# 2+1D exactly soluble tensor network model with quantized Hall conductance

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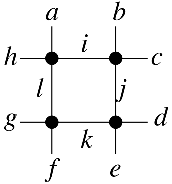
DeMarco Wen, arXiv:2102.13057

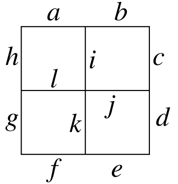
Simons Collaboration on Ultra-Quantum Matter

# Tensor network and spacetime path integral

- Most general **quantum many-body systems** well defined and local can be described by **tensor network** on a spacetime lattice.

$$Z_{\text{path integral}} = \sum_{i,j,k,\dots} T_{abij} T_{dejk} = \mathrm{Tr}\, TTTT\dots$$
$$= \mathrm{Tr}\, T^{(1)} T^{(1)} \dots$$



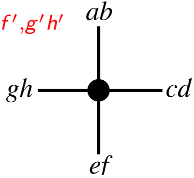



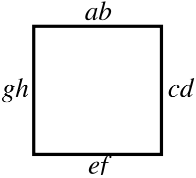

- **Tensor network renormalization**:

$T_{abcd} \to \widetilde{T}^{(1)}_{ab,cd,ef,gh}$

$\widetilde{T}^{(1)}_{ab,cd,ef,gh} \to T^{(1)}_{abcd}$

$T^{(1)}_{abcd} = U_a^{a'b'} U_b^{c'd'} U_c^{e'f'} U_d^{g'h'} \widetilde{T}^{(1)}_{a'b',c'd',e'f',g'h'}$



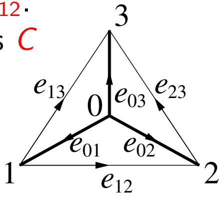

- **Fixed-point tensor**

$$T^{\text{fixed}} \propto \lim_{n\to\infty} T^{(n)}$$

We will concentrate on fixed-point tensors.

# A general tensor network on spacetime complex

- Degrees of freedom on vertices $v$, edges $e_{01}$, faces $v_{012}$. Tensor on vertices $w_v$, edges, $d_{e_{01}}^{v_0,v_1}$, and tetrahedrons $C$

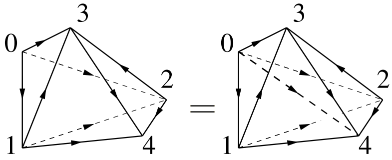



$$Z_{2+1\text{D}} = \sum_{v_0,\cdots;e_{01},\cdots;\phi_{012},\cdots} \prod_{\text{vertex}} \underbrace{w_{v_0}}_{>0} \prod_{\text{edge}} \underbrace{d_{e_{01}}^{v_0 v_1}}_{>0} \times \prod_{\text{tetra}} [\underbrace{C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0 v_1 v_2 v_3;\phi_{013}\phi_{123}}}_{\text{rank-14 complex tensor}}]^{s_{0123}}$$

$s_{0123} = 1, *$ (for orientations of $\langle 0123 \rangle$)

- **Re-triangulation invariance**

  $\rightarrow$ the fixed-point tensor



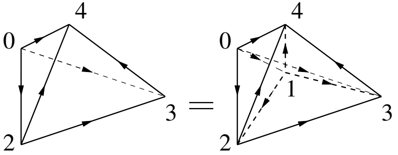

$$\sum_{\phi_{123}} C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0 v_1 v_2 v_3;\phi_{013}\phi_{123}} C^{e_{12}e_{13}e_{14}e_{23}e_{24}e_{34};\phi_{123}\phi_{134}}_{v_1 v_2 v_3 v_4;\phi_{124}\phi_{234}}$$

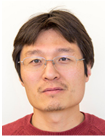
$$= \sum_{e_{04}} d_{e_{04}}^{v_0 v_4} \sum_{\phi_{014}\phi_{024}\phi_{034}} C^{e_{01}e_{02}e_{04}e_{12}e_{14}e_{24};\phi_{012}\phi_{024}}_{v_0 v_1 v_2 v_4;\phi_{014}\phi_{124}} C^{*\, e_{01}e_{03}e_{04}e_{13}e_{14}e_{34};\phi_{013}\phi_{034}}_{v_0 v_1 v_3 v_4;\phi_{014}\phi_{134}}$$

$$C^{e_{02}e_{03}e_{04}e_{23}e_{24}e_{34};\phi_{023}\phi_{034}}_{v_0 v_2 v_3 v_4;\phi_{024}\phi_{234}}.$$

