.$$
Tropical cones

- $C \subseteq (\mathbb{R}_{\text{max}})^n$ is a tropical (convex) cone if $\forall x, y \in C, \forall \lambda \in \mathbb{R}_{\text{max}}$:
  $$\lambda + x \in C \text{ and } \max(x, y) \in C.$$

- For a set $\mathcal{V} \subseteq (\mathbb{R}_{\text{max}})^n$, the tropical cone generated by $\mathcal{V}$:
  $$\text{Span}(\mathcal{V}) := \{ \sup_{\nu \in \mathcal{V}} (\lambda \nu + \nu) | \lambda \nu \in \mathbb{R}_{\text{max}} \}.$$

- The tropical projective space $\mathbb{P}(\mathbb{R}_{\text{max}})^n$ is the quotient
  $$(\mathbb{R}_{\text{max}}^n \setminus \{-\infty\}) / \sim,$$
  where $x \sim y$ iff $x - y$ is a constant vector.
Tropical cones

- $C \subseteq (\mathbb{R}_{\text{max}})^n$ is a tropical (convex) cone if $\forall x, y \in C, \forall \lambda \in \mathbb{R}_{\text{max}}:$
  $$\lambda + x \in C \text{ and } \max(x, y) \in C.$$

- For a set $\mathcal{V} \subseteq (\mathbb{R}_{\text{max}})^n$, the tropical cone generated by $\mathcal{V}$:
  $$\text{Span}(\mathcal{V}) := \{ \sup_{v \in \mathcal{V}} (\lambda_v + v) \mid \lambda_v \in \mathbb{R}_{\text{max}} \}.$$  

- The tropical projective space $\mathbb{P}(\mathbb{R}_{\text{max}})^n$ is the quotient
  $$(\mathbb{R}_{\text{max}}^n \setminus \{-\infty\}) / \sim,$$
  where $x \sim y$ iff $x - y$ is a constant vector.

- A natural metric on $\mathbb{P}(\mathbb{R}^n)$ is induced by Hilbert’s seminorm:
  $$\|Z\|_H := (\max_{i \in [n]} z_i) - (\min_{i \in [n]} z_i).$$
Tropical cones

- $C \subseteq (\mathbb{R}_{\text{max}})^n$ is a tropical (convex) cone if $\forall x, y \in C, \forall \lambda \in \mathbb{R}_{\text{max}}$: $
\lambda + x \in C$ and $
\max(x, y) \in C.$

- For a set $\mathcal{V} \subseteq (\mathbb{R}_{\text{max}})^n$, the tropical cone generated by $\mathcal{V}$:

  $\text{Span}(\mathcal{V}) := \{ \sup_{\nu \in \mathcal{V}} (\lambda \nu + \nu) \mid \lambda \nu \in \mathbb{R}_{\text{max}} \}.$

- The tropical projective space $\mathbb{P}(\mathbb{R}_{\text{max}})^n$ is the quotient $
(\mathbb{R}_{\text{max}}^n \setminus \{-\infty\}) / \sim,$

  where $x \sim y$ iff $x - y$ is a constant vector.

- A natural metric on $\mathbb{P}(\mathbb{R}^n)$ is induced by Hilbert’s seminorm:

  $\|z\|_H := (\max_{i \in [n]} z_i) - (\min_{i \in [n]} z_i).$

- Given $A, B \subseteq \mathbb{P}(\mathbb{R}_{\text{max}})^n$, the one-sided Hausdorff distance:

  $d_H(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|_H.$
Given \( a \in \mathbb{P}(\mathbb{R}_{\text{max}})^n \), we define the \textbf{tropical hyperplane}:

\[
H_a := \{ x \in (\mathbb{R}_{\text{max}})^n \mid \max_{i \in [n]} (a_i + x_i) \text{ achieved at least twice} \} .
\]
Tropical hyperplanes

