Spectral gap in PEPS

David Pérez-García

M. Kastoryano, A. Lucia, DPG, Commun. Math. Phys. (2019) 366: 895

DPG, A. Pérez-Hernández, arXiv:2004.10516

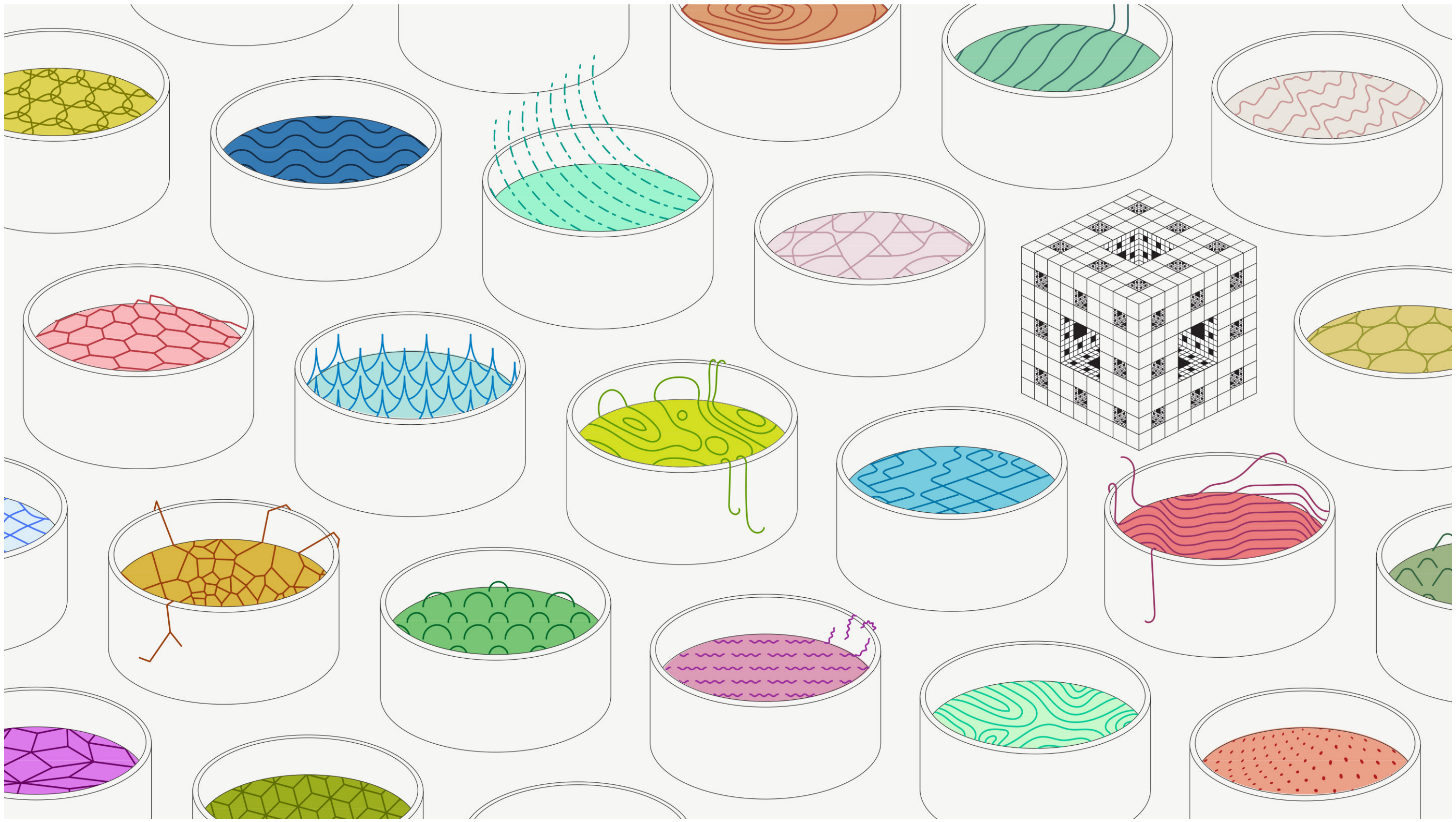
A. Lucia, DPG, A. Pérez-Hernández, in preparation

GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA Y COMPETITIVIDAD
QUITEMAD
Quantum Information Technologies in Madrid
Project 648913

Question:

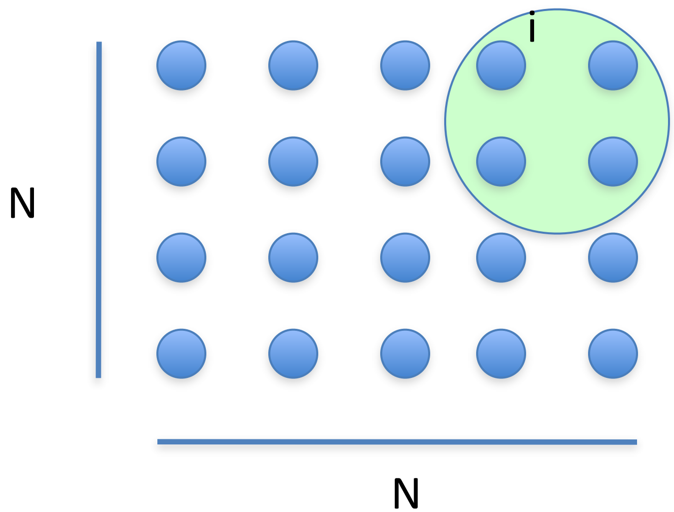
Do self-correcting quantum memories exist in 2D?

If so, it is due to topological order.



Picture from Quanta Magazine

Quantum many-body systems and their spectral gap



Spin s particles = finite dim Hilbert space

Total Hilbert space = tensor product of local ones

Translational invariant finite range interaction

Hamiltonian:

$$H = \sum_i h_i \otimes Id_{\text{rest}}$$

Eigenstates associated to $\lambda_0(N)$ = ground states

Eigenstates associated to $\lambda_1(N)$ = excited states

Thermal (or Gibbs) state: $\rho_\beta = \dfrac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}$

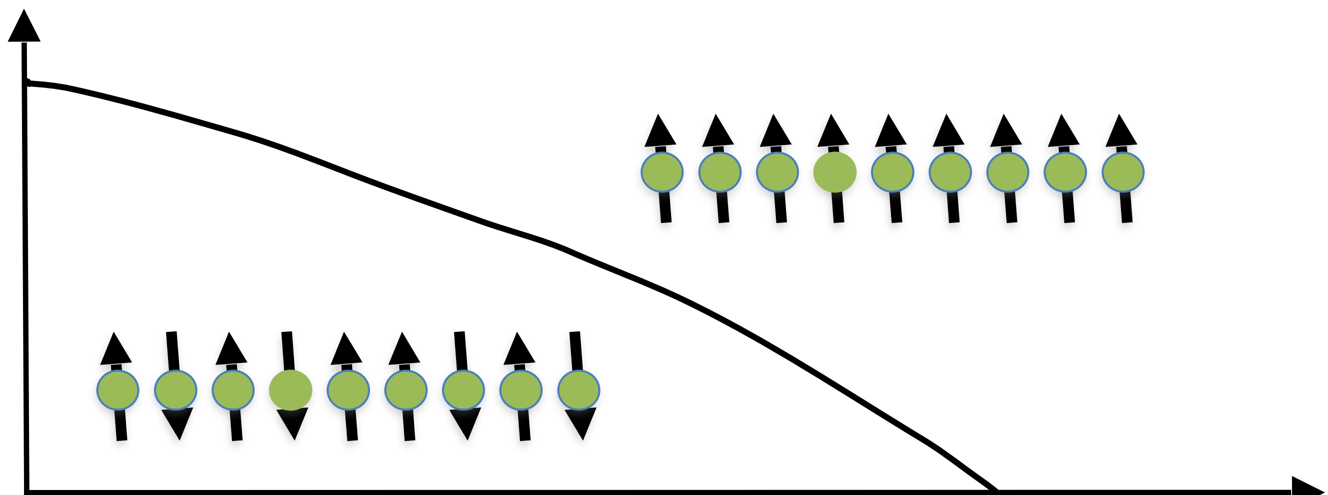
Spectral gap: $\gamma_N = \lambda_1(N) - \lambda_0(N)$

The system has gap if there exists c>0 such that $\Delta_N > c$ for all N?

Spectral gap: a central concept

Spectral Gap in **condensed matter physics**:

- It defines the concept of quantum ***phase***, *phase transition, phase diagram, …*



Spectral Gap in **quantum information and computation**:

- It measures the efficiency in adiabatic quantum computation and quantum state engineering

Topological phases

1. Degeneracy of the ground state in Hamiltonian depends on topology
2. All ground states are indistinguishable locally
3. Excitations behave like quasiparticles with anyonic statistics.
4. To move between ground states: non-local operator.

CANDIDATES TO BE GOOD QUANTUM MEMORIES

Protected space = ground space

Errors need to accumulate in a non-local pattern to change the protected information. This is unlikely.

