Low Rank Tucker Approximation of a Tensor from Streaming Data

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Based on joint work with
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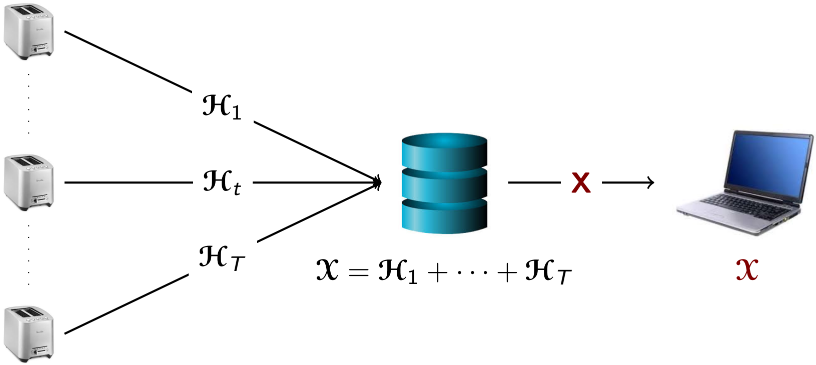
Big data, small laptop

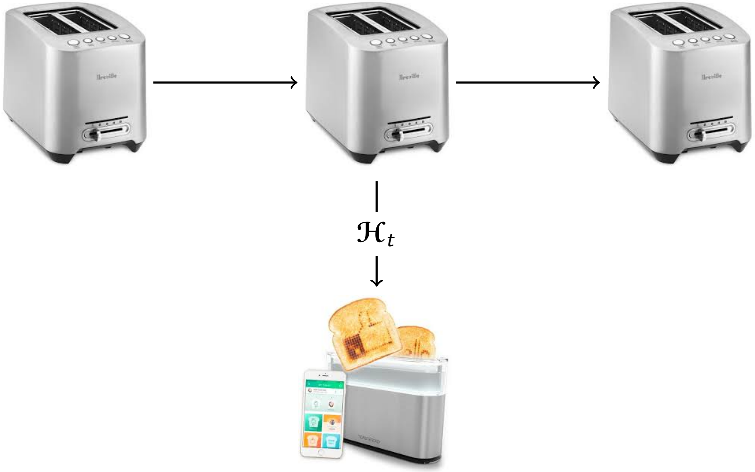
 $\xrightarrow{\quad \mathcal{H}_T \quad}$  $\xrightarrow{\quad \mathcal{X} \quad}$ 

$$\mathcal{X} = \mathcal{H}_1 + \cdots + \mathcal{H}_T \qquad\qquad \mathcal{X}$$

Distributed data



Streaming data



$$\mathcal{X}^{(t)} = \mathcal{H}_1 + \cdots + \mathcal{H}_t$$

Streaming multilinear algebra

turnstile model:

$$\mathcal{X} \;\; = \;\; \mathcal{H}_1 + \cdots + \mathcal{H}_T$$

- tensor $\mathcal{X}$ presented as sum of smaller, simpler tensors $\mathcal{H}_t$
- must discard $\mathcal{H}_t$ after it is processed
- **Goal:** without storing $\mathcal{X}$, approximate $\mathcal{X}$ after seeing all updates (with guaranteed accuracy)

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applications:

- scientific simulation
- sensor measurements
- memory- or communication-limited computing
- low memory optimization

Linear sketch

$$\begin{aligned} \mathcal{X} &= \mathcal{H}_1 + \cdots + \mathcal{H}_T \\ \mathcal{L}(\mathcal{X}) &= \mathcal{L}(\mathcal{H}_1) + \cdots + \mathcal{L}(\mathcal{H}_T) \end{aligned}$$

Linear sketch

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- select a linear map $\mathcal{L}$ independent of $\mathcal{X}$
- sketch $\mathcal{L}(\mathcal{X})$ is much smaller than input tensor $\mathcal{X}$
- use randomness so sketch works for an arbitrary input
- essentially the only way to handle the turnstile model [Li et al. 2014]

examples:

- $\mathcal{L}(\mathcal{X}) = \mathcal{X} \times_n \mathbf{\Omega}$ for some matrix $\mathbf{\Omega}$
- $\mathcal{L}(\mathcal{X}) = \{\mathcal{X} \times_n \mathbf{\Omega}_n\}_{n \in [N]}$ for some matrices $\{\mathbf{\Omega}_n\}_{n \in [N]}$

Main idea

sketch suffices for (Tucker) approximation:

- compute (randomized) linear sketch of tensor
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- (optional) improve approximation by revisiting data

Big data, small laptop: sketch

 $\xrightarrow{\mathcal{H}_T}$  $\xrightarrow{\mathcal{L}(\mathcal{X})}$ 

$\mathcal{X} = \mathcal{H}_1 + \cdots + \mathcal{H}_T$ $\qquad$ $\mathcal{L}(\mathcal{X})$

- (+) reduced communication
- (+) sketch of data fits on laptop

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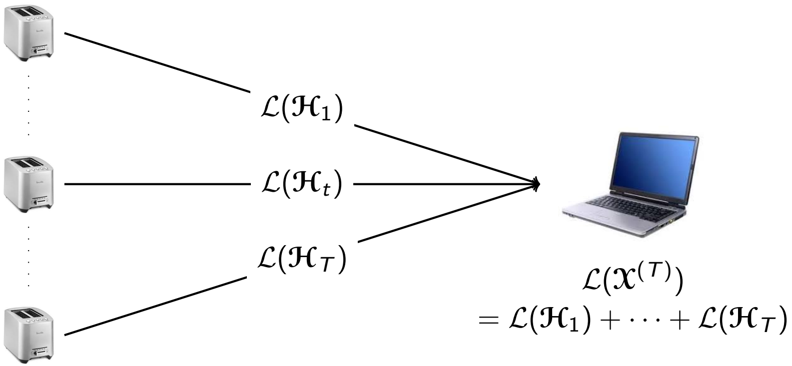

- (+) reduced communication
- (+) no PITI (personally identifiable toast information)
- (+) sketch of data fits on laptop

Streaming data: sketch

 $\longrightarrow$  $\longrightarrow$ 

$$\Big| \quad \mathcal{L}(\mathcal{H}_t) \quad \downarrow$$

$$\mathcal{L}(\mathcal{X}^{(t)}) = \mathcal{L}(\mathcal{H}_1 + \cdots + \mathcal{H}_{t-1}) + \mathcal{L}(\mathcal{H}_t)$$

- $(+)$ even a toaster can form sketch

Notation

tensor to compress:

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- mode n product: for $\mathcal{A} \in \mathbf{R}^{k \times I_n}$,
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- unfolding $\mathbf{X}^{(n)} \in \mathbf{R}^{I_n \times I_{(-n)}}$ stacks mode-n fibers of $\mathcal{X}$ as columns of matrix

Tucker factorization

rank $\mathbf{r} = (r_1, \ldots, r_N)$ **Tucker factorization** of $\mathcal{X} \in \mathbf{R}^{I_1 \times \cdots \times I_N}$:

$$\mathcal{X} \;\; = \;\; \mathcal{G} \times_1 \mathbf{U}_1 \cdots \times_N \mathbf{U}_N =: [\![\mathcal{G}; \mathbf{U}_1, \ldots, \mathbf{U}_N]\!]$$

where

- $\mathcal{G} \in \mathbf{R}^{r_1 \times \cdots \times r_N}$ is the **core matrix**
- $\mathbf{U}_n \in \mathbf{R}^{I_n \times r_n}$ is the **factor matrix** for each mode $n \in [N]$

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Tucker is useful for compression: when N is small,

- Tucker stores $O(rNI)$ numbers for rank r^3 approximation
- CP stores $O(rNI)$ numbers for rank r approximation

Computing Tucker: HOSVD

Algorithm Higher order singular value decomposition (HOSVD)
[De Lathauwer, De Moor & Vandewalle 2000, Tucker 1966]