Given \( a \in \mathbb{P}(\mathbb{R}_{\text{max}})^n \), we define the tropical hyperplane:

\[
H_a := \{ x \in (\mathbb{R}_{\text{max}})^n \mid \max_{i \in [n]} (a_i + x_i) \text{ achieved at least twice} \}.
\]

Figure: The hyperplane \( H_a \) with \( a = (0, 0, 1)^\top \) and in \( \mathbb{P}(\mathbb{R}_{\text{max}})^3 \).
Tropical hyperplanes

Given $a \in \mathbb{P}(\mathbb{R}_{\text{max}})^n$, we define the tropical hyperplane:

$$H_a := \{ x \in (\mathbb{R}_{\text{max}})^n \mid \max_{i \in [n]} (a_i + x_i) \text{ achieved at least twice} \}.$$

![Figure: The hyperplane $H_a$ with $a = (0, 0, 1)^T$ and in $\mathbb{P}(\mathbb{R}_{\text{max}})^3$.](image)

Given a set $\mathcal{V} \subseteq \mathbb{R}_{\text{max}}^n$, we solve the tropical linear regression problem:

$$\inf_{a \in \mathbb{P}(\mathbb{R}_{\text{max}})^n} d_H(\mathcal{V}, H_a).$$
For a subset $\mathcal{V}$ of $(\mathbb{R}_{\text{max}})^n$, the inner radius of $\mathcal{V}$ is:

$$\text{in-rad}(\mathcal{V}) := \sup\{ r \geq 0 \mid \exists b \in \mathbb{R}^n, B(b, r) \subseteq \text{Span}(\mathcal{V}) \}.$$
Inner radius

For a subset $\mathcal{V}$ of $\mathbb{R}_{\text{max}}^n$, the *inner radius* of $\mathcal{V}$ is:

$$\text{in-rad}(\mathcal{V}) := \sup\{ r \geq 0 \mid \exists b \in \mathbb{R}^n, B(b, r) \subseteq \text{Span}(\mathcal{V}) \}.$$ 

**Theorem (Akian–Gaubert–Q.–Saadi)**

$$\inf_{a \in \mathbb{P}(\mathbb{R}_{\text{max}}^n)} d_H(\mathcal{V}, H_a) = \text{in-rad}(\mathcal{V}).$$

**Question:** how to compute $\text{in-rad}(\mathcal{V})$?
Mean Payoff Games

For $V \in (\mathbb{R}_{\text{max}})^{n \times p}$, the zero-sum two-player deterministic game:

- Two players Min and Max
- Starting from a state $i$, Min chooses $k \in [p]$ s.t. $V_{ik} \neq -\infty$
- Then Max chooses the next state $j \neq i$
- $-V_{ik} + V_{jk}$ is the instantaneous payment made by Min to Max

After $\ell$ turns, under a strategy $\sigma$ (resp. $\tau$) of Min (resp. Max), if the sequence of actions is $i, k_1, i_1, k_2, \ldots, k_\ell, i_\ell$, the total payment:

$$R_{\ell i}(\sigma, \tau) = -V_{ik_1} + V_{i_1k_1} - V_{i_1k_2} + \cdots + V_{i_\ell k_\ell}$$

The value $v_{\ell i}$ of the game at horizon $\ell$ starting from $i$:

$$v_{\ell i} := \min_{\sigma} \max_{\tau} R_{\ell i}(\sigma, \tau) = \max_{\tau} \min_{\sigma} R_{\ell i}(\sigma, \tau).$$
For $V \in (\mathbb{R}_{\text{max}})^{n \times p}$, the \textit{zero-sum two-player deterministic game}:

- Two players Min and Max
Mean Payoff Games

For $V \in (\mathbb{R}_{\text{max}})^{n \times p}$, the *zero-sum two-player deterministic game*:

- Two players Min and Max
- Starting from a state $i$, Min chooses $k \in [p]$ s.t. $V_{ik} \neq -\infty$
Mean Payoff Games