This is proven true in 4D. What about 2D and 3D? Here we will focus on 2D

How to construct topologically ordered systems. PEPS

They approximate well GS of local Hamiltonians
(Hastings)

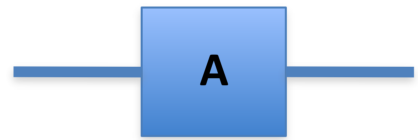
Basics in TNS. Box-leg notation for tensors

Each leg = one index

vector

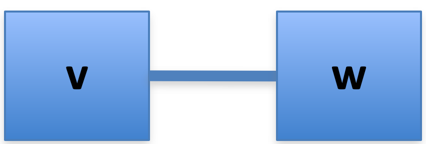
$$= \sum_i v_i |i\rangle$$

matrix



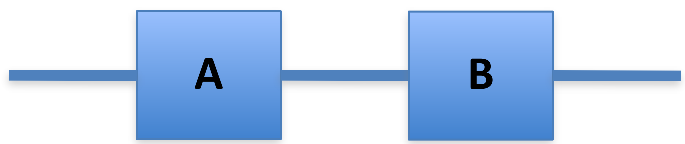
$$= \sum_{i,j} A_{i,j} |i\rangle |j\rangle$$

Joining leg = tensor contraction



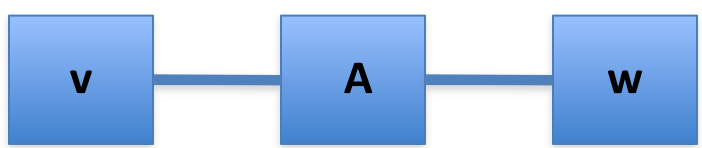
Scalar product

$$\sum_i v_i w_i$$



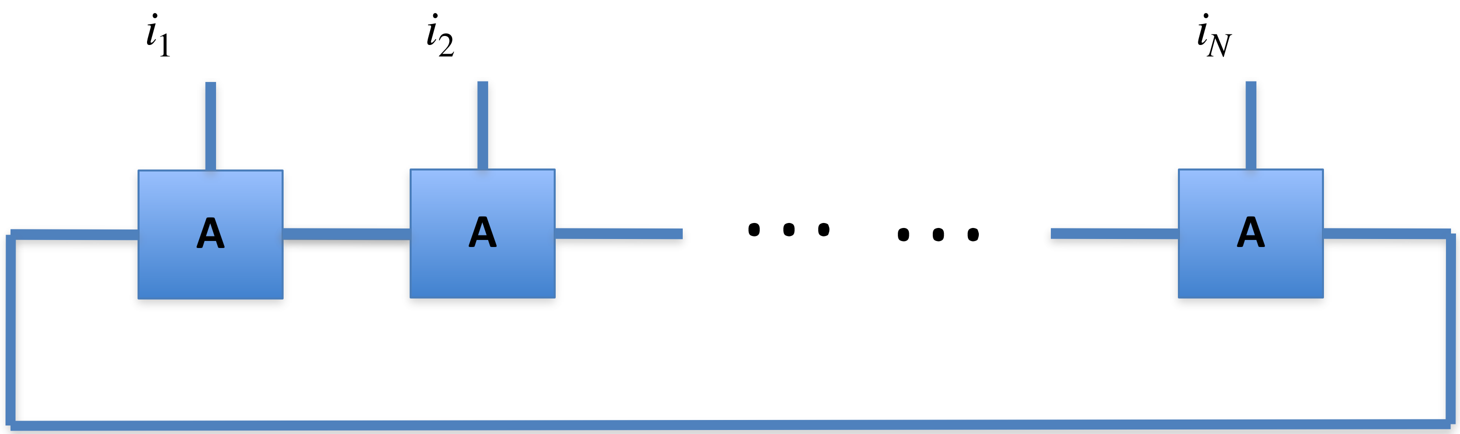
$$= \sum_i A_{ij} B_{jk} |i\rangle |k\rangle = AB$$

Matrix Multiplication



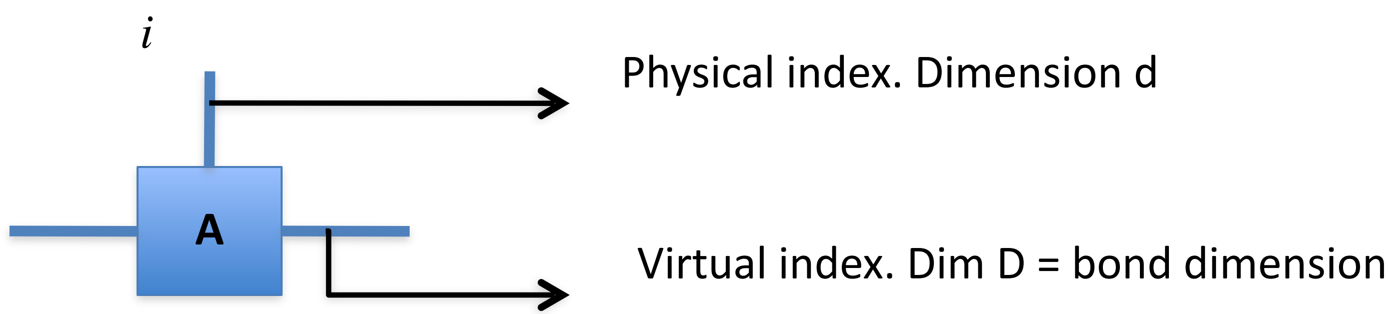
$$= \sum_i A_{ij} v_i w_j$$

1D PEPS = MPS

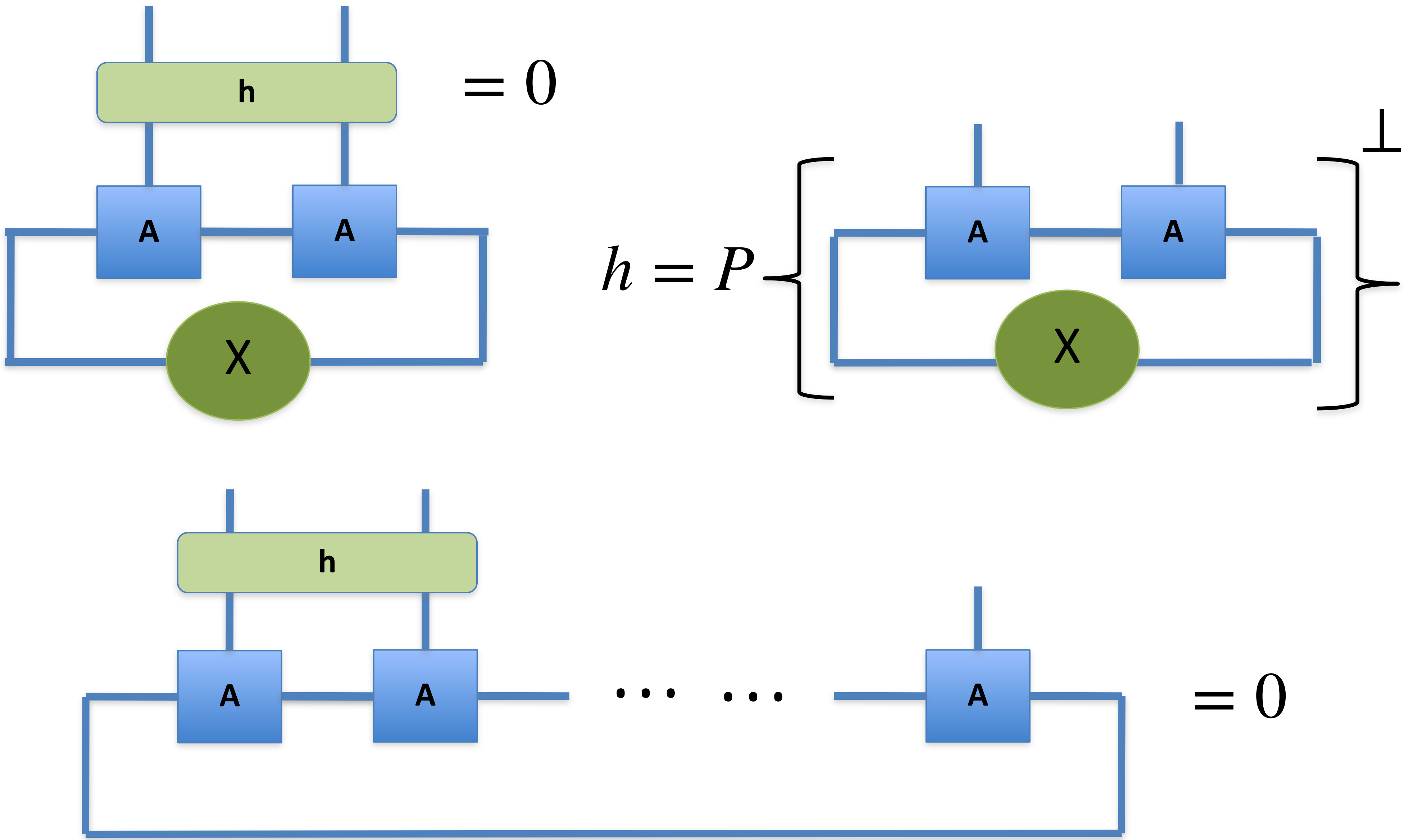


$$= |\mathrm{MPS}\rangle = \sum_{i_1,\ldots,i_n} \mathrm{tr}(A_{i_1}A_{i_2}\cdots A_{i_N})\,|\,i_1 i_2 \cdots i_N\rangle$$

$(A_i)_i =$



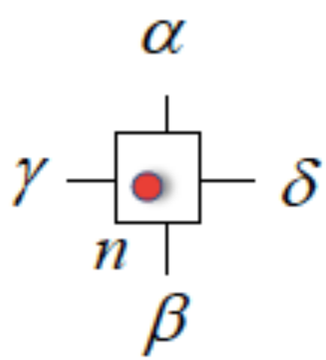
Parent Hamiltonian

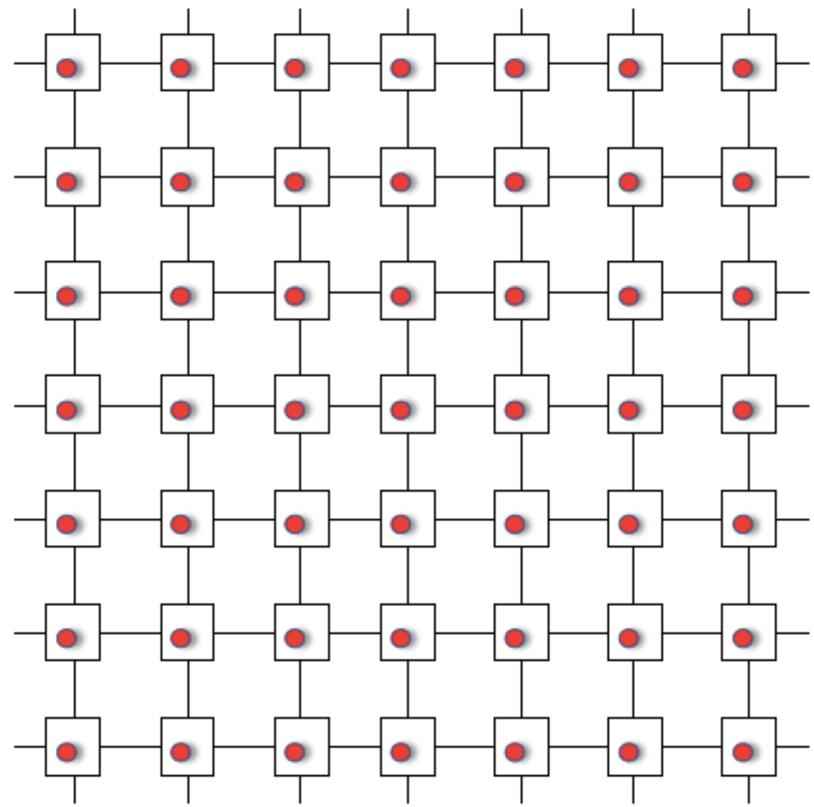


Parent Hamiltonian

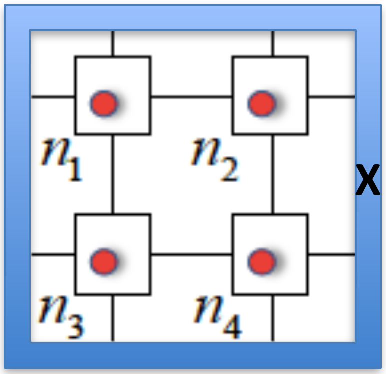
$H = \sum_i h_i$ $H \geq 0$ $H|\mathrm{MPS}\rangle = 0$ MPS is GS of H

