Spectral gap in PEPS

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A. Lucia, DPG, A. Pérez-Hernández, in preparation
Question:
Do self-correcting quantum memories exist in 2D?
If so, it is due to topological order.
Quantum many-body systems and their spectral gap

Spin $s$ particles = finite dim Hilbert space

Total Hilbert space = tensor product of local ones

Translational invariant finite range interaction

Hamiltonian:

$$H = \sum_i h_i \otimes I_{\text{rest}}$$

Eigenstates associated to $\lambda_0(N) = \text{ground states}$

Eigenstates associated to $\lambda_1(N) = \text{excited states}$

Thermal (or Gibbs) state: $\rho_\beta = \frac{e^{-\beta H}}{\text{tr}(e^{-\beta H})}$

Spectral gap: $\gamma_N = \lambda_1(N) - \lambda_0(N)$

The system has gap if there exists $c > 0$ such that $\Delta_N > c$ for all $N$?
Spectral gap: a central concept

Spectral Gap in **condensed matter physics:**
- It defines the concept of quantum *phase, phase transition, phase diagram,* ...

Spectral Gap in **quantum information and computation:**
- It measures the efficiency in adiabatic quantum computation and quantum state engineering
Topological phases

1. Degeneracy of the ground state in Hamiltonian depends on topology
2. All ground states are indistinguishable locally
4. To move between ground states: non-local operator.

CANDIDATES TO BE GOOD QUANTUM MEMORIES

Protected space = ground space

Errors need to accumulate in a non-local pattern to change the protected information. This is unlikely.

This is proven true in 4D. What about 2D and 3D? Here we will focus on 2D
How to construct topologically ordered systems. PEPS

They approximate well GS of local Hamiltonians (Hastings)
Basics in TNS. Box-leg notation for tensors

Each leg = one index

Vector:
\[ V = \sum_i v_i |i\rangle \]

Matrix:
\[ A = \sum_{i,j} A_{i,j} |i\rangle |j\rangle \]

Joining leg = tensor contraction

Scalar product:
\[ \sum_i v_i w_i \]

Matrix Multiplication:
\[ AB = \sum_i A_{ij} B_{jk} |i\rangle |k\rangle \]

\[ = \sum_i A_{ij} v_i w_j \]
1D PEPS = MPS

\[ |\text{MPS} \rangle = \sum_{i_1, \ldots, i_n} \text{tr}(A_{i_1}A_{i_2}\cdots A_{i_N}) |i_1i_2\cdots i_N \rangle \]

Physical index. Dimension \( d \)

Virtual index. Dim \( D = \text{bond dimension} \)
Parent Hamiltonian

$$h = 0$$

$$h = P$$

$$= 0$$

Parent Hamiltonian

\[ H = \sum_i h_i \quad H \geq 0 \quad H |\text{MPS}\rangle = 0 \quad \text{MPS is GS of } H \]

The same in 2D

\[ A_{\alpha,\beta,\gamma,\delta}^n = \gamma \quad n \quad \beta \]

|\text{PEPS}\rangle =
Parent Hamiltonian

\[ h = P \]

\[ H = \sum_i h_i \]

\[ H \geq 0 \]

\[ H|\text{PEPS}\rangle = 0 \]
Topology in PEPS. Gauge symmetry

$G$ any finite group. For example $G = \mathbb{Z}_2 = \{1, Z\}$
Topology in PEPS. Gauge symmetry

Contractible loops of $\mathbb{Z}$ vanish.

What about not contractible loops?
Non contractible loops can be arbitrarily deformed but they do not vanish.
Non contractible loops can be arbitrarily deformed but they do not vanish. New ground states of the parent Hamiltonian (which are locally equal).
Excitations = open strings

Open strings can be arbitrarily deformed except for the extreme points (quasi-particles).
All of them have the same energy (=2). Quasi-particles can move freely.
Anyonic statistics (G non-abelian)

Moving one excitation around another one has a non-trivial effect.
Topological phases

1. Degeneracy of the ground state in Hamiltonian depends on topology
2. All ground states are indistinguishable locally
4. To move between ground states: non-local operator.

Those models are exactly Kitaev’s quantum double models (the case of $G = \mathbb{Z}_2$ is the Toric Code)

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Lifetime of topological quantum memories
Quantum memories

Take a 2D topological model with Hamiltonian $H_{\text{top}}$

E.g Kitaev’s quantum double of a group $G$ (Toric code for $G = \mathbb{Z}_2$).

Assume thermal noise (weak coupling limit). Evolution given by Linbladian:

$$\rho_t = e^{t \mathcal{L}_\beta} (\rho_0)$$

How long does it take to reach $\rho_\infty = e^{-\beta H_{\text{top}}}$?

