Exactly solved models of many body quantum chaos

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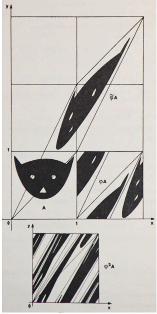
Menu

Goal: Find 'Baker and Cat maps' of many body quantum physics!

1. A proof of random-matrix spectral form factor for a kicked spin chain `PRL` **`121`**`, 264101 (2018); arXiv:2012.12254`
2. Exact local dynamical correlation functions in **dual-unitary** models: *An example of exact ergodic hierarchy of quantum many-body dynamics* `PRL` **`123`**`, 210601 (2019),`
3. Dynamical complexity (entanglement entropy `PRX` **`9`**`, 021033 (2019)`, operator entropy `SciPost Phys.` **`8`**`, 067 (2020)`), and structural / perturbative stability of quantum ergodicity `PRX` **`11`**`, 011022 (2021).`

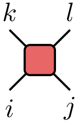


Consider a unitary gate on a two-qudit system $U \in U(d^2)$ and define the following duality transformation

$$\sim : U \longmapsto \tilde{U} \,,$$

via reshuffling of basis states

$$\langle j| \otimes \langle \ell|\tilde{U}|i\rangle \otimes |k\rangle = \langle k| \otimes \langle \ell|U|i\rangle \otimes |j\rangle \,.$$



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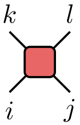


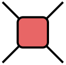
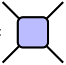
We call a gate dual-Unitary, if not only U is unitary, i.e.

$$UU^\dagger = U^\dagger U = \mathbb{1},$$

but also $\tilde{U}$ is unitary

$$\tilde{U}\tilde{U}^\dagger = \tilde{U}^\dagger \tilde{U} = \mathbb{1}.$$

Diagrammatic expression of dual unitarity

$U =$  , $U^\dagger =$  .

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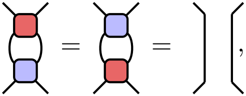
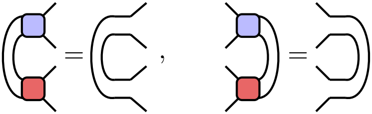


forward fusion rules

$$\tilde{U}\tilde{U}^\dagger = \tilde{U}^\dagger \tilde{U} = \mathbb{1}.$$



dual fusion rules

Dual unitary quantum circuits

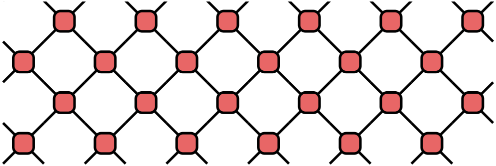
One step of a quantum circuit is a unitary over $(\mathbb{C}^d)^{\otimes 2L}$

$$\mathbb{U} = \mathbb{U}^{\mathrm{o}}\mathbb{U}^{\mathrm{e}}$$

where

$$\mathbb{U}^{\mathrm{e}} = U^{\otimes L}, \quad \mathbb{U}^{\mathrm{o}} = \Pi_{2L}\mathbb{U}^{\mathrm{e}}\Pi_{2L}^{\dagger}$$

and Π_ℓ is a periodic translation $\Pi_\ell|i_1\rangle \otimes |i_2\rangle \cdots |i_\ell\rangle \equiv |i_2\rangle \otimes |i_3\rangle \cdots |i_\ell\rangle \otimes |i_1\rangle$.



(here $t = 2$ and $L = 6$)

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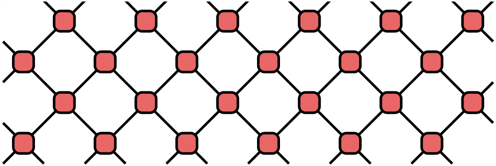

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Similarly we define dual quantum circuit propagator over $(\mathbb{C}^d)^{\otimes 2t}$

$$\tilde{\mathbb{U}} = \tilde{U}^{\otimes t}\Pi_{2t}\tilde{U}^{\otimes t}\Pi_{2t}^{\dagger}.$$

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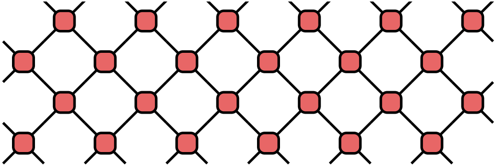