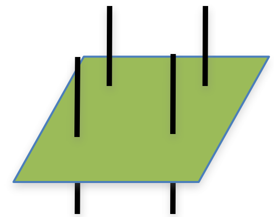
The same in 2D

$A^n_{\alpha,\beta,\gamma,\delta} =$ 

$|\mathrm{PEPS}\rangle =$ 

Parent Hamiltonian

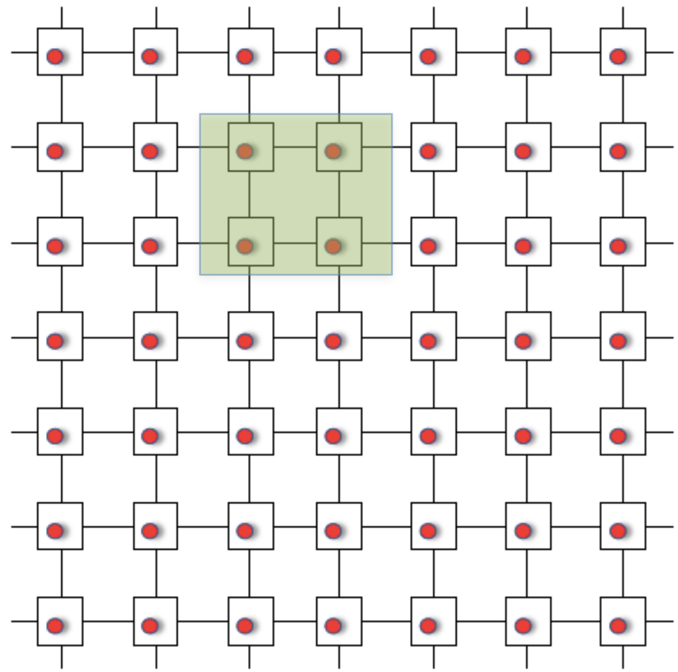
$h = P\Big\{$ 

$\Big\}^{\perp}$ = 

$$H = \sum_i h_i$$

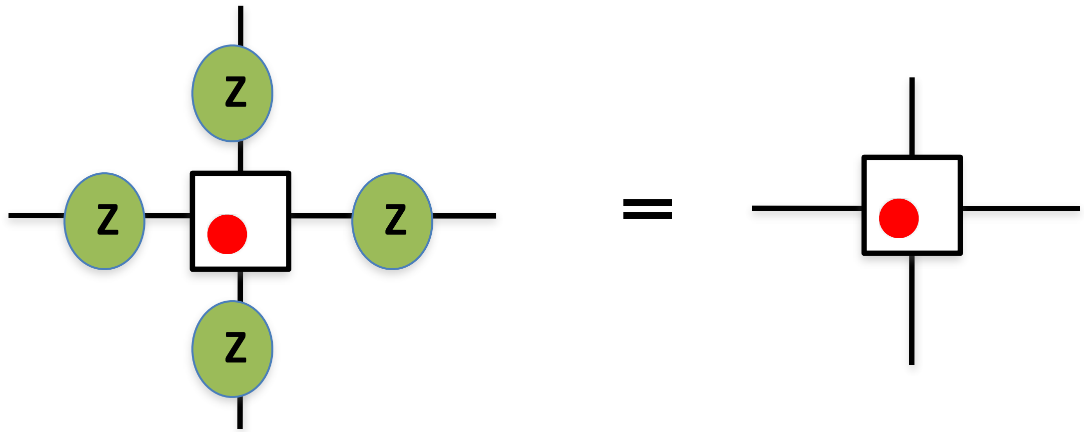
$$H \geq 0$$

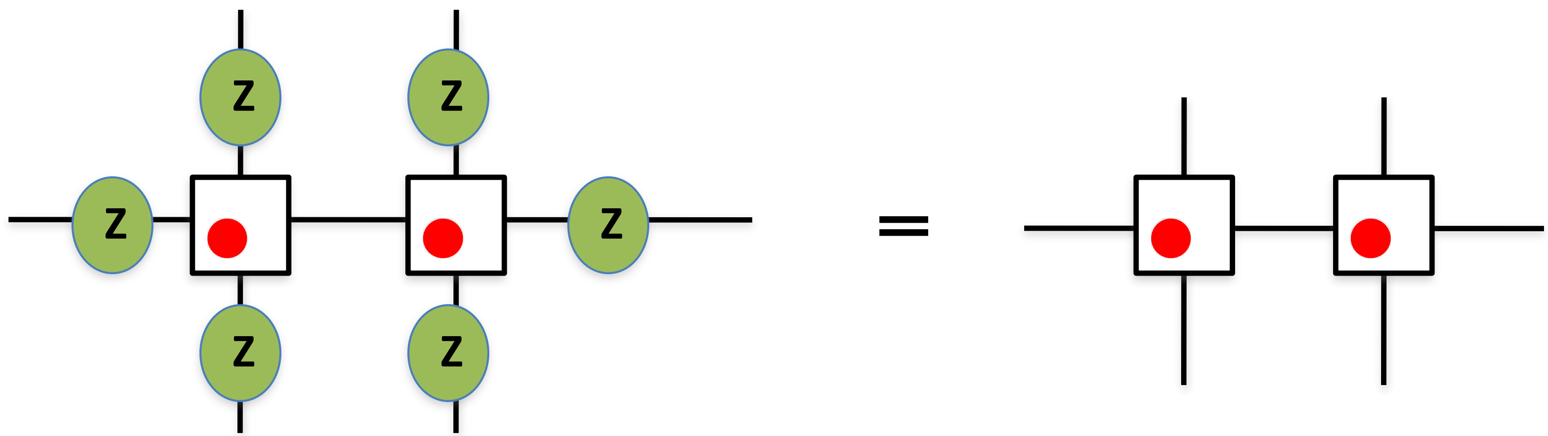
$$H|\mathrm{PEPS}\rangle = 0$$

 $= 0$

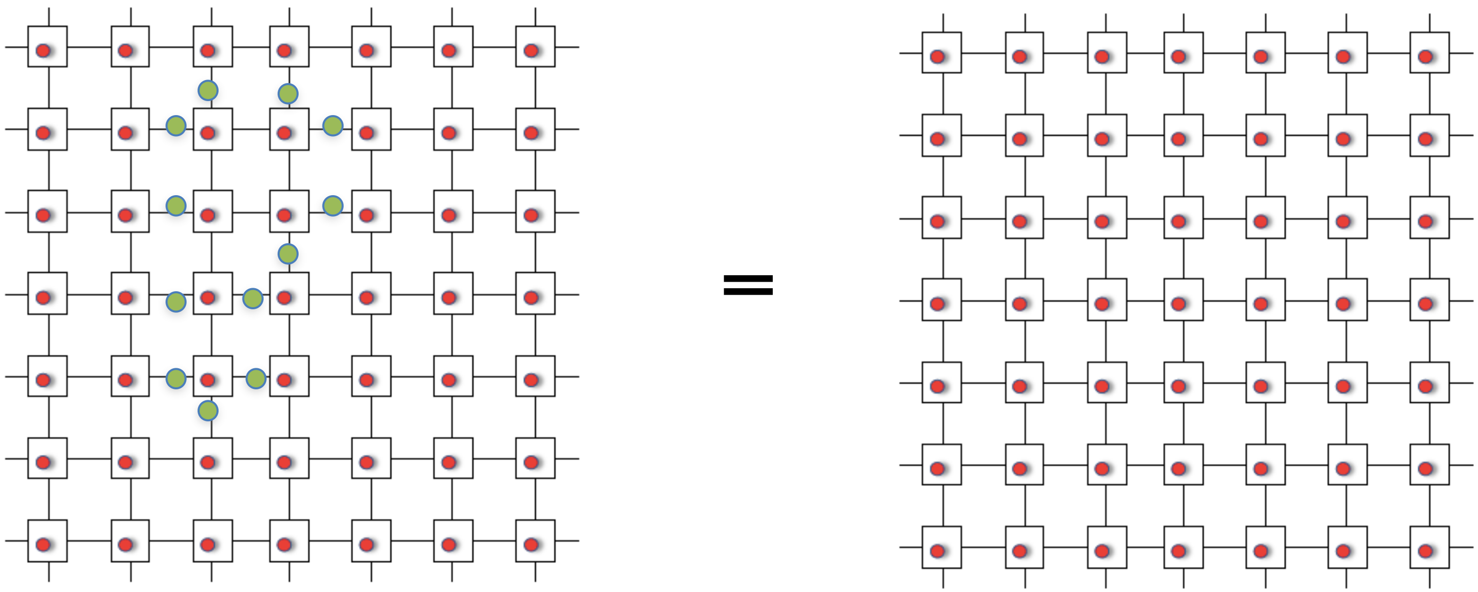
Topology in PEPS. Gauge symmetry