Short memory time $\Leftrightarrow \text{Gap}(\mathcal{L}_\beta) \geq c_\beta > 0$, for all $\beta$
Quantum memories

Previous results for 2D quantum memories:

Alicki, Fannes, Horodecki 2007. For the Toric code: \( \text{Gap}(\mathcal{L}_\beta) \geq c_\beta > 0 \), for all \( \beta \)

Komar, Landon-Cardinal, Temme 2016. For abelian models \( \text{Gap}(\mathcal{L}_\beta) \geq c_\beta > 0 \), for all \( \beta \)

What about the non-abelian case?

**Theorem** (A. Lucia, DPG, A. Pérez-Hernández, in preparation):
For all (even **non abelian**) quantum double models \( \text{Gap}(\mathcal{L}_\beta) \geq c_\beta > 0 \), for all \( \beta \)
Quantum memories

**Theorem** (A. Lucia, DPG, A. Pérez-Hernández, in preparation):  
For all (even non abelian) quantum double models $\text{Gap}(\mathcal{L}_\beta) \geq c_\beta > 0$, for all $\beta$

**Proof:** Consider $e^{-\beta H_{\text{top}}}$

At each site we do a partial transposition: $| \cdot \rangle\langle \cdot | \rightarrow | \cdot \rangle| \cdot \rangle$

We obtain a PEPS, called the *thermofield double* $| \text{TMD}_\beta \rangle$

\[
\text{Gap} \left( H_{\text{TMD}_\beta} \right) = \text{Gap} \left( \mathcal{L}_\beta \right)
\]

The problem boils down to estimate the gap of a PEPS parent Hamiltonian

**Theorem** (Scarpa et al PRL 2020): The existence of spectral gap is an UNDECIDABLE problem, even for parent Hamiltonians of PEPS.
Solution in this case via bulk-boundary correspondence in PEPS

Poilblanc et al 2013.
It is a mixed 1D state living on the virtual d.o.f.
*Mediates the correlations in the system*
*Defines the parent Hamiltonian of the state*
Spectral gap via boundary state

Spectral gap in PEPS

Conjecture Poilblanc et al. 2013 (numerical evidence): the parent Hamiltonian of the PEPS has gap if and only if the boundary state is the Gibbs state of a short-range Hamiltonian.

Intuition. Araki’s theorem: Gibbs state of finite range 1D Hamiltonians have exponentially decaying correlations

Remember that boundary states mediate the correlations in a PEPS.
Spectral gap in PEPS

**Theorem 1**: If the boundary state is approximately factorizable, then the bulk Hamiltonian is gapped.

A 1D state is approximately factorizable if \( \rho_{ABC} \approx \Lambda_{AB} \Omega_{BC} \)

The case of *exact* factorization implies that the Hamiltonian terms commute with each other and hence the system is gapped. *(Remember boundary states define the Hamiltonian terms)*

The approximate case reduces to the martingale condition of Nachtergaele (1995)

Martingale condition is equivalent to gap (Lucia, Kastoryano 2018)
Spectral gap in PEPS

**Theorem 1**: If the boundary state is approximately factorizable, then the bulk Hamiltonian is gapped.

**Theorem 2**: Gibbs states of Hamiltonians with fast decaying interactions are approximately factorizable.

Imagine $e^{-H} \rightarrow e^{-iH} \rightarrow \text{finite depth circuit} \rightarrow \Lambda_{AB} \Omega_{BC}$
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The boundary state of \( | \text{TMD}_{\beta} \rangle \) is approximately factorizable. QED.
Thank you for your attention
Boundary state properties. Illustration in 1D

\[ |\psi\rangle = \]

\[ \rho_A = \]
\[ \rho_A = \]

\[ \rho_{\partial A^c} = \]

Boundary state
Lives on the virtual d.o.f connecting \( A \) & \( A^c \)
Encodes the correlations of the system
Boundary state properties. Illustration in 1D

\[
\rho_{\partial A} = P_A = \rho_{\partial A}^{-1}
\]
Boundary state properties. Illustration in 1D

\[ \rho_{\partial A} = \]  

\[ P_A = \]

Orthogonal projector

\[ P_A \psi = \]

Orthogonal projector
Boundary state properties. Illustration in 1D

\[
\rho_{\partial A} = \rho_{\partial A}^{-1} \partial A - \rho_{\partial A} \rho_{\partial A}^{-1} \partial A
\]

\[
P_A = \rho_{\partial A}^{-1}
\]

\[
P_A |\psi\rangle = \rho_{\partial A}
\]

Orthogonal projector
Boundary state properties. Illustration in 1D

\[ P_A = \rho_{\partial A}^{-1} \]

\[ P_A |\psi\rangle = \]

Orthogonal projector
Boundary state properties. Illustration in 1D

\[ \rho_{\partial A} = \frac{\partial A}{1 - \rho_{\partial A}} \]

\[ P_A = \sum_i (1 - P_i) \]

Orthogonal projector

\[ P_A |\psi\rangle = |\psi\rangle \]