For $V \in (\mathbb{R}_{\text{max}})^{n \times p}$, the \textit{zero-sum two-player deterministic game}:

- Two players Min and Max
- Starting from a state $i$, Min chooses $k \in [p]$ s.t. $V_{ik} \neq -\infty$
- Then Max chooses the next state $j \neq i$
Mean Payoff Games

For $V \in (\mathbb{R}_{\text{max}})^{n \times p}$, the zero-sum two-player deterministic game:

- Two players Min and Max
- Starting from a state $i$, Min chooses $k \in [p]$ s.t. $V_{ik} \neq -\infty$
- Then Max chooses the next state $j \neq i$
- $-V_{ik} + V_{jk}$ is the instantaneous payment made by Min to Max
For $V \in (\mathbb{R}_{\text{max}})^{n \times p}$, the zero-sum two-player deterministic game:

- Two players Min and Max
- Starting from a state $i$, Min chooses $k \in [p]$ s.t. $V_{ik} \neq -\infty$
- Then Max chooses the next state $j \neq i$
- $-V_{ik} + V_{jk}$ is the instantaneous payment made by Min to Max

After $\ell$ turns, under a strategy $\sigma$ (resp. $\tau$) of Min (resp. Max), if the sequence of actions is $i, k_1, i_1, \cdots, k_\ell, i_\ell$, the total payment:

$$R_i^{\ell}(\sigma, \tau) = -V_{ik_1} + V_{i_1k_1} - V_{i_1k_2} - \cdots + V_{i_\ell k_\ell}$$

The value $v_i^{\ell}$ of the game at horizon $\ell$ starting from $i$:

$$v_i^{\ell} := \min_{\sigma} \max_{\tau} R_i^{\ell}(\sigma, \tau) = \max_{\tau} \min_{\sigma} R_i^{\ell}(\sigma, \tau) .$$
Shapley operator

**Shapley operator** $T : (\mathbb{R}_{\text{max}})^n \rightarrow (\mathbb{R}_{\text{max}})^n,$

$$T_i(x) = \min_{k \in [p], V_{ik} \neq -\infty} \left[ -V_{ik} + \max_{j \in [n], j \neq i} (V_{jk} + x_j) \right], \ i \in [n],$$

**Theorem (Shapley)**

$$\nu^\ell = T^\ell(0).$$
**Shapley operator**

*Shapley operator* $T : (\mathbb{R}_{\text{max}})^n \rightarrow (\mathbb{R}_{\text{max}})^n$,

$$T_i(x) = \min_{k \in [p], V_{ik} \neq -\infty} \left[ -V_{ik} + \max_{j \in [n], j \neq i} (V_{jk} + x_j) \right], \quad i \in [n],$$

**Theorem (Shapley)**

$$\nu^\ell = T^\ell(0).$$

In the infinite horizon case, the *mean payoff vector*:

$$\chi(T) := \lim_{\ell \to \infty} T^\ell(0)/\ell.$$
Shapley operator

Shapley operator $T : (\mathbb{R}_{\text{max}})^n \rightarrow (\mathbb{R}_{\text{max}})^n$,

$$T_i(x) = \min_{k \in [p], V_{ik} \neq -\infty} \left[ -V_{ik} + \max_{j \in [n], j \neq i} (V_{jk} + x_j) \right], \quad i \in [n],$$

Theorem (Shapley)

$$\nu^\ell = T^\ell(0).$$

In the infinite horizon case, the mean payoff vector:

$$\chi(T) := \lim_{{\ell \to \infty}} T^\ell(0)/\ell.$$

- Complexity: NP $\cap$ coNP.
**Shapley operator**

*Shapley operator* $T : (\mathbb{R}_{\text{max}})^n \rightarrow (\mathbb{R}_{\text{max}})^n$, 

$$T_i(x) = \min_{k \in [p], V_{ik} \neq -\infty} \left[ -V_{ik} + \max_{j \in [n], j \neq i} (V_{jk} + x_j) \right], \quad i \in [n],$$