G any finite group. For example $G = \mathbb{Z}_2 = \{1, Z\}$





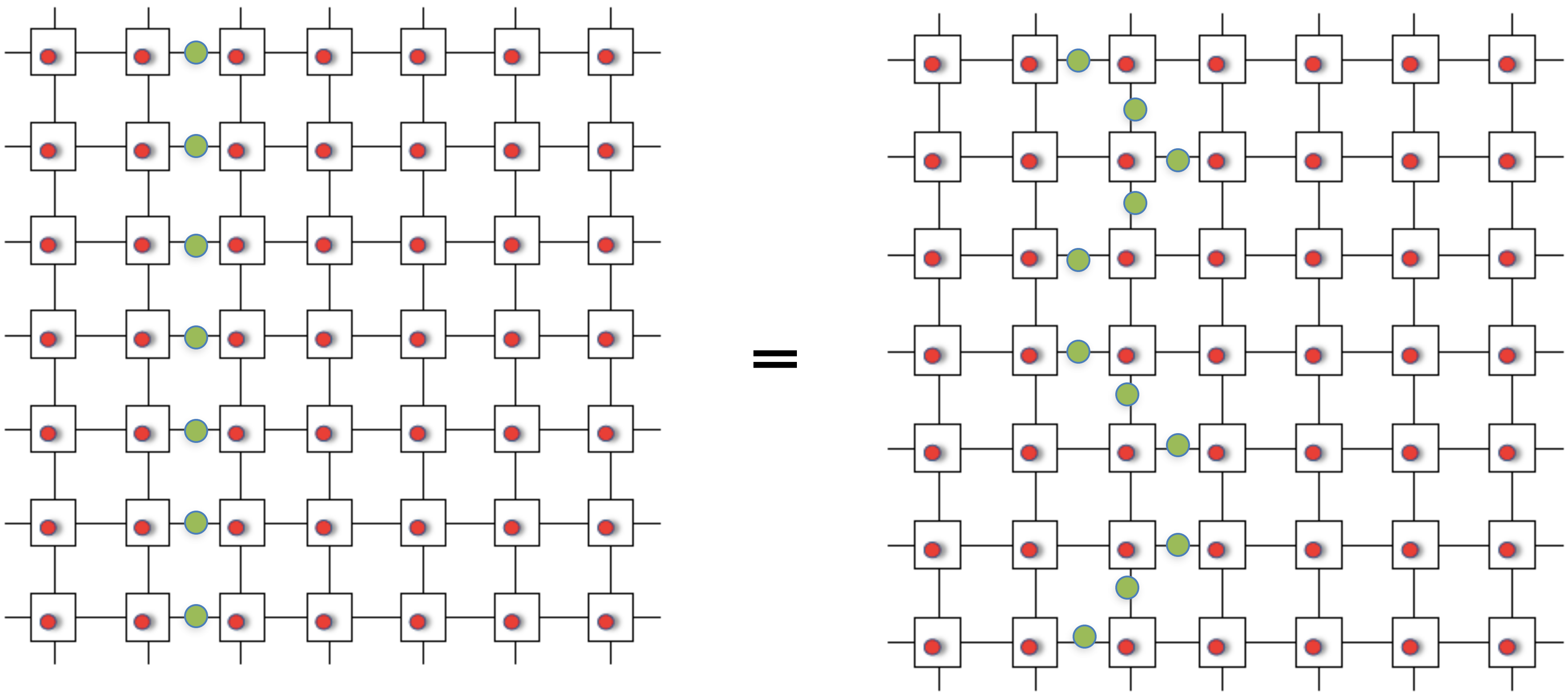
Topology in PEPS. Gauge symmetry



Contractible loops of Z vanish.

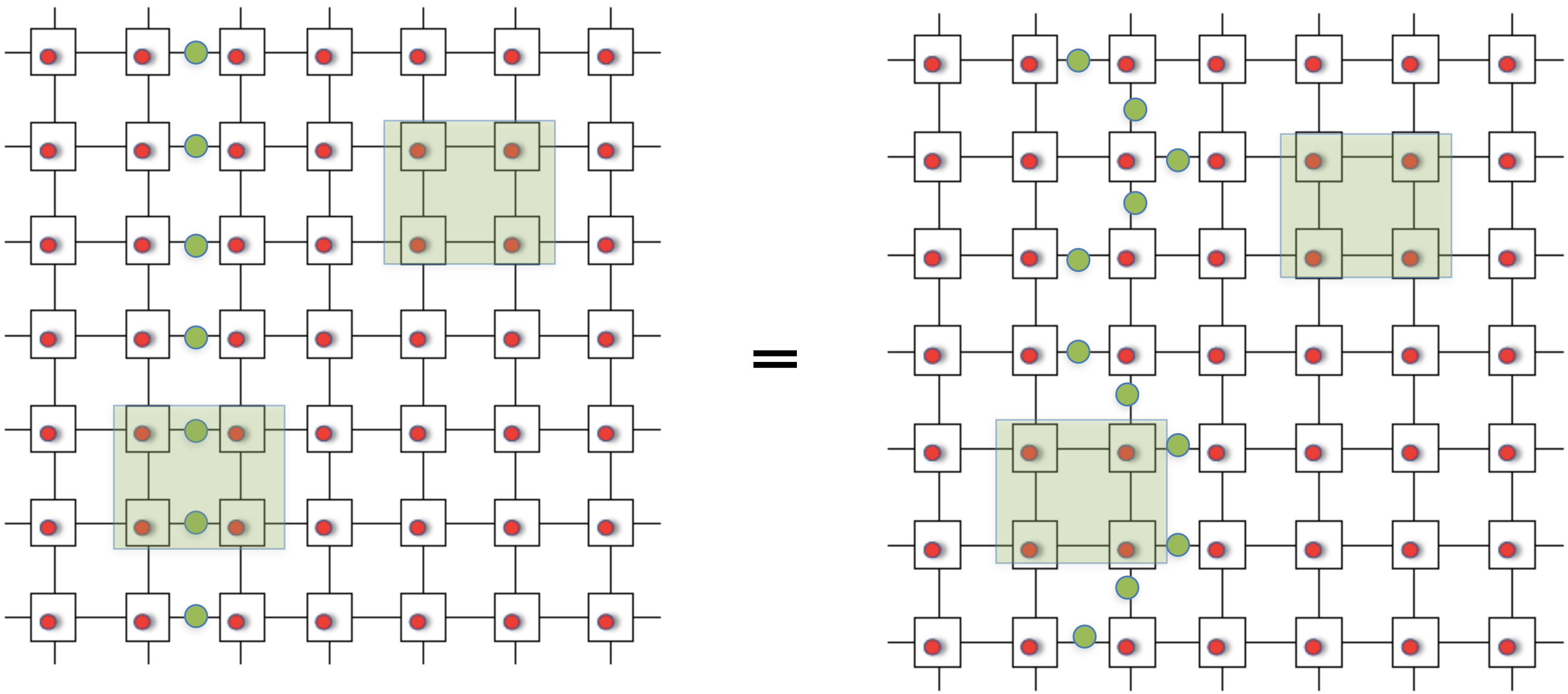
What about not contractible loops?

Topology in PEPS. Gauge symmetry



Non contractible loops can be arbitrarily deformed but they do not vanish.

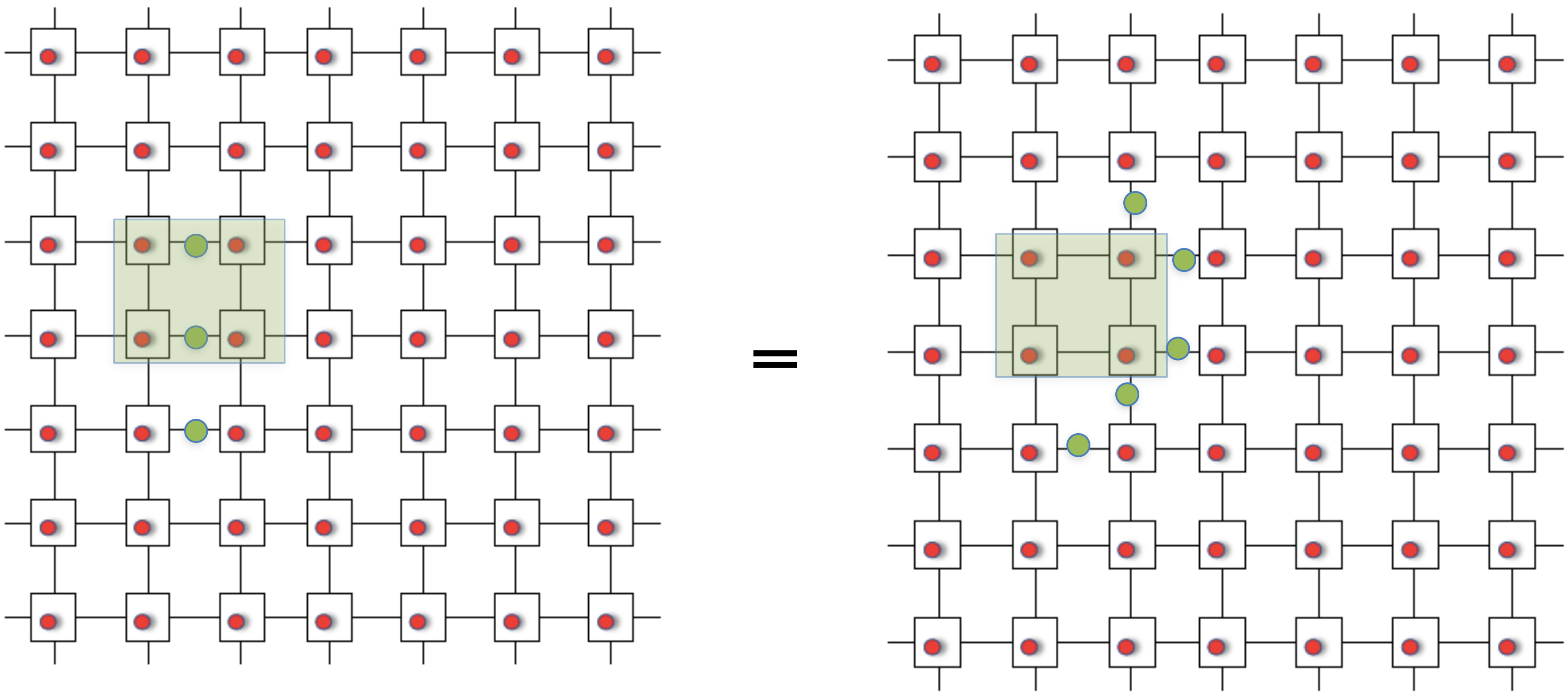
Topology in PEPS. Gauge symmetry



Non contractible loops can be arbitrarily deformed but they do not vanish.

