Algebraic methods to construct tensors

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Tensors

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are linear spaces of matrices:

$$W_3^* \hookrightarrow Hom(W_1^*, W_2)$$

Matrix multiplication

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$$\mathrm{Mat}_n \times \mathrm{Mat}_n \to \mathrm{Mat}_n$$

$$M_n \in (\mathrm{Mat}_n)^* \otimes (\mathrm{Mat}_n)^* \otimes \mathrm{Mat}_n$$

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$$A = \begin{pmatrix} a_1 & \dots & a_n \\ a_{n+1} & \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n^2-n+1} & \dots & a_{n^2} \end{pmatrix}$$

$$M_n = \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & 0 & \dots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & \dots & A \end{pmatrix}$$

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$$\mathrm{rk}\, M_n \leq n^3$$

Strassen's algorithm

$$\operatorname{rk} M_2 = 7 < 8$$

$$M_2 = (e^{21}+e^{22})\otimes e^{11}\otimes(e_{21}+e_{22})+(e^{11}+e^{22})\otimes(e^{11}+e^{22})\otimes(e_{11}+e_{22})+$$
$$e^{11}\otimes(e^{12}-e^{22})\otimes(e_{12}+e_{22})+e^{22}\otimes(e^{21}-e^{11})\otimes(e_{11}+e_{21})+$$
$$(e^{11}+e^{12})\otimes e^{22}\otimes(e_{12}-e_{11})+(e^{21}-e^{11})\otimes(e^{11}+e^{12})\otimes e_{22}+$$
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$(e^{11}+e^{12})\otimes e^{22}\otimes(e_{12}-e_{11})+(e^{21}-e^{11})\otimes(e^{11}+e^{12})\otimes e_{22}+$
$(e^{12}-e^{22})\otimes(e^{21}+e^{22})\otimes e_{11}$

$$M_a \boxtimes M_b = M_{ab}$$

$$\omega =$$

$$\inf\{\tau : \text{ the complexity of multiplication of two } n\times n \text{ matrices is } O(n^\tau)\}$$
$$= \inf\{\tau : \text{rank of } M_n = O(n^\tau)\} = \inf\{\tau : \text{border rank of } M_n = O(n^\tau)\}.$$

Coppersmith-Winograd

$$CW_k = \left(\begin{array}{c|ccc|c} a_0 & a_1 & \dots & a_k & a_{k+1} \\ \hline a_1 & 0 & \dots & a_{k+1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_k & a_{k+1} & \dots & 0 & 0 \\ \hline a_{k+1} & 0 & \dots & 0 & 0 \end{array}\right).$$

$$\omega < 2.38$$

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Conjecture

$\omega = 2$

Bad news

- CW should not give $\omega < 2.3$ [Ambainis, Filmus, Le Gall]
- Fixed smoothable algebra should not give $\omega = 2$ [Lysikov, Blaeser, Zuiddam, Christandl, Buergisser]

What can we do?

We probably need:

- new, good tensors
- families of tensors

Try to look around CW

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- families of tensors

Try to look around CW
CW comes from:

- smoothable local algebra
- highest weight vector

Cubics on matrices

$S^3(Mat_n)$ decomposes [Seynnaeve]
Every irreducible representation contains a special hwv
CW are among the hwv
We analyzed one that:

- has minimal border rank
- gives $\omega < 2.45$

Symmetrized M_n is also one of hwv
Conjecturally, they all have not too big border rank

$$
\left(\begin{array}{cc|ccc|ccc|ccc|ccc}
b_1 & b_0 & y_1 & \dots & y_m & x_1 & \dots & x_m & 0 & \dots & 0 & a_2 & a_1 & 0 \\
b_0 & b_2 & z_1 & \dots & z_m & 0 & \dots & 0 & x_1 & \dots & x_m & a_1 & 0 & a_2 \\
\hline
y_1 & z_1 & & & & a_1 & & & a_2 & & & & & \\
\vdots & \vdots & & & & & \ddots & & & \ddots & & & & \\
y_m & z_m & & & & & & a_1 & & & a_2 & & & \\
\hline
x_1 & 0 & a_1 & & & & & & & & & & & \\
\vdots & \vdots & & \ddots & & & & & & & & & & \\
x_m & 0 & & & a_1 & & & & & & & & & \\
\hline
0 & x_1 & a_2 & & & & & & & & & & & \\
\vdots & \vdots & & \ddots & & & & & & & & & & \\
0 & x_m & & & a_2 & & & & & & & & & \\
\hline
a_2 & a_1 & & & & & & & & & & & & \\
a_1 & 0 & & & & & & & & & & & & \\
0 & a_2 & & & & & & & & & & & &
\end{array}\right).
$$