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$$\operatorname{tr}\mathbb{U}^t = \operatorname{tr}\tilde{\mathbb{U}}^L\,.$$

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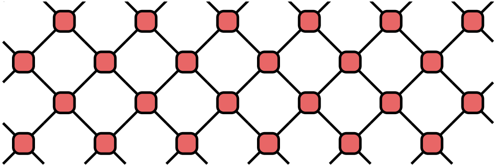

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DUC generalize the *self-dual kicked Ising model* `PRL` **`121`**`, 264101 (2018)` where exact RMT expression for the **spectral form factor** was derived. See also [`Gopalakrishnan and Lamacraft, PRB` **`100`**`, 064309 (2019)`]

The **spectrum** $\{\varphi_n\}$ of a unitary one-period propagator $U = \mathcal{T}\exp(-i\int_0^1 H(t)\mathrm{d}t)$ as a **gas** in one dimension

Spectral density:

$$\rho(\varphi) = \frac{2\pi}{\mathcal{N}} \sum_n \delta(\varphi - \varphi_n), \quad \mathcal{N} = 2^L.$$

Spectral pair correlation function (2-point function):

$$r(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \rho(\varphi + \tfrac{1}{2}\vartheta)\rho(\varphi - \tfrac{1}{2}\vartheta) - 1.$$

Spectral Form Factor (SFF) (Fourier transform of 2-point function):

$$\begin{aligned} K(t) &= \frac{\mathcal{N}^2}{2\pi}\int_0^{2\pi} \mathrm{d}\vartheta r(\vartheta) e^{it\vartheta} = \sum_{m,n} e^{it(\varphi_m - \varphi_n)} - \mathcal{N}^2 \delta_{t,0} \\ &= \left|\mathrm{tr}\, U^t\right|^2 - \mathcal{N}^2 \delta_{t,0}, \quad t \in \mathbb{Z}. \end{aligned}$$

Spectral Form Factor in Floquet Systems

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Caveat: SFF is not self-averaging! Consider instead $\bar{K}(t) = \mathbb{E}[K(t)]$.

RMT (No time reversal symmetry):

$$K_{\mathrm{CUE}}(t) = t, \quad t < \mathcal{N}.$$

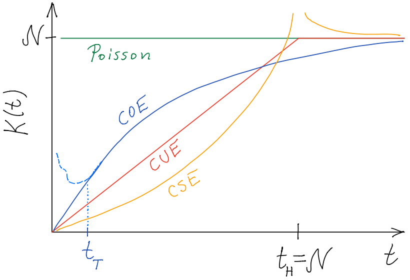
RMT (With time teversal symmetry):

$$K_{\mathrm{COE}}(t) = 2t - t\log(1 + 2t/\mathcal{N}), \quad t < \mathcal{N}.$$

Random (uncorrelated, Poissonian) spectrum $\{\varphi_n\}$:

$$K_{\mathrm{Poisson}} \equiv \mathcal{N}.$$

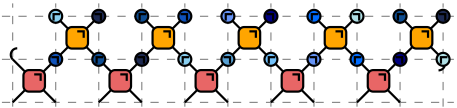
RMT vs Real System:



$$\mathbb{E}[K(t)] = \mathbb{E}\left[\sum_{m,n} e^{i(\varphi_m - \varphi_n)}\right].$$

Saturation $\bar{K}(t) \sim \mathcal{N}$ beyond Heisenberg time $t > t_{\mathrm{H}} = \mathcal{N} = 1/\Delta\varphi$.

Non-universal (system-specific) behaviour below Ehrenfest/Thouless time $t < t_{\mathrm{T}}$.

$\mathbb{U} =$ 

$\cdots \quad -\frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 1 \quad \frac{3}{2} \quad 2 \quad \frac{5}{2} \quad 3 \quad \cdots$

$U =$ , $\quad u =$ .

$$K(t,L) = \mathbb{E}_u[|\mathrm{tr}\,\mathbb{U}^t|^2] = \mathbb{E}_u[\mathrm{tr}\,(\mathbb{U}^\dagger \otimes \mathbb{U}^T)^t] = \mathrm{tr}[(\mathbb{E}_u[\tilde{\mathbb{U}}^\dagger \otimes \tilde{\mathbb{U}}^T])^L]$$

Theorem [Bertini, Kos, P, arXiv:2012.12254]:

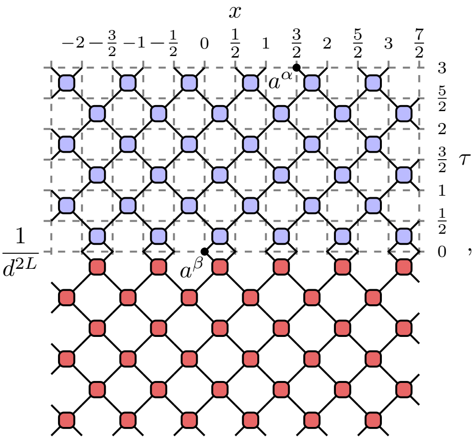
For i.i.d. local 1-qubit gates u, with nonvanishing probability density in arbitrary small ball in $SU(2)$ around the identity $u = \mathbb{1}$, and for any dual unitary 2-qubit gates U other than the SWAP, we have

$$\lim_{L\to\infty} K(t) = \dim \left\{ \sum_{\tau=0}^{t-1} \sigma^{\alpha}_{\tau+\frac{\iota}{2}}, \sum_{\tau=0}^{t-1} \sigma^{\alpha}_{\tau+\frac{\iota}{2}} \sigma^{\beta}_{\tau+\frac{\iota+1}{2}}; \alpha,\beta \in \{x,y,z\}, \iota \in \{0,1\} \right\}' \\ = t$$

$$\sigma^{\alpha}_{\tau} = \mathbb{1}_{2\tau} \otimes \sigma^{\tau} \otimes \mathbb{1}_{2t-2\tau-1} \in \mathrm{End}((\mathbb{C}^2)^{\otimes 2t}), \quad \tau \in \frac{1}{2}\mathbb{Z}_{2t}$$

Writing the orthonormal set of local observables as a^α, $\alpha = 0, \ldots, d^2 - 1$, $\mathrm{tr}\,\left[(a^\alpha)^\dagger a^\beta\right] = d\,\delta_{\alpha,\beta}$ and choose $a^0 = \mathbb{1}$, so all other a^α are traceless, we shall be interested in the following space-time correlator

$$D^{\alpha\beta}(x,y,t) \equiv \frac{1}{d^{2L}}\mathrm{tr}\left[a_x^\alpha \mathbb{U}^{-t} a_y^\beta \mathbb{U}^t\right] = \begin{cases} C_-^{\alpha\beta}(x-y,t) & 2y \text{ even} \\ C_+^{\alpha\beta}(x-y,t) & 2y \text{ odd} \end{cases},$$

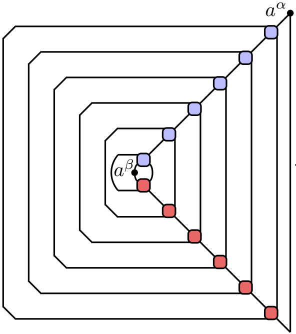


Property 1

If U is dual-unitary, the dynamical correlations are non-zero for $t \leq L/2$ only on the edges of a lightcone spreading at speed ± 1

$$C_\nu^{\alpha\beta}(x,t) = \delta_{x,\nu t} C_\nu^{\alpha\beta}(\nu t, t)\,, \qquad \nu = \pm,\ \alpha, \beta \neq 0\,.$$

$$C_+^{\alpha\beta}(t,t) = \frac{1}{d^{2t+1}}$$

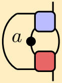


Property 2

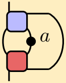
The light cone correlations $C_{+}^{\alpha\beta}(t,t)$ and $C_{-}^{\alpha\beta}(-t,t)$ are given by

$$C_{\nu}^{\alpha\beta}(\nu t, t) = \frac{1}{d}\mathrm{tr}\left[\mathcal{M}_{\nu}^{2t}(a^{\beta})a^{\alpha}\right],$$

where we introduced the linear maps over $\mathrm{End}(\mathbb{C}^{d})$

$$\mathcal{M}_{+}(a) = \frac{1}{d}\mathrm{tr}_1\left[U^{\dagger}(a\otimes \mathbb{1})U\right] = \frac{1}{d}$$ 

,

$$\mathcal{M}_{-}(a) = \frac{1}{d}\mathrm{tr}_2\left[U^{\dagger}(\mathbb{1}\otimes a)U\right] = \frac{1}{d}$$ 

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$\mathrm{tr}_i[A]$ denote partial traces over i-th site ($i = 1, 2$).