New ground states of the parent Hamiltonian (which are locally equal).

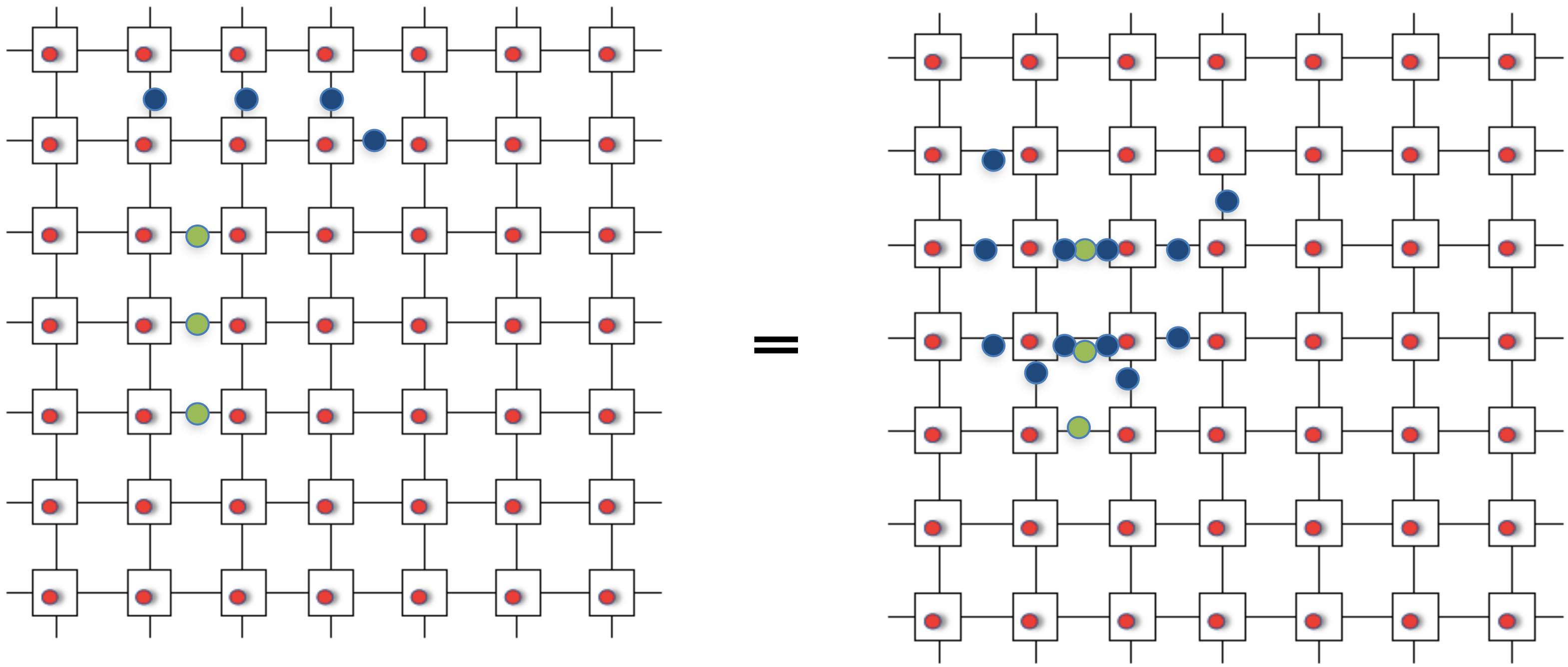
Excitations = open strings



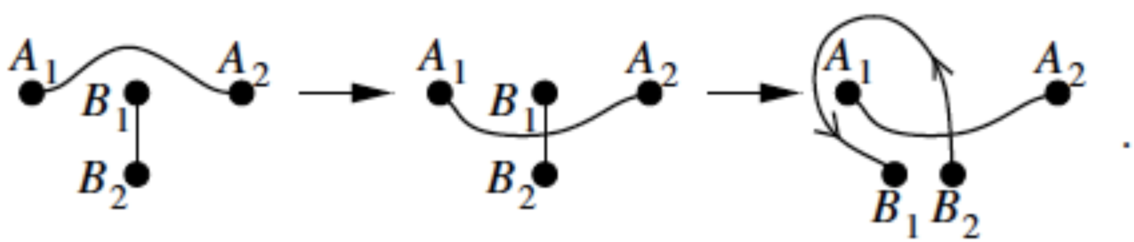
Open strings can be arbitrarily deformed except for the extreme points (quasi-particles).

All of them have the same energy (=2). Quasi-particles can move freely.

Anyonic statistics (G non-abelian)



Moving one excitation around another one has a non-trivial effect.



Topological phases

1. Degeneracy of the ground state in Hamiltonian depends on topology ✓
2. All ground states are indistinguishable locally ✓
3. Excitations behave like quasiparticles with anyonic statistics. ✓
4. To move between ground states: non-local operator. ✓

Those models are exactly Kitaev's quantum double models (the case of $G = \mathbb{Z}_2$ is the *Toric Code*)

CANDIDATES TO BE GOOD QUANTUM MEMORIES

Protected space = ground space

Errors need to accumulate in a non-local pattern to change the protected information. This is unlikely.

This is proven true in 4D. What about 2D and 3D? Here we will focus on 2D

Lifetime of topological quantum memories

Quantum memories

Take a **2D** topological model with Hamiltonian H_{top}

E.g Kitaev's quantum double of a group G (Toric code for $G = \mathbb{Z}_2$).

Assume thermal noise (weak coupling limit). Evolution given by Linbladian:

$$\boxed{\rho_t = e^{t\,\mathscr{L}_\beta}(\rho_0)}$$

How long does it take to reach $\rho_\infty = e^{-\beta H_{\text{top}}}$?

Information is lost

Short memory time $\Leftrightarrow$ $\text{Gap}(\mathscr{L}_\beta) \geq c_\beta > 0$, for all β

Quantum memories

Previous results for **2D** quantum memories:

Alicki, Fannes, Horodecki 2007. For the Toric code: $\mathrm{Gap}(\mathscr{L}_\beta) \geq c_\beta > 0$, for all β

Komar, Landon-Cardinal, Temme 2016.
For abelian models $\mathrm{Gap}(\mathscr{L}_\beta) \geq c_\beta > 0$, for all β

What about the non-abelian case?

Theorem (A. Lucia, DPG, A. Pérez-Hernández, in preparation):
For all (even **non abelian**) quantum double models $\mathrm{Gap}(\mathscr{L}_\beta) \geq c_\beta > 0$, for all β

Quantum memories

Theorem (A. Lucia, DPG, A. Pérez-Hernández, in preparation):
For all (even **non abelian**) quantum double models $\mathrm{Gap}(\mathscr{L}_\beta) \geq c_\beta > 0$, for all β

Proof:

Consider $e^{-\beta H_{\mathrm{top}}}$

At each site we do a partial transposition: $|\cdot\rangle\langle\cdot| \to |\cdot\rangle|\cdot\rangle$

We obtain a PEPS, called the *thermofield double* $|\mathrm{TMD}_\beta\rangle$

$$\mathrm{Gap}\left(H_{TMD_\beta}\right) = \mathrm{Gap}\left(\mathscr{L}_\beta\right)$$

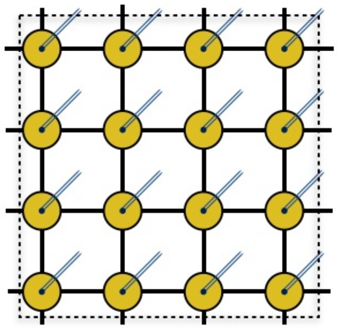
The problem boils down to estimate the gap of a PEPS parent Hamiltonian

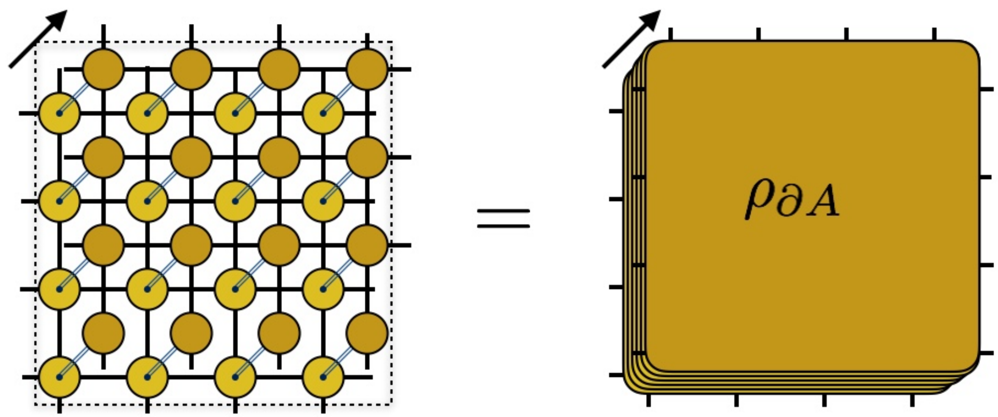
Theorem (Scarpa et al PRL 2020): The existence of spectral gap is an UNDECIDABLE problem, even for parent Hamiltonians of PEPS.

Solution in this case via bulk-boundary correspondence in PEPS

Poilblanc et al 2013.

Boundary state





It is a mixed 1D state living on the virtual d.o.f.