# Re-triangulation invariant = exactly soluble $\to$ renormalization group (RG) fixed point

- Motivated by Turaev-Viro state sum:



$$C^{e_{02}e_{03}e_{04}e_{23}e_{24}e_{34};\phi_{023}\phi_{034}}_{v_0v_2v_3v_4;\phi_{024}\phi_{234}} = \sum_{e_{01}e_{12}e_{13}e_{14},v_1} w_{v_1} d^{v_0v_1}_{e_{01}} d^{v_1v_2}_{e_{12}} d^{v_1v_3}_{e_{13}} d^{v_1v_4}_{e_{14}}$$

$$\sum_{\phi_{012}\phi_{013}\phi_{014}\phi_{123}\phi_{124}\phi_{134}} C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}} C^{*\,e_{01}e_{02}e_{04}e_{12}e_{14}e_{24};\phi_{012}\phi_{024}}_{v_0v_1v_2v_4;\phi_{014}\phi_{124}}$$

$$C^{e_{01}e_{03}e_{04}e_{13}e_{14}e_{34};\phi_{013}\phi_{034}}_{v_0v_1v_3v_4;\phi_{014}\phi_{134}} C^{e_{12}e_{13}e_{14}e_{23}e_{24}e_{34};\phi_{123}\phi_{134}}_{v_1v_2v_3v_4;\phi_{124}\phi_{234}}$$

- The $2+1$D fixed-point tensors (the finite-dimension solution of above non-linear algebraic equations) $\to$ **fusion categories**.
- The $n+1$D fixed-point tensors $\to$ **fusion $n-1$-categories**.
- **They classify $n+1$D topo. orders with gapped boundary**

Kong Wen, arXiv:1405.5858

# Exactly soluble models from re-triangulation invariant (RG-fixed-point) tensor network


- With only edge variables $e_{ij} = g_{ij} \in G$ and $C = 0, 1$
  $\rightarrow$ lattice gauge theory
- With only edge variables $e_{ij} = g_{ij} \in G$ and $C = 0, \mathrm{e}^{\mathrm{i}\theta}$
  $\rightarrow$ Dijkgraaf-Witten gauge theory

Dijkgraaf Witten, CMP, **129** 393 (90)

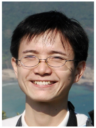
- With only generic edge and face variables $e_{ij}, \phi_{ijk}$, as well as non-trivial tensors $d, C$
  $\rightarrow$ String-net models.

Levin Wen, cond-mat/0404617

- With only vertex variable $v_i = g_i \in G$ and $C = \mathrm{e}^{\mathrm{i}\theta}$
  $\rightarrow$ exactly soluble models realizing **symmetry-protected topological (SPT) orders**

Chen Gu Liu Wen, arXiv:1106.4772





# Group cohomology theory of SPT states

- With only vertex variable, the tensor $C$ in $2+1$D becomes $C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23};\phi_{012}\phi_{023}}_{v_0v_1v_2v_3;\phi_{013}\phi_{123}} \to C_{g_0g_1g_2g_3}$, where $g_i \in G$.
- The linear algebraic equations become (with $w_v = d^{v_0v_1}_{e_{01}} = 1$)

$$C_{g_1g_2g_3g_4}C_{g_0g_1g_3g_4}C_{g_0g_1g_2g_3} = C_{g_0g_2g_3g_4}C_{g_0g_1g_2g_4}$$

  which has no summation and becomes a linear equation after taking log. So it is easy to solve.
- if $C_{g_0g_1g_2g_3}$ is a solution, then the following

$$\tilde{C}_{g_0g_1g_2g_3} = C_{g_0g_1g_2g_3}\frac{B_{g_0g1g2}B_{g_0g2g3}}{B_{g_0g1g3}B_{g1g2g3}}$$

  is also a solution, is said to be equivalent to $C_{g_0g_1g_2g_3}$.
- This leads to group cohomology theory and the solution is the **group cocycle** $\to$ **Group cohomology theory of SPT orders** (which are classified by equivalent classes of group cocycles).

Chen Gu Liu Wen, arXiv:1106.4772

# Infinite dimensional tensor network from continuous group

- For continuous group such as $G = U(1), SU(2)$, we should not assume the solutions (the fixed-point tensors) $C_{g_0 g_1 g_2 g_3}$ to be continuous function of $g_i$'s.