Algebra as tensor

Algebra: polynomial ring modulo and ideal
$n := \dim_{\mathbb{C}} A < \infty$

$$A \times A \to A$$

is 2-linear
T_A the associated tensor

Theorem

T_A is symmetric if and only if A is Gorenstein

A is smoothable $\Leftrightarrow \operatorname{brk} T_A = n$ [Blaeser, Lysikov, Landsberg]

Apolarity

$S \subset K[Y] = K[y_1, \ldots, y_n]$
$S^\perp \subset K[X] = K[x_1 = \partial y_1, \ldots, x_n = \partial y_n]$
$S^\perp$ is an ideal and defines a finite local algebra $A_S := K[X]/S^\perp$
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Example

$S = x_1x_4 + x_2x_3,\ S^\perp = \langle y_1^2, y_2^2, y_3^2, y_4^2, y_1y_2, y_1y_3, y_2y_4, y_3y_4, y_1y_4 - y_2y_3 \rangle$
$K[Y]/S^\perp$ has basis: $1, y_1, y_2, y_3, y_4, m = y_1y_4$
Multiplication in this basis:

$$\begin{pmatrix} 1 & y_1 & y_2 & y_3 & y_4 & m \\ y_1 & 0 & 0 & 0 & m & 0 \\ y_2 & 0 & 0 & m & 0 & 0 \\ y_3 & 0 & m & 0 & 0 & 0 \\ y_4 & m & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Two quadrics

Theorem (Cartwright, Erman, Velasco, Viray)

If S consists of two quadrics, then $S^{\perp}$ is smoothable.

We may use two letters in CW

$$
\left(\begin{array}{c|ccc|ccc|ccc|cc}
\lambda_0 & \lambda_1 & \dots & \lambda_m & \lambda_{m+1} & \dots & \lambda_{2m} & \lambda_{2m+1} & \dots & \lambda_{3m} & \lambda_{3m+1} & \lambda_{3m+2} \\
\hline
\lambda_1 & & & & \lambda_{3m+1} & & & \lambda_{3m+2} & & & & \\
\vdots & & & & & \ddots & & & \ddots & & & \\
\lambda_m & & & & & & \lambda_{3m+1} & & & \lambda_{3m+2} & & \\
\hline
\lambda_{m+1} & \lambda_{3m+1} & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & \\
\lambda_{2m} & & & \lambda_{3m+1} & & & & & & & & \\
\hline
\lambda_{2m+1} & \lambda_{3m+2} & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & \\
\lambda_{3m} & & & \lambda_{3m+2} & & & & & & & & \\
\hline
\lambda_{3m+1} & & & & & & & & & & & \\
\lambda_{3m+2} & & & & & & & & & & &
\end{array}\right).
$$

$$
\left(\begin{array}{c|ccc|ccc|ccc|cc}
\lambda_0 & \lambda_1 & \dots & \lambda_m & \lambda_{m+1} & \dots & \lambda_{2m} & \lambda_{2m+1} & \dots & \lambda_{3m} & \lambda_{3m+1} & \lambda_{3m+2} \\
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\vdots & & & & & \ddots & & & \ddots & & & \\
\lambda_m & & & & & & \lambda_{3m+1} & & & \lambda_{3m+2} & & \\
\hline
\lambda_{m+1} & \lambda_{3m+1} & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & \\
\lambda_{2m} & & & \lambda_{3m+1} & & & & & & & & \\
\hline
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\lambda_{3m+1} & & & & & & & & & & & \\
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$$

$$\omega < 2.431$$

Why algebras?

Theorem

Fix $\epsilon > 0$. There exists a local smoothable algebra A of length l, degenerating to M_n, where $n^{\omega+\epsilon} > l$.

$S = \sum x_{ij}x_{jk}x_{ik}$ gives nonsmoothable Gorenstein algebra A.
Want: B smoothable that surjects to A

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- graded algebras: good for CW
- monomial algebras?

Thank you!