Mediates the correlations in the system

Defines the parent Hamiltonian of the state

Spectral gap via boundary state

M. Kastoryano, A. Lucia, DPG, Commun. Math. Phys. (2019) 366: 895

Spectral gap in PEPS

Conjecture Poilblanc et al. 2013 (numerical evidence): the parent Hamiltonian of the PEPS has gap if and only if the boundary state is the Gibbs state of a short-range Hamiltonian.

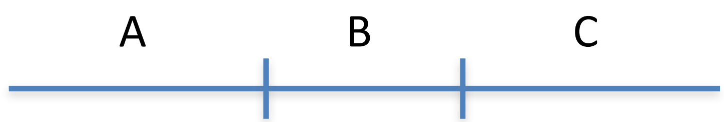
Intuition. Araki's theorem: Gibbs state of *finite range* 1D Hamiltonians have exponentially decaying correlations

Remember that boundary states mediate the correlations in a PEPS.

Spectral gap in PEPS

Theorem 1: If the boundary state is approximately factorizable, then the bulk Hamiltonian is gapped.

A 1D state is approximately factorizable if $\rho_{ABC} \approx \Lambda_{AB}\, \Omega_{BC}$



The case of *exact* factorization implies that the Hamiltonian terms commute with each other and hence the system is gapped. (**Remember boundary states define the Hamiltonian terms**)

The approximate case reduces to the martingale condition of Nachtergaele (1995)

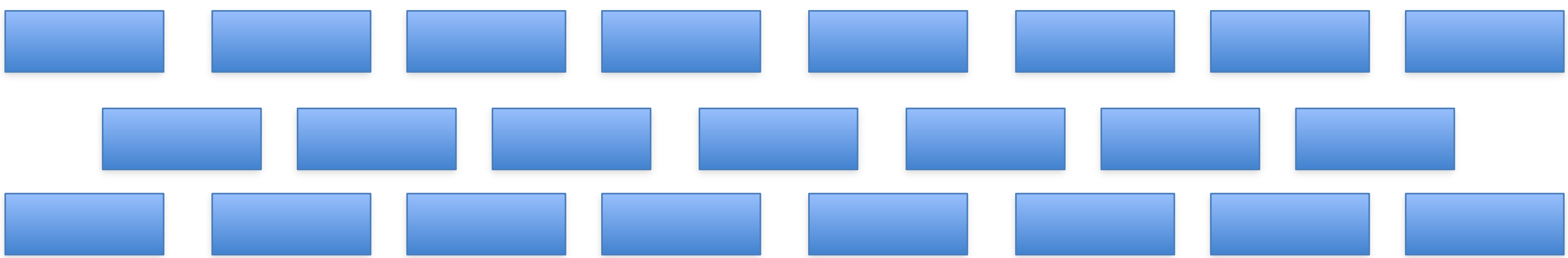
Martingale condition is equivalent to gap (Lucia, Kastoryano 2018)

Spectral gap in PEPS

Theorem 1: If the boundary state is approximately factorizable, then the bulk Hamiltonian is gapped.

Theorem 2: Gibbs states of Hamiltonians with fast decaying interactions are approximately factorizable.

Imagine $e^{-H} \to e^{-iH} \xrightarrow[\textbf{locality}]{}$ **finite depth circuit** $\to \Lambda_{AB}\,\Omega_{BC}$

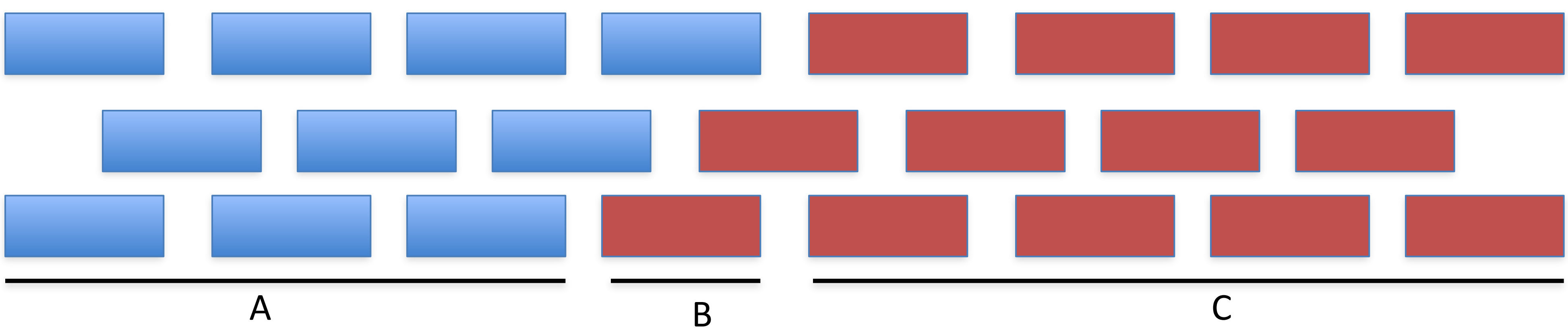


Spectral gap in PEPS

Theorem 1: If the boundary state is approximately factorizable, then the bulk Hamiltonian is gapped.

Theorem 2: Gibbs states of Hamiltonians with fast decaying interactions are approximately factorizable.

Imagine $e^{-H} \to e^{-iH} \xrightarrow[\textbf{locality}]{}$ **finite depth circuit** $\to \Lambda_{AB}\ \Omega_{BC}$



Back to quantum memories

Theorem 1: If the boundary state is approximately factorizable, then the bulk Hamiltonian is gapped.

Theorem (A. Lucia, DPG, A. Pérez-Hernández, in preparation):
For all (even **non abelian**) quantum double models $\mathrm{Gap}(\mathscr{L}_\beta) \geq c_\beta > 0$, for all β

Proof:

Consider $e^{-\beta H_{\mathrm{top}}}$

At each site we do a partial transposition: $|\cdot\rangle\langle\cdot| \to |\cdot\rangle|\cdot\rangle$

We obtain a PEPS, called the *thermofield double* $|\mathrm{TMD}_\beta\rangle$

$$\mathrm{Gap}\left(H_{TMD_\beta}\right) = \mathrm{Gap}\left(\mathscr{L}_\beta\right)$$

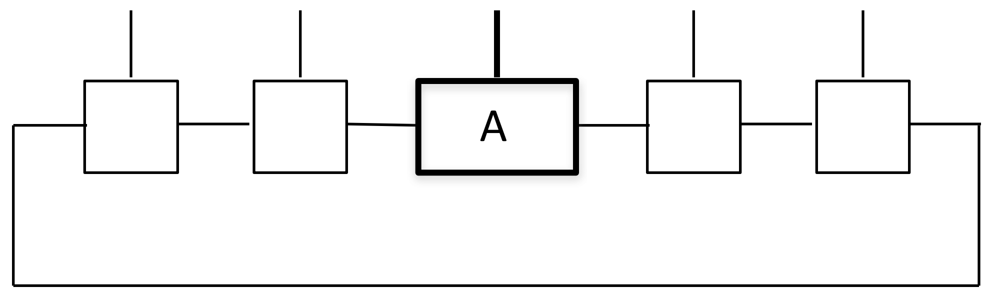
The problem boils down to estimate the gap of a PEPS parent Hamiltonian

The boundary state of $|\mathrm{TMD}_\beta\rangle$ is approximately factorizable. QED.

