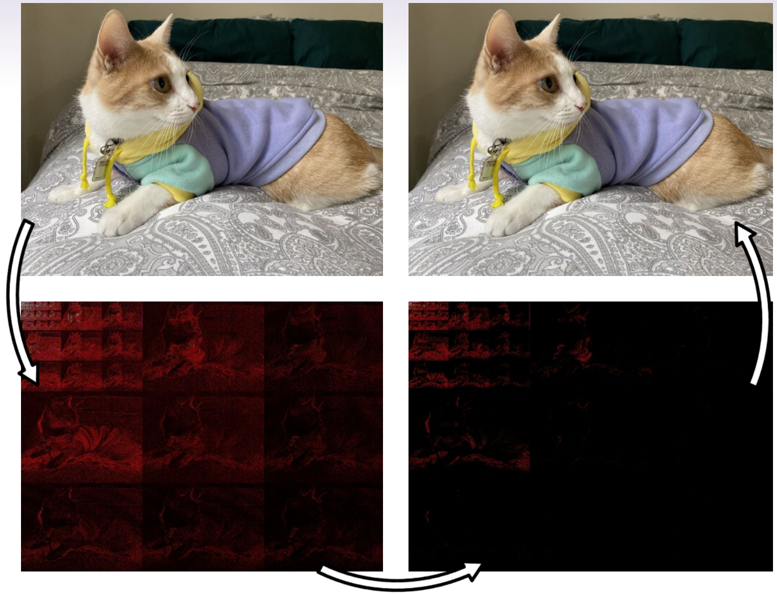
IPAM Workshop II: Tensor Network States and Applications
April 2021

Using Tensor Networks to Design Improved Wavelets for Image Compression

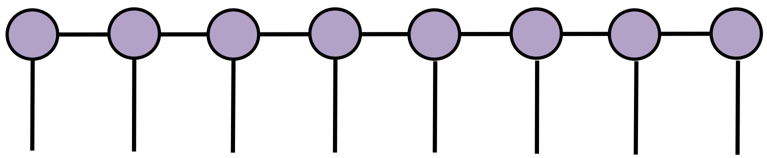


Glen Evenbly

Georgia Tech

Georgia Tech QUANTUM ALLIANCE
Institute for Electronics and Nanotechnology

Introduction



Tensor Network

||

Compressed representation of a quantum many-body wavefunction

Do tensor networks have useful applications outside of physics (i.e. in math and data science)? Yes!

Machine Learning

Data Completion

"Reconstruction of Ground-state Wavefunctions using Tensor Completion", Aaron Stahl and **G. E.**, *in preparation.*

Data Compression