Dynamical correlations tamed!

Decay of correlations is given in terms of $d^2 - 1$ eigenvalues $\lambda_{+,\alpha}$ of single qudit channel ($d^2 \times d^2$ matrix) $\mathcal{M}_+$, and $d^2 - 1$ eigenvalues $\lambda_{-,\alpha}$ of $\mathcal{M}_-$.

$$D^{\alpha\beta}(x,y,t) = \begin{cases} \delta_{y-x,t} \sum_{\gamma=1}^{d^2-1} c^{\alpha\beta}_{-,\gamma} (\lambda_{-,\gamma})^{2t} & 2y \text{ even} \\ \delta_{x-y,t} \sum_{\gamma=1}^{d^2-1} c^{\alpha\beta}_{+,\gamma} (\lambda_{+,\gamma})^{2t} & 2y \text{ odd} \end{cases}$$

(One eigenvalue is always $\lambda_{\nu,0} = 1$, with eigenoperator $a^0 = \mathbb{1}$.)

Classification of ergodic behaviours:

1. Non-interacting dynamics:
   all $\lambda_{\nu,\gamma} = 1$ (example: SWAP $U|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle$)
2. Non-ergodic (and generically *non-integrable*) behavior:
   $\exists$ additional eigenvalue equal to one $\lambda_{\nu,\gamma} = 1$, $\gamma \neq 0$.
3. Ergodic but non-mixing behavior:
   all $\lambda_{\nu,\gamma\neq 0} \neq 1$, but $\exists \gamma \neq 0, |\lambda_{\nu,\gamma}| = 1$.
4. Ergodic and mixing behavior:
   all $|\lambda_{\nu,\gamma\neq 0}| < 1$.

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We can provide a complete classification only for $d = 2$:

$$U = e^{i\phi}(u_+ \otimes u_-) \cdot V[J] \cdot (v_- \otimes v_+) \,,$$

where $\phi, J \in \mathbb{R}$, $u_\pm, v_\pm \in \mathrm{SU}(2)$ and

$$V[J] = \exp\left[-i\left(\frac{\pi}{4}\sigma^x \otimes \sigma^x + \frac{\pi}{4}\sigma^y \otimes \sigma^y + J\sigma^z \otimes \sigma^z\right)\right].$$

Relevant examples:

1. SWAP gate $U = V[\pi/4]$.
2. One parameter line of the trotterized XXZ chain

$$U_{\mathrm{XXZ}} = V[J] \,,$$

3. The maximally chaotic self-dual kicked Ising (SDKI) chain

$$U_{\mathrm{SDKI}} = e^{-ih\sigma^z} e^{i\frac{\pi}{4}\sigma^x} \otimes e^{i\frac{\pi}{4}\sigma^x} \cdot V[0] \cdot e^{i\frac{\pi}{4}\sigma^y} e^{-ih\sigma^z} \otimes e^{i\frac{\pi}{4}\sigma^y}.$$

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$$U = e^{i\phi}(u_+ \otimes u_-) \cdot V[J] \cdot (v_- \otimes v_+)\,,$$

where $\phi, J \in \mathbb{R}$, $u_\pm, v_\pm \in \mathrm{SU}(2)$ and

$$V[J] = \exp\left[-i\left(\frac{\pi}{4}\sigma^x \otimes \sigma^x + \frac{\pi}{4}\sigma^y \otimes \sigma^y + J\sigma^z \otimes \sigma^z\right)\right].$$

Relevant examples:

1. SWAP gate $U = V[\pi/4]$.
2. One parameter line of the trotterized XXZ chain

$$U_{\mathrm{XXZ}} = V[J]\,,$$

3. The maximally chaotic self-dual kicked Ising (SDKI) chain

$$U_{\mathrm{SDKI}} = e^{-ih\sigma^z} e^{i\frac{\pi}{4}\sigma^x} \otimes e^{i\frac{\pi}{4}\sigma^x} \cdot V[0] \cdot e^{i\frac{\pi}{4}\sigma^y} e^{-ih\sigma^z} \otimes e^{i\frac{\pi}{4}\sigma^y}.$$

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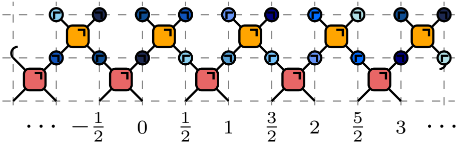
See [Claeys & Lamacraft, PRL **126**, 100603 (2021)] for generalization (not complete classification!) to higher d, and [Gutkin, Braun, Akila, Waltner, Guhr, arXiv:2001.01298] for generalization of KI model to higher d.