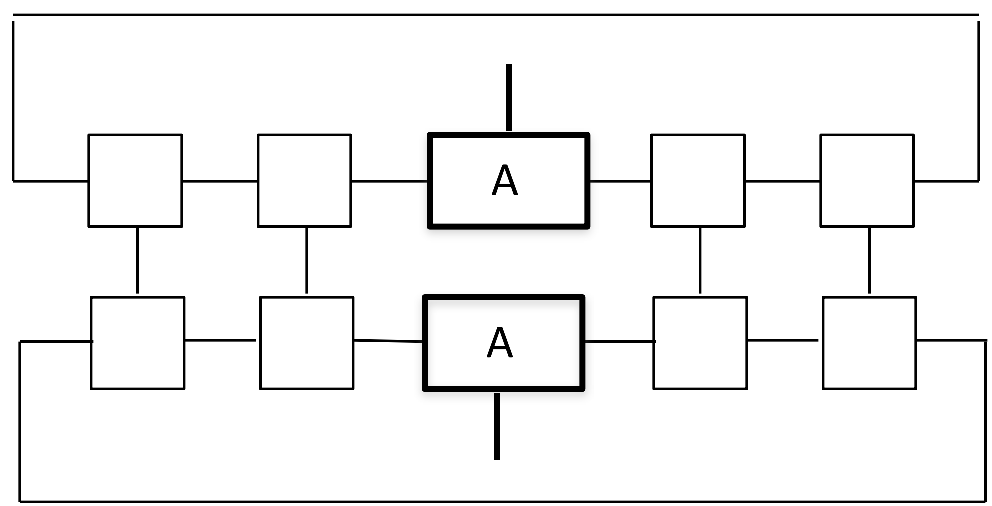
Thank you for your attention

Boundary state properties. Illustration in 1D

$|\psi\rangle =$

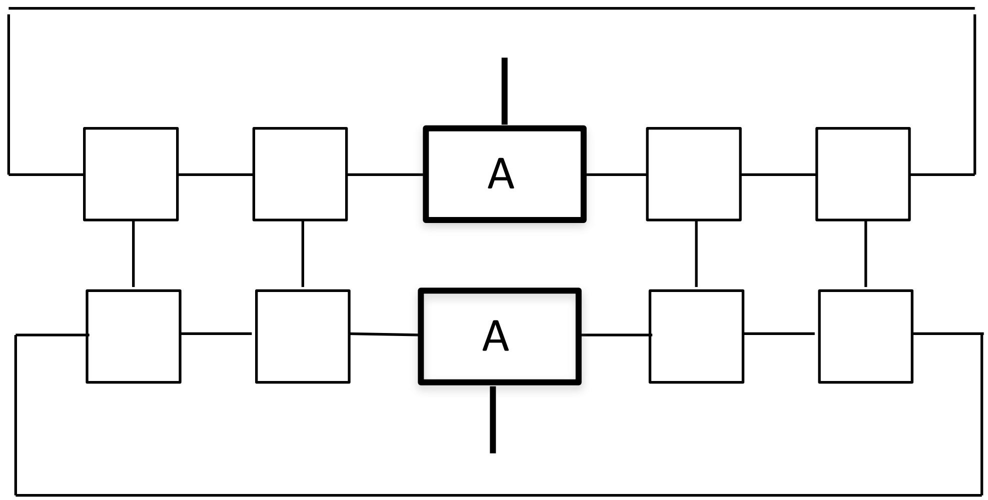


$\rho_A =$

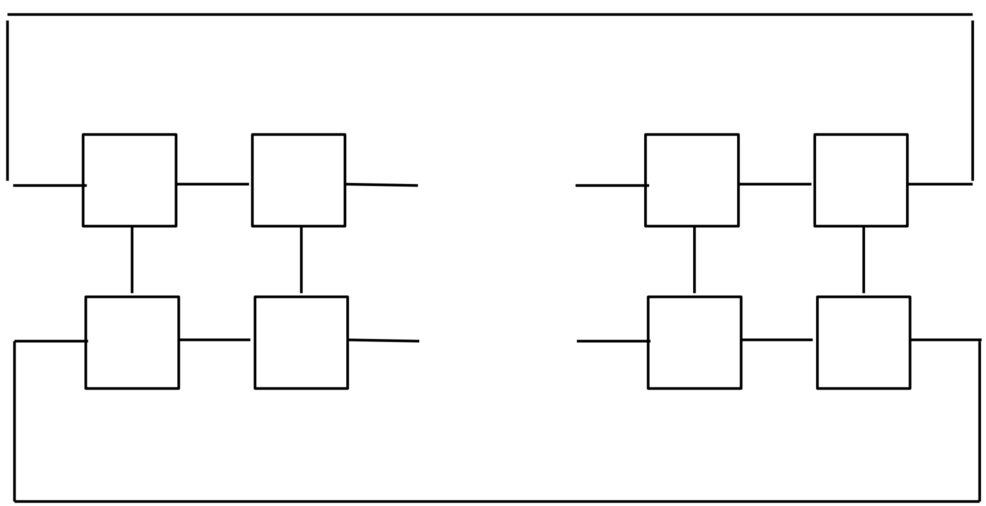


Boundary state properties. Illustration in 1D

$\rho_A =$



$\rho_{\partial A^c} =$

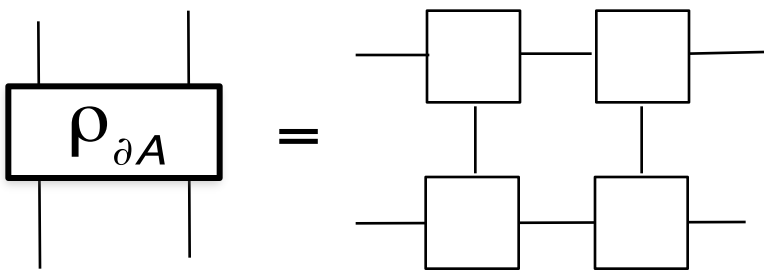


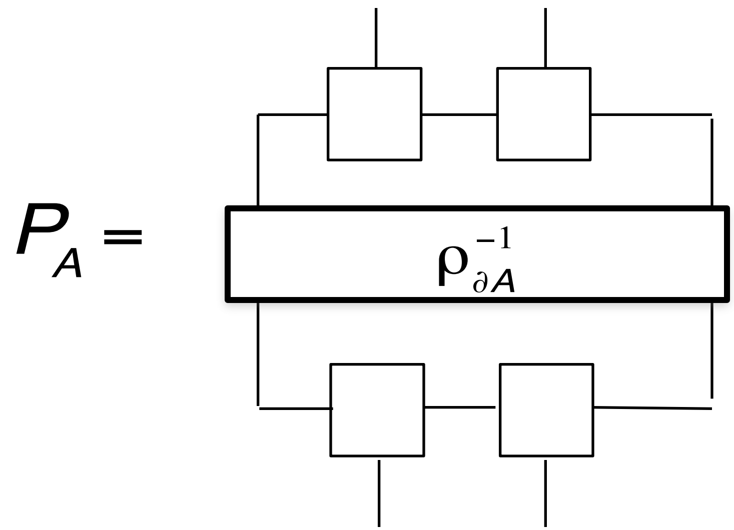
Boundary state

Lives on the virtual d.o.f connecting A & A^c

Encodes the correlations of the system

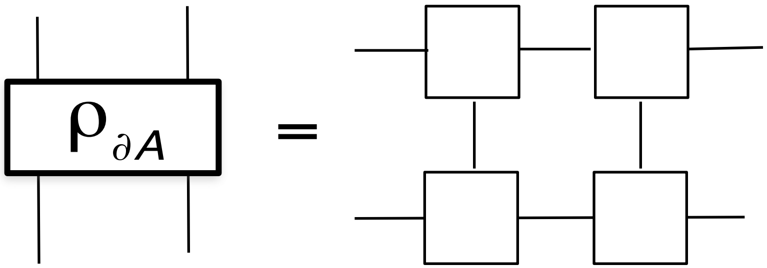
Boundary state properties. Illustration in 1D