Given: tensor $\mathcal{X}$, target rank $\mathbf{r} = (r_1, \ldots, r_N)$

1. *Factors.* Compute top r_n left singular vectors $\mathbf{U}_n$ of the unfolding $\mathbf{X}^{(n)}$ for each $n \in [N]$.
2. *Core.* Contract these with $\mathcal{X}$ to form the core

$$\mathcal{G} = \mathcal{X} \times_1 \mathbf{U}_1^T \cdots \times_N \mathbf{U}_N^T.$$

Return: Tucker approximation $\mathcal{X}_{\mathrm{HOSVD}} = [\![\mathcal{G}; \mathbf{U}_1, \ldots, \mathbf{U}_N]\!]$

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  - if $\mathbf{rank}(\boldsymbol{\Omega}) \geq \mathbf{rank}(\mathbf{X}^{(n)})$, then whp for random $\boldsymbol{\Omega}$,

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1. compute sketch $\mathcal{L}(\mathcal{X}) = \{\mathbf{X}^{(n)}\mathbf{\Omega}_n\}_{n \in [N]}$
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  2. use QR on sketch to approximate $\mathbf{range}(\mathbf{X}^{(n)})$
- **Core.** Computation is linear in $\mathcal{X}$:

$$\mathcal{G} = \mathcal{X} \times_1 \mathbf{U}_1^T \cdots \times_N \mathbf{U}_N^T.$$

Source: [Halko et al. 2011, Zhou et al. 2014, Battaglino et al. 2019]

Computing Tucker: HOOI

Algorithm Higher order orthogonal iteration (HOOI) [De Lathauwer et al. 2000]

Given: tensor $\mathcal{X}$, target rank $\mathbf{r} = (r_1, \ldots, r_N)$
Initialize: compute $\mathcal{X} \approx [\![\mathcal{G}; \mathbf{U}_1, \ldots, \mathbf{U}_N]\!]$ using HOSVD
Repeat:

1. *Factors.* For $n \in [N]$,

$$\mathbf{U}_n \leftarrow \underset{\mathbf{U_n}}{\operatorname{argmin}} \|[\![\mathcal{G}; \mathbf{U}_1, \ldots, \mathbf{U}_N]\!] - \mathcal{X}\|_F^2 ,$$

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- core update has closed form $\mathcal{G} \leftarrow \mathcal{X} \times_1 \mathbf{U}_1^\top \cdots \times_N \mathbf{U}_N^\top$

Previous work: one pass algorithm via HOOI

[Malik & Becker 2018]:

- (+) sketch design matrix to reduce size of HOOI subproblems
- (+) exploit Tucker structure of design matrix
- (-) expensive slow reconstruction (via iterative optimization)
- (-) no error guarantees for one pass algorithm

Background: randomized sketches

idea: random matrix $\mathbf{\Omega}$ is not orthogonal to range of interest (whp)

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examples of DRMS: multiplication by random matrix $\mathbf{\Omega}$ that is

- gaussian
- sparse [Achlioptas 2003, Li et al. 2006]
- SSFRT [Woolfe et al. 2008]
- tensor random projection (TRP) [Sun et al. 2018]
- ...

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- **Factor sketch (k)**. For each $n \in [N]$,
  fix random DRM $\boldsymbol{\Omega}_n \in \mathbb{R}^{I_{(-n)} \times k_n}$ and compute the sketch

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- define $(\mathcal{H}, \mathbf{V}_1, \ldots, \mathbf{V}_N) = \textsc{Sketch}\left(\mathcal{X}; \{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n \in [N]}\right)$

Low memory DRMs

factor sketch DRMs are big!

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how to store?