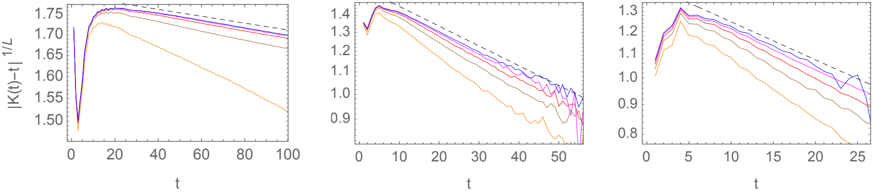
SFF in general (non-dual-unitary) brickwork circuits

Circuit models with local quench disorder:

$\mathbb{U} =$ 

$U =$ , $u =$ .

$$K(t,L) = \mathbb{E}_u[|\mathrm{tr}\,\mathbb{U}^t|^2] = \mathbb{E}_u[\mathrm{tr}\,(\mathbb{U}^\dagger \otimes \mathbb{U}^T)^t] = \mathrm{tr}[(\mathbb{E}_u[\tilde{\mathbb{U}}^\dagger \otimes \tilde{\mathbb{U}}^T])^L]$$



Distance to nearest dual-unitary gate to U *decreases* from left to right plot.

Data for $L = 8, 10, 12, 14, 16$ suggest the conjecture

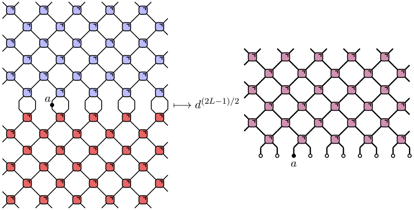
$$K(t) - t \leq ALe^{-Bt}, \quad A, B > 0.$$

[unpublished, c.f. Garratt and Chalker, arXiv:2008.01697]

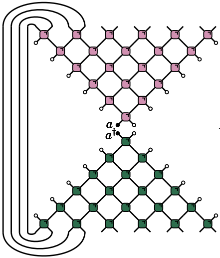
Analytic computation of Renyi-2 operator entanglement entropy for spreading of local operators [Bertini, Kos & P, SciPost Phys. 2020]:

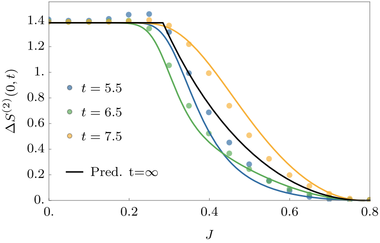
$$E_{op}(t) = \alpha t$$

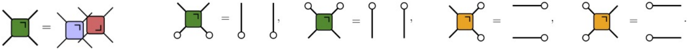
where $\alpha = 2 \log d$ signals maximal chaos.

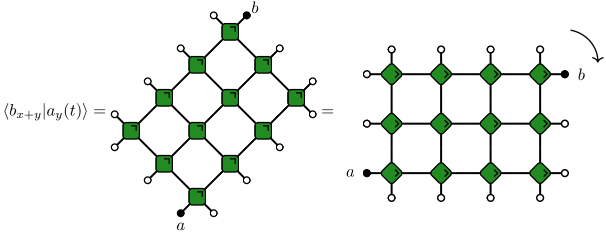


$\rho_A(t, y; a) = \mathrm{tr}_{\bar{A}}[|a_y(t)\rangle\langle a_y(t)|] =$



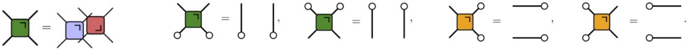






$\langle b_{x+y}|a_y(t)\rangle =$
b
a
=
b
a

Structural (perturbative) stability of DUC [PRX 11, 011022 (2021)]