**Theorem (Shapley)**

$$v^\ell = T^\ell(0).$$

In the infinite horizon case, the *mean payoff vector*:

$$\chi(T) := \lim_{\ell \rightarrow \infty} T^\ell(0)/\ell.$$

- Complexity: $\text{NP} \cap \text{coNP}$.
**Shapley operator**

Shapley operator $T : (\mathbb{R}_{\text{max}})^n \rightarrow (\mathbb{R}_{\text{max}})^n$,

$$T_i(x) = \min_{k \in [p], V_{ik} \neq -\infty} \left[ -V_{ik} + \max_{j \in [n], j \neq i} (V_{jk} + x_j) \right], \ i \in [n],$$

**Theorem (Shapley)**

$$v^\ell = T^\ell(0).$$

In the infinite horizon case, the *mean payoff vector*:

$$\chi(T) := \lim_{\ell \to \infty} T^\ell(0)/\ell.$$

- Complexity: NP $\cap$ coNP.
- Our case: $\chi(T) \leq 0$. 
Equivalence between tropical linear regression and MPG

The spectral radius of $T$ is defined as

$$\rho(T) = \sup\{\lambda \in \mathbb{R} \cup \{-\infty\} \mid \exists u \in (\mathbb{R}_{\max})^n, u \neq -\infty, T(u) = \lambda + u\}.$$ 

Theorem (Akian–Gaubert–Q.–Saadi)

$$\min_{a \in P(\mathbb{R}_{\max})^n} d_H(V, H_a) = -\rho(T) = \text{in-rad}(V)$$

Moreover,

- if $T(a) \geq \rho(T) + a$, then $H_a$ is optimal
- if $T(b) \leq \rho(T) + b$, then $B(-b, -\rho(T))$ is optimal
The spectral radius of $T$ is defined as

$$\rho(T) = \sup\{\lambda \in \mathbb{R} \cup \{-\infty\} | \exists u \in (\mathbb{R}_{\max})^n, u \neq -\infty, \ T(u) = \lambda + u\}.$$ 

**Theorem (Akian–Gaubert–Q.–Saadi)**

$$\min_{a \in \mathcal{P}(\mathbb{R}_{\max})^n} d_H(V, H_a) = -\rho(T) = \text{in-rad}(V)$$

Moreover,

- if $T(a) \geq \rho(T) + a$, then $H_a$ is optimal
- if $T(b) \leq \rho(T) + b$, then $B(-b, -\rho(T))$ is optimal

**Corollary**

The tropical linear regression problem is polynomial-time equivalent to the problem of solving a mean payoff game.
Figure: A tropical cone $\text{Sp}(\mathcal{V})$ with an optimal regression hyperplane $H_a$ and an optimal inner ball $B(-a, 1)$, where $a = (0, 0, 1)^\top$ satisfies $T(a) = -1 + a$. 

\[ x_1 \cdot 1 + x_2 \cdot 2 + x_3 \cdot 3 + x_4 \cdot 4 = 0 \]
Revisit to equilibrium in ITT

Equilibrium:

$$\min_{i \in [q]} p_{ij} f_i^{-1}$$

is achieved twice at least

By letting $V_{ij} = -\log(p_{ij})$ and $a_i = \log(f_i)$, the equilibrium is:

$$\max_{i \in [n]} (V_{ij} + a_i)$$

is achieved at least twice,

namely,

$$\forall j \in [q], \quad V_{.j} \in H_a$$

In practice, we solve the **tropical linear regression problem:**

$$\min_{b \in \mathbb{P}(\mathbb{R}_{\max})^n} d_H(V, H_b).$$
Algorithm

- **Goal:** \( T(\nu) = \rho(T) + \nu \)
- **Algorithm:** projective Krasnoselkii-Mann value iteration algorithm
  - Given \( \epsilon > 0 \), start with \( \nu^0 = (0, \cdots, 0)\top \).
  - If \( \| T(\nu^k) - \nu^k \| \geq \epsilon \), let
    \[
    \tilde{\nu}^{k+1} = T(\nu^k) - (\max_{i \in [n]} T(\nu^k)_i) e,
    \nu^{k+1} = (1 - \beta)\nu^k + \beta \tilde{\nu}^{k+1},
    \]
    where \( e = (1, \cdots, 1)\top \in \mathbb{R}^n \) and \( \beta \in (0, 1) \) fixed.
Suppose that $V \in \mathbb{R}^{n \times p}$ is finite, and let

$$W := \max_{k \in [p]} \| V \cdot e_k \|_H.$$

Then, an $\epsilon$-approximation of $\text{in-rad}(V)$, and vectors $v, z \in \mathbb{R}^n$ satisfying

$$B_H(v, \text{in-rad}(V) - \epsilon) \subseteq \text{Span}(V)$$

and

$$d_H(\text{Span}(V), H_z) \leq \text{in-rad}(V) + \epsilon$$

can be obtained in $O(npW/\epsilon)$ arithmetic operations.
For a subset $\mathcal{V}$ of $\mathbb{R}_{\text{max}}^n$, the outer radius of $\mathcal{V}$ is:

$$\text{out-rad}(\mathcal{V}) := \inf\{ r \geq 0 \mid \exists b \in \mathbb{R}^n, B(b, r) \supseteq \text{Span}(\mathcal{V}) \}.$$ 

**Theorem (Akian–Gaubert–Q.–Saadi)**

For any matrix $V \in \mathbb{R}^{n \times p}$, we have

$$\min_{A \in \mathbb{R}^{n \times p}, \text{rank } A = 1} \| V - A \|_{\infty} = \frac{1}{2} \text{out-rad}(\text{Span } V).$$

Question: how to compute $\text{out-rad}(\text{Span } V)$?
Given $A \in (\mathbb{R}_{\text{max}})^{n \times n}$, define

$$A^* = I \oplus A \oplus A^2 \oplus \cdots$$

For $V \in \mathbb{R}^{n \times p}$, let $H = V \odot (-V^\top) \in \mathbb{R}^{n \times n}$, where

$$H_{ik} = \max_{j \in [p]} (V_{ij} - V_{kj}), \quad i, k \in [n].$$

**Theorem (Akian–Gaubert–Q.–Saadi)**

$H$ has a unique eigenvalue, which equals $\text{out-rad}(\text{Span } V)$. Moreover, the set of centers of all Hilbert outer balls of $\text{Span } V$ is the column space of $(-\lambda + H)^*$. 
Kernel approximations

\( X \): compact metric, \( Y \): nonempty, \( \mathcal{C}(X) \): the space of continuous functions on \( X \), \( \mathcal{B}(Y) \): the space of bounded functions on \( Y \).

**Theorem (Akian–Gaubert–Q.–Saadi)**

If \( V : X \times Y \to \mathbb{R} \) bounded and \( \{ V(\cdot, y) \}_{y \in Y} \) equicontinuous, then

\[
\inf_{f \in \mathcal{C}(X), g \in \mathcal{B}(Y)} \sup_{x \in X, y \in Y} | V(x, y) - f(x) - g(y) |
\]

achieves an optimal solution, which is equal to one half of the tropical eigenvalue of \( H \), where

\[
H(x, z) = \sup_{y \in Y} (V(x, y) - V(z, y))
\]

and \( f \) is a tropical eigenvector of \( H \).
Tropical linear regression is equivalent to finding inner radius, which is polynomial-time equivalent to mean payoff game.

Finding a best tropical rank-one approximation is equivalent to finding outer radius, which is equivalent to eigenvalue problem.

Thank you for your attention!