- don't store DRMS; instead, use pseudorandom number generator to generate (parts of) DRMs as needed.
- use structured DRM:
  - TRP generates DRM as Khatri-Rao product of simpler, smaller DRMs
  - behaves approximately like a Gaussian sketch

Source: [Sun et al. 2018, Rudelson 2012]

Recovery: factor matrices

- compute QR factorization of each factor sketch $\mathbf{V}_n$:

$$\mathbf{V}_n = \mathbf{Q}_n \mathbf{R}_n$$

where $\mathbf{Q}_n$ is orthonormal and $\mathbf{R}_n$ is triangular

Two pass algorithm

Algorithm Two Pass Sketch and Low Rank Recovery

Given: tensor $\mathcal{X}$, DRMs $\{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n\in[N]}$ with parameters $\mathbf{k}$ and $\mathbf{s} \geq \mathbf{k}$

1. *Sketch.* $(\mathcal{H}, \mathbf{V}_1, \dots, \mathbf{V}_N) = \textsc{Sketch}\left(\mathcal{X}; \{\mathbf{\Phi}_n, \mathbf{\Omega}_n\}_{n\in[N]}\right)$
2. *Recover factor matrices.* For $n \in [N]$,

$$(\mathbf{Q}_n, \sim) \leftarrow \mathrm{QR}(\mathbf{V}_\mathrm{n})$$

3. *Recover core.*

$$\mathcal{W} \leftarrow \mathcal{X} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N$$

Return: Tucker approximation $\tilde{\mathcal{X}} = [\![\mathcal{W}; \mathbf{Q}_1, \dots, \mathbf{Q}_N]\!]$ with rank $\leq \mathbf{k}$

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accesses $\mathcal{X}$ twice: 1) to sketch 2) to recover core

Intuition: one pass core recovery

- we want to know $\mathcal{W}$:
  compression of $\mathcal{X}$ using factor range approximations $\mathbf{Q}_n$
- we observe $\mathcal{H}$:
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we can solve for $\mathcal{W}$: $\mathbf{s} > \mathbf{k}$, so each $\mathbf{\Phi}_n^\top \mathbf{Q}_n$ has a left inverse (whp):

$$\mathcal{W} \approx \mathcal{H} \times_1 (\mathbf{\Phi}_1^\top \mathbf{Q}_1)^\dagger \times \cdots \times_N (\mathbf{\Phi}_N^\top \mathbf{Q}_N)^\dagger$$

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- *Recover core.*

$$\mathcal{W} \leftarrow \mathcal{H} \times_1 (\mathbf{\Phi}_1^\top \mathbf{Q}_1)^\dagger \times \cdots \times_N (\mathbf{\Phi}_N^\top \mathbf{Q}_N)^\dagger$$

Return: Tucker approximation $\hat{\mathcal{X}} = [\![\mathcal{W}; \mathbf{Q}_1, \ldots, \mathbf{Q}_N]\!]$

accesses $\mathcal{X}$ only once, to sketch

Source: [Sun et al. 2019]

Fixed rank approximation

to truncate reconstruction to rank $\mathbf{r}$, truncate core:

Lemma

For a tensor $\mathcal{W} \in \mathbb{R}^{k_1 \times \cdots \times k_N}$, orthogonal matrices $\mathbf{Q}_n \in \mathbb{R}^{k_n \times r_n}$,

$$[\![\mathcal{W} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N]\!]_{\mathbf{r}} = [\![\mathcal{W}]\!]_{\mathbf{r}} \times_1 \mathbf{Q}_1 \cdots \times_N \mathbf{Q}_N,$$

where $[\![\cdot]\!]$ denotes the best rank $\mathbf{r}$ Tucker approximation.

Fixed rank approximation

to truncate reconstruction to rank **r**, truncate core:

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where $[\![\cdot]\!]$ denotes the best rank **r** Tucker approximation.

$\implies$ compute fixed rank approximation using, e.g., HOOI on (small) core approximation $\mathcal{W}$

Tail energy

For each unfolding $\mathbf{X}^{(n)}$, define its ρ*th tail energy* as

$$(\tau_\rho^{(n)})^2 := \sum_{k>\rho}^{\min(I_n, I_{(-n)})} \sigma_k^2(\mathbf{X}^{(n)}),$$

where $\sigma_k(\mathbf{X}^{(n)})$ is the kth largest singular value of $\mathbf{X}^{(n)}$.