- We may assume $C_{g_0 g_1 g_2 g_3}$ to be **measurable** functions (the limits of continuous function and can do path integral)

Chen Gu Liu Wen, arXiv:1106.4772

- We may assume $C_{g_0 g_1 g_2 g_3}$ to be **patch-wise continuous** function

- The above two very different setups give rise to the same set of equivalent classes (the same cohomology group $H^{n+1}(G; U(1))$)

*How to write down patch-wise continuous 3-cocycles $C_{g_0 g_1 g_2 g_3}$ for $g_i \in U(1)$ → 2+1D $U(1)$ SPT states.*

Using cochain-cocycle theory in algebraic topology

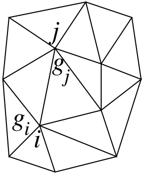
# Cochain-cocycle theory

- 0-cochain $g_i$ is a field living on vertices $i$: $g_i : \{i\} \to G$, where $G$ is the value of 1-cochain.
- 1-cochain $a_{ij}$ is a field living on edges $\langle ij \rangle$: $a_{ij} : \{\langle ij \rangle\} \to G$.
- **Derivative** $d$: From an 0-cochain $g_i$, we can construct a 1-cochain: $a_{ij} = g_i - g_j$ or $a = \mathrm{d}g$.
  From an 1-cochain $a_{ij}$, we can construct a 2-cochain (living on faces $\langle ijk \rangle$): $b_{ijk} = a_{ij} - a_{ik} + a_{jk}$ or $b = \mathrm{d}a$.
- **Cup product** $\smile$: From a $m$-cochain $p$ and a a $n$-cochain $q$, we can construct a $m+n$-cochain $s$: $s_{i_0,\dots,i_{m+n}} = p_{i_0,\dots,i_m} q_{i_m,\dots,i_{m+n}}$, which is written as $s = p \smile q$ or $s = pq \neq qp$.

  For example $c_{i_0 i_1 i_2 i_3} = a_{i_0 i_1} b_{i_1 i_2 i_3}, \quad c = a \smile b = ab$.
- **Integration**: $\int_{M^{2+1}} c = \sum_{\langle i_0 i_1 i_2 i_3 \rangle} \pm c_{i_0 i_1 i_2 i_3}$, where $\pm$ depends on the orientations of $\langle i_0 i_1 i_2 i_3 \rangle$.

# A 2+1D $U(1)$ rotor model via spacetime path integral (*ie* tensor network)

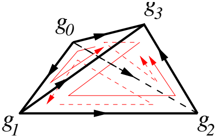
DeMarco Wen, arXiv:2102.13057




- On each vertex of spacetime complex, we have an "angle" $g_i \in (-\frac{1}{2}, \frac{1}{2}]$, $g_i \sim g_i + 1$ → a rotor model.

$$Z = \sum_{\{g_i\}} \prod_{\langle ijkl \rangle} (C_{ijkl})^{s_{ijkl}} = \mathrm{e}^{\mathrm{i} 2\pi k \sum_{\langle ijkl \rangle} \mathrm{sgn}_{ijkl} c_{ijkl}}$$

$$= \int [\prod_i \mathrm{d}g_i] \mathrm{e}^{\mathrm{i} 2\pi k \int_{M^{2+1}} c}, \quad C_{ijkl} = \mathrm{e}^{\mathrm{i} 2\pi k\, c_{ijkl}}.$$




$$c = \mathrm{d}g\, \mathrm{d}\lfloor \mathrm{d}g \rceil, \quad c_{0123} = (g_0 - g_1)(\lfloor g_1 - g_2 \rceil - \lfloor g_1 - g_3 \rceil + \lfloor g_2 - g_3 \rceil)$$

where $\lfloor x \rceil$ = nearest integer of $x$. $c$ is a 3-cocycle.

- The tensor $C_{ijkl}$ has a $\mathbb{Z}$-**gauge invariance**: $g_i \to g_i + n_i$, $n_i \in \mathbb{Z}$ and a global $U(1)$ **symmetry (invariance)**: $g_i \to g_i + h$.
- Exactly soluble on closed spacetime $\mathrm{e}^{\mathrm{i} 2\pi k \int_{M^{2+1}} c} = 1$ if $\partial M^{2+1} = \emptyset$.

# Why the path integral with $k \neq 0$ describes a gapped non-trivial $U(1)$ SPT state?

- On spacetime with boundary $\partial M^{2+1} = B$:

$$Z = \int [\prod_i \mathrm{d}g_i] \mathrm{e}^{\mathrm{i}2\pi k \int_{M^{2+1}} \mathrm{d}g\, \mathrm{d}\lfloor \mathrm{d}g \rceil} = \int [\prod_i \mathrm{d}g_i] \mathrm{e}^{\mathrm{i}2\pi k \int_B g\, \mathrm{d}\lfloor \mathrm{d}g \rceil}$$

  where we have used $\mathrm{d}g\, \mathrm{d}\lfloor \mathrm{d}g \rceil = \mathrm{d}(g\, \mathrm{d}\lfloor \mathrm{d}g \rceil)$. The path integral describes an low energy effective boundary theory. But the boundary effective theory (with action-amplitude $\mathrm{e}^{\mathrm{i}2\pi k \int_B g\, \mathrm{d}\lfloor \mathrm{d}g \rceil}$ ) is not $U(1)$ invariant under $g_i \to g_i + h$ if $\partial B \neq \emptyset$.

- The effective $1+1$D theory has $U(1)$ symm. if space has no boundary, but break the $U(1)$ symm. if the space has boundary.
- Add an $1+1$D term $\mathrm{e}^{-\mathrm{i}2\pi k \int_B \mathrm{d}(g \lfloor \mathrm{d}g \rceil)}$ to fix the $U(1)$ symmetry:

$$\widetilde{Z} = \int [\prod_i \mathrm{d}g_i] \mathrm{e}^{\mathrm{i}2\pi k \int_B g\, \mathrm{d}\lfloor \mathrm{d}g \rceil - \mathrm{d}(g \lfloor \mathrm{d}g \rceil)} = \int [\prod_i \mathrm{d}g_i] \mathrm{e}^{-\mathrm{i}2\pi k \int_B \mathrm{d}g \lfloor \mathrm{d}g \rceil}$$

  But $\mathrm{e}^{-\mathrm{i}2\pi k \int_B \mathrm{d}(g \lfloor \mathrm{d}g \rceil)}$ is not $\mathbb{Z}$-gauge invariant (when $\partial B \neq \emptyset$).

# The ground state wavefunction

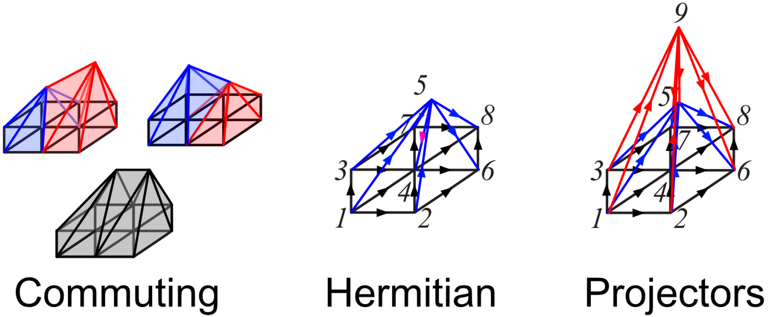
- If we view the spacetime boundary $B = \partial M^{2+1}$ as the space at a time slice, then the boundary term $\mathrm{e}^{\mathrm{i}2\pi k \int_B g\, \mathrm{d}\lfloor \mathrm{d}g \rceil} = \Phi(\{g_i\})$ actually give rise to a ground state wave function.
- Consider ground state wavefunction $\Phi(\{g_i\};\theta_x,\theta_y)$ on a torus, with $U(1)$ twisted boundary condition $g_{x,y} = g_{x+L_x,y} + \theta_x = g_{x,y+L_y} + \theta_y$
- The ground state wavefunction $\Phi(\{g_i\};\theta_x,\theta_y)$ define a complex line-bundle over $(\theta_x,\theta_y)$-space (a torus). The Chern number of the line-bundle (the Berry phase of the ground states) is $2k \to$ the Hall conductance is $\sigma_{xy} = 2k\frac{e^2}{h}$ DeMarco Wen, arXiv:2102.13057
- **The exactly soluble path integral**

$$Z = \int [\prod_i \mathrm{d}g_i] \mathrm{e}^{\mathrm{i}2\pi k \int_{M^{2+1}} \mathrm{d}g\, \mathrm{d}\lfloor \mathrm{d}g \rceil}$$

**realizes a $U(1)$-SPT state with Hall conductance $\sigma_{xy} = 2k\frac{e^2}{h}$**

# Retrianglation invariant → commuting projector $H$

- Start with a time slice, a time evolution is obtained by add a piece of spacetime and do the path integral.




Two ways to add pieces of spacetime have the same surface. $\rightarrow$ Two way to apply Hamiltonian terms induce the same time evolution.

# Can a commuting-projector Hamiltonian really gives rise to a Hall conductance?

- Kapustin and Fidkowski (arXiv:1810.07756):
  *"We prove that neither Integer nor Fractional Quantum Hall Effects with nonzero Hall conductivity are possible in gapped systems described by Local Commuting Projector Hamiltonians."*
- Our result: DeMarco Wen, arXiv:2102.13057
  **Commuting projector Hamiltonian of rotors can give rise to non-zero Hall conductance** $\sigma_{xy} = 2k\frac{e^2}{h}$.

- The two results are consistent since in our model, each site has an **infinite dimensional Hilbert space**. This makes it possible to use a commuting projector Hamiltonian to realize a non-zero Hall conductance.