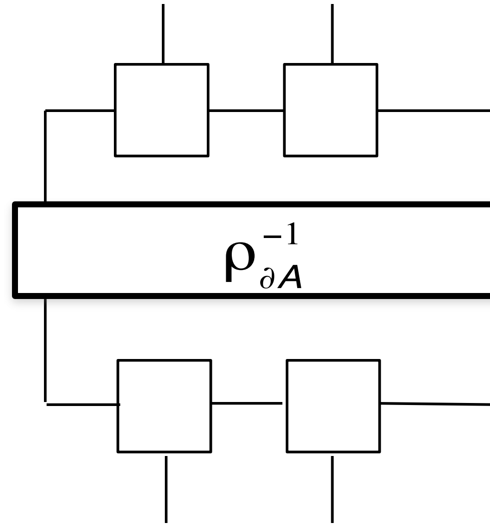


Orthogonal projector

Boundary state properties. Illustration in 1D

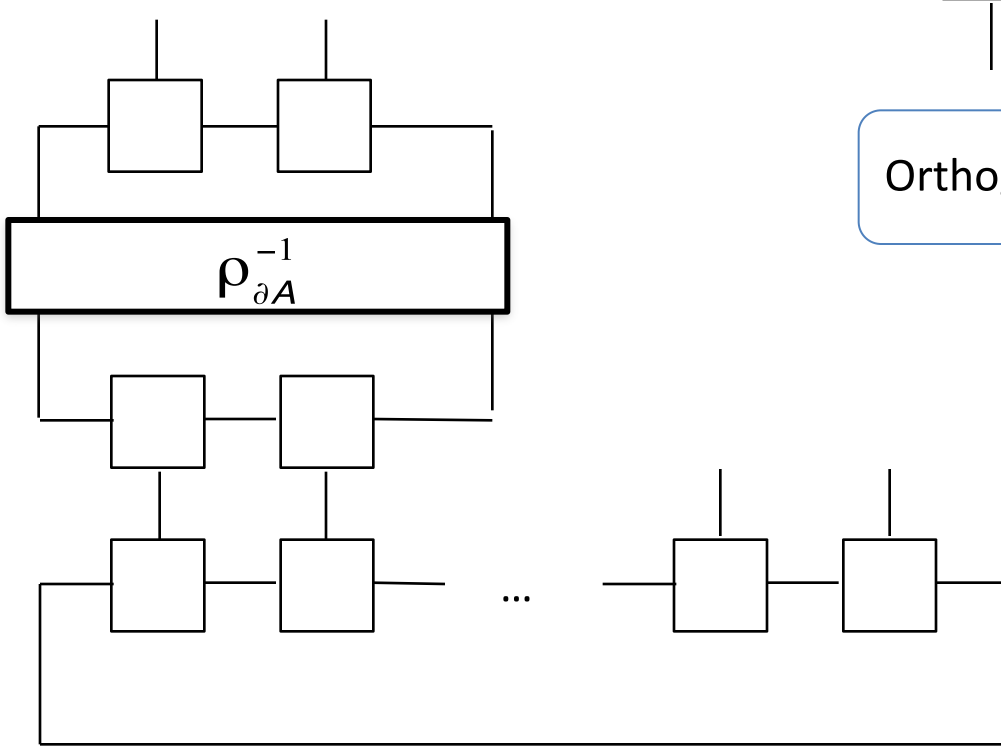


$P_A =$

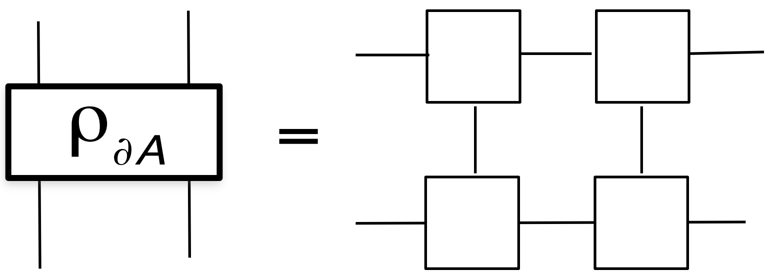


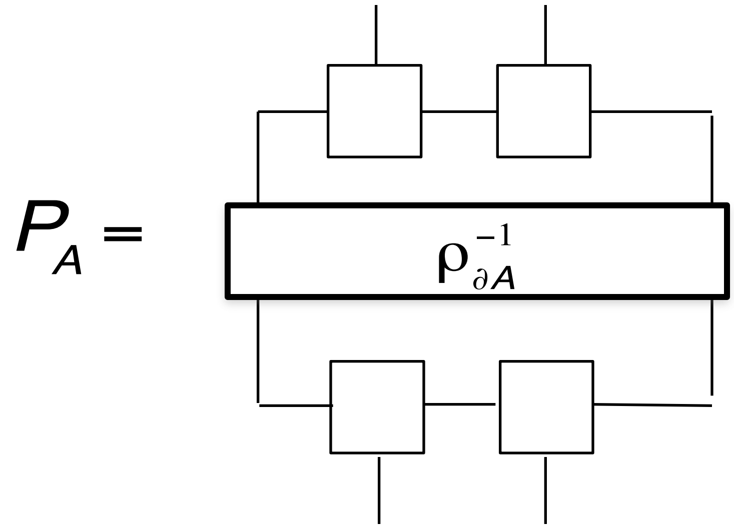
Orthogonal projector

$P_A|\psi\rangle =$

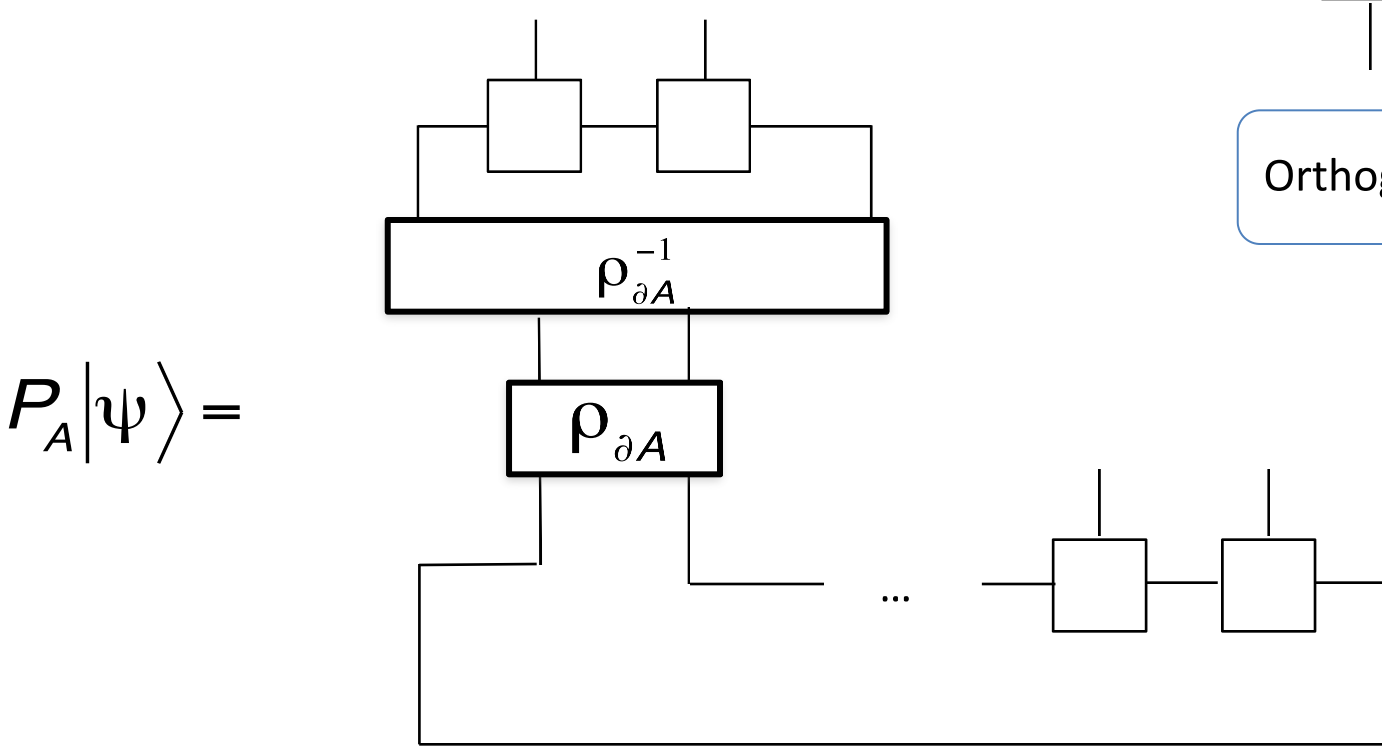


Boundary state properties. Illustration in 1D



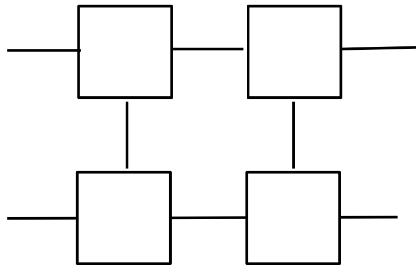


Orthogonal projector

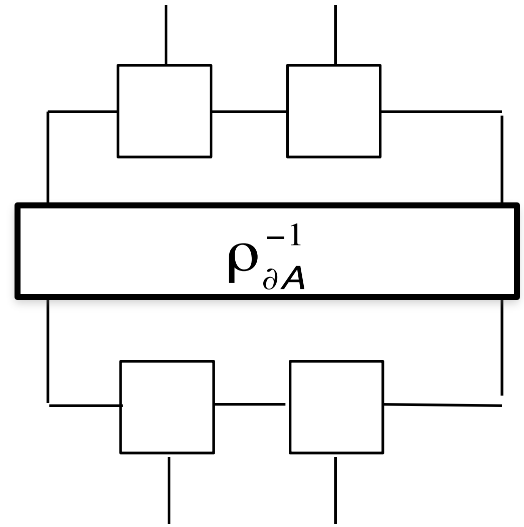


Boundary state properties. Illustration in 1D

$\rho_{\partial A}$ =

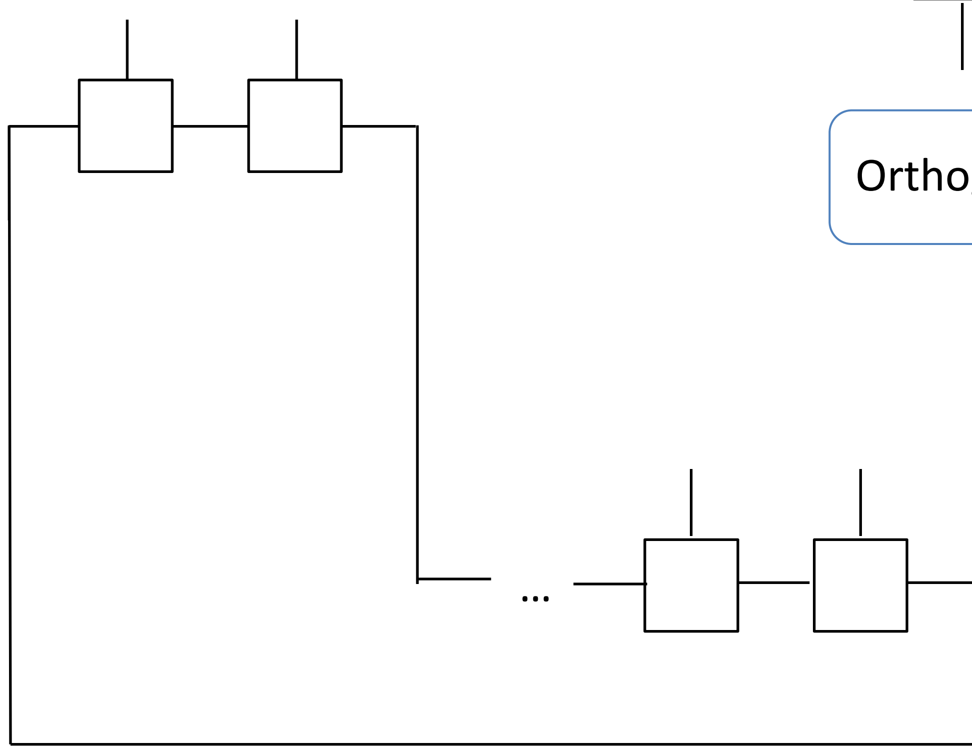


$P_A =$

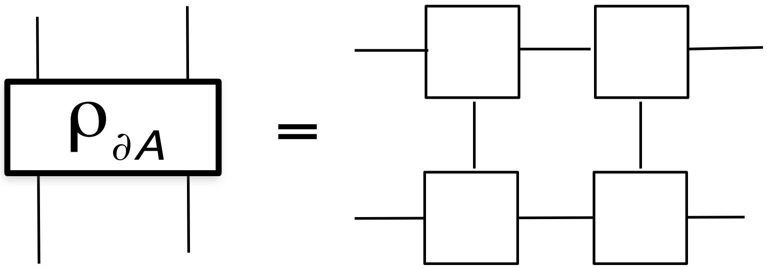


Orthogonal projector

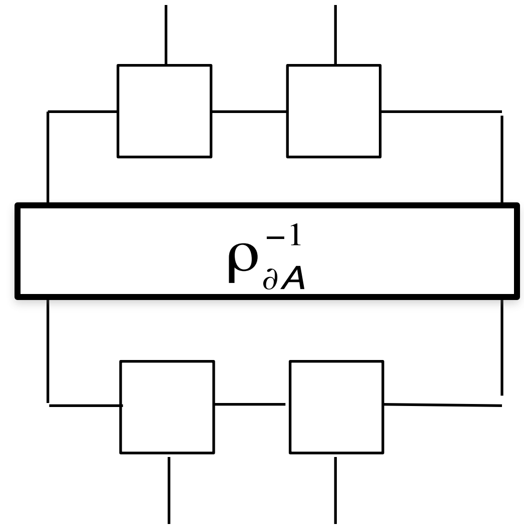
$P_A\left|\psi\right\rangle =$



Boundary state properties. Illustration in 1D

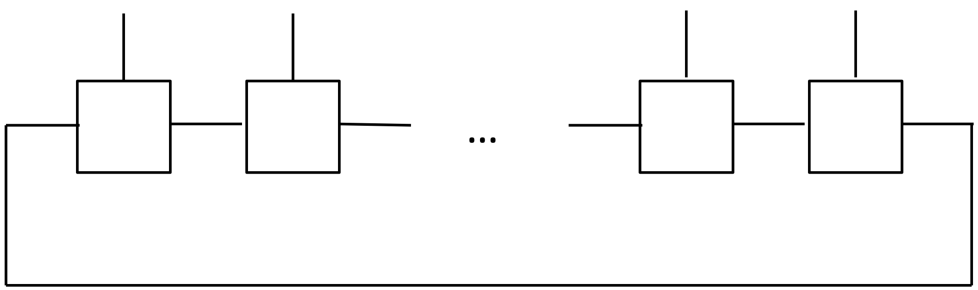


$P_A =$



Orthogonal projector

$P_A|\psi\rangle =$



$= |\psi\rangle$

$$H = \sum_i (1 - P_i)$$