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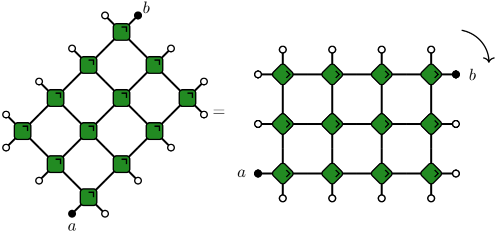
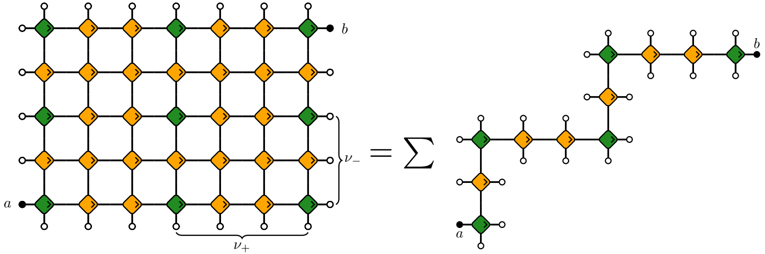


$$U_\eta = U_{\rm DU}\cdot e^{i\eta P}$$

$\langle b_x|a_0(t)\rangle =$

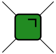


Reduced gates/circuits

The U(1)-noise averaged dynamical correlations

$$c_{ab}(x,t) = \mathbb{E}_{\{h_{j,t}\}} C_{ab}(x,t), \quad U_{j,j+1} \to U_{j,j+1} e^{i h_{j,t} \sigma^z_j + i h_{j+1,t} \sigma^z_{j+1}}$$

can be formulated in terms of classical bistochastic brickwork Markov circuits in the basis of diagonal operators $|\mathbb{1}\rangle$, $|\sigma^z\rangle$ with elementary 2-gate

$$w :=$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & a & b \\ 0 & c & \varepsilon_2 & d \\ 0 & e & f & g \end{pmatrix},$$

$\varepsilon_1 = \varepsilon_2 = 0$ corresponds to dual-unutary/dual-bistochastic circuit.

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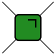
$w :=$  $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & a & b \\ 0 & c & \varepsilon_2 & d \\ 0 & e & f & g \end{pmatrix},$

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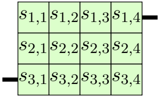
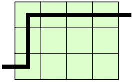
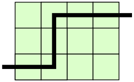
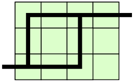
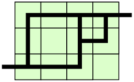
Tilling representation of dynamical correlations:

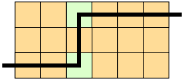
$\langle \bullet_x | \bullet_0 (t) \rangle = \sum_{s_{ij} \in \text{tiles}}$ 

$=$  $+$  $+ \ldots$

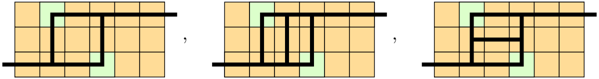
$+$  $+$  $+ \ldots .$

 $= 1$  $= a$  $= b$  $= c$  $= d$  $= e$  $= f$  $= g$  $= \varepsilon_1$  $= \varepsilon_2$

To fixed, say 2nd order in $\varepsilon_1, \varepsilon_2$, we get contributions from the no-loop (skeleton) diagram



as well as from higher, loop diagrams



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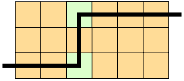

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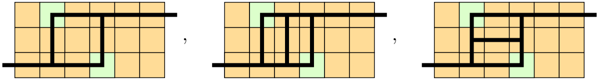

However, if

$$|a| > a^2 + \frac{|bf|}{1-\alpha}, \qquad \text{or} \qquad |c| > c^2 + \frac{|de|}{1-\beta}$$

where α and β are, respectively, the largest singular values of

$$\begin{pmatrix} c & e \\ d & g \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} a & f \\ b & g \end{pmatrix},$$

then the tile-sum can be explicitly evaluated and shown to be equal to sum over skeleton diagrams. *Convergence proven* in ‘low density’ regime, while conjectured at any density of perturbed gates.

- First exact results on spectral statistics of extended quantum lattice systems, when thermodynamic limit taken first
- Exact results on spatio-temporal correlation functions: *from regular to ergodic and mixing dynamics*
- Strong indication that the results are **structurally stable** to perturbations

The main challenges for future work:

- Exact results in finite systems, finite size corrections?
- Statements about eigenstates: *dual unitary circuits as models where ETH can be proven?*
- Exactly solvable chaotic driven/dissipative chaos: Dual quantum bistochastic Kraus cricuits?

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