Guarantees (I)

Theorem (Recommended parameters [Sun et al. 2019])

Sketch $\mathcal{X}$ with Gaussian DRMs of parameters $\mathbf{k} = \mathbf{r} + 1$, $\mathbf{s} = 2\mathbf{k} + 1$. Form a rank $\mathbf{r}$ Tucker approximation $\hat{\mathcal{X}}$ using the one pass algorithm. Then

$$\mathbb{E}\|\mathcal{X} - \hat{\mathcal{X}}\|_F^2 \leq 4 \sum_{n=1}^{N} (\tau_{r_n}^{(n)})^2.$$

If $\mathcal{X}$ is truly rank $\mathbf{r}$, we obtain the true Tucker factorization!

Guarantees (II)

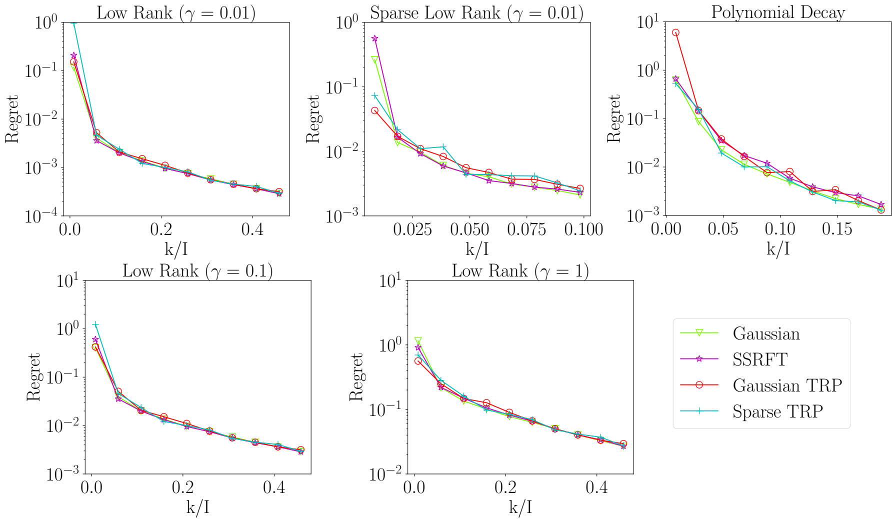
Theorem (Detailed guarantee [Sun et al. 2019])

Sketch $\mathfrak{X}$ with Gaussian DRMs of parameters $\mathbf{k}$, $\mathbf{s} \geq 2\mathbf{k} + 1$. Form a rank $\mathbf{r}$ Tucker approximation $\hat{\mathfrak{X}}$ using the one pass algorithm. Then

$$\mathbb{E}\|\mathfrak{X} - \hat{\mathfrak{X}}\|_F^2 \leq (1 + \Delta) \min_{1 \leq \rho_n < k_n - 1} \sum_{n=1}^{N} \left(1 + \frac{\rho_n}{k_n - \rho_n - 1}\right) (\tau_{\rho_n}^{(n)})^2$$

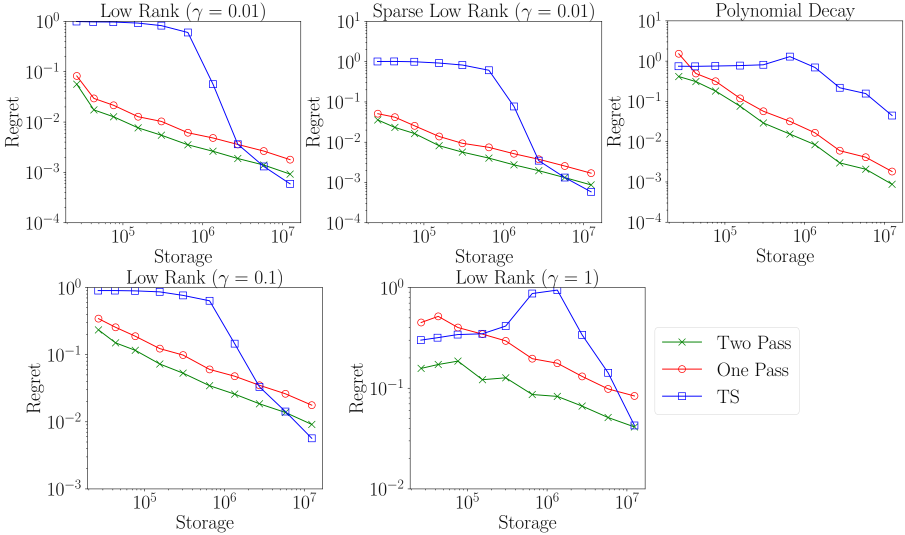
where $\Delta = \max_{n=1}^{N} k_n/(s_n - k_n - 1)$

Different DRMs perform similarly



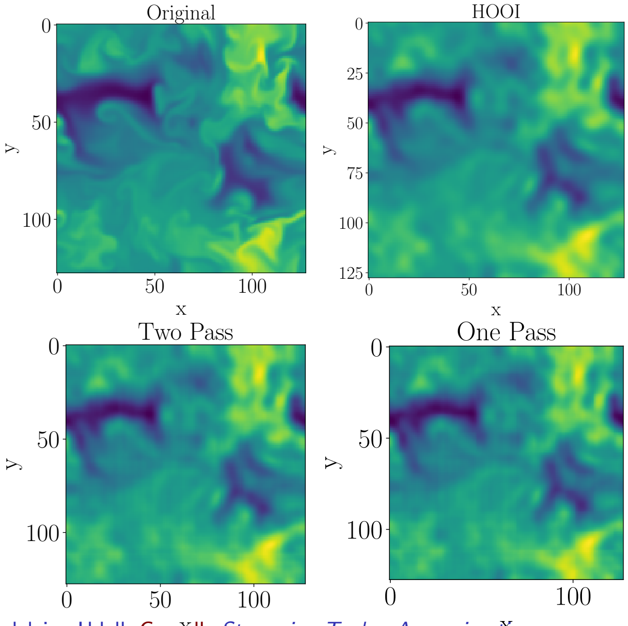
Comments: Synthetic data, $I = 600$ and $\mathbf{r} = (5, 5, 5)$. $k/I = .4 \implies 20\times$ compression.

Sensible reconstruction at practical compression level



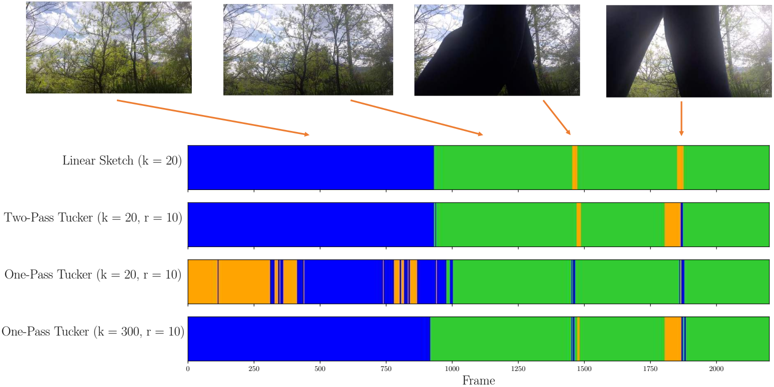
Comments: Error of fixed-rank approximation relative to HOOI for $r = 10$, $I = 300$ using TRP. Total memory use is $((2k+1)^N + kIN)$ and $(Kr^{2N} + K * r^{2N-2})$. Low-rank data uses $\gamma = 0.01, 0.1, 1$.

Combustion simulation



Comments:
$1408 \times 128 \times 128$ simulated combustion data from [Lapointe, Savard & Blanquart 2015].

Video scene classification



Comments: Video data $2200 \times 1080 \times 1980$. Classify scenes using k-means on: 1) linear sketch along the time dimension $k = 20$ (Row 1); 2) The Tucker factor along the time dimension, computed via our two pass (Row 2) and one pass (Row 3) sketching algorithm $(r, k, s) = (10, 20, 41)$. 3) The Tucker factor along the time dimension, computed via our one pass (Row 4) sketching algorithm $(r, k, s) = (10, 300, 601)$.

Summary

Streaming Tucker approximation compresses tensor without storing it.

useful for:

- streaming data
- distributed data
- low memory compute

key ideas:

- form linear sketch of tensor and recover from sketch
- random projection of tensor preserves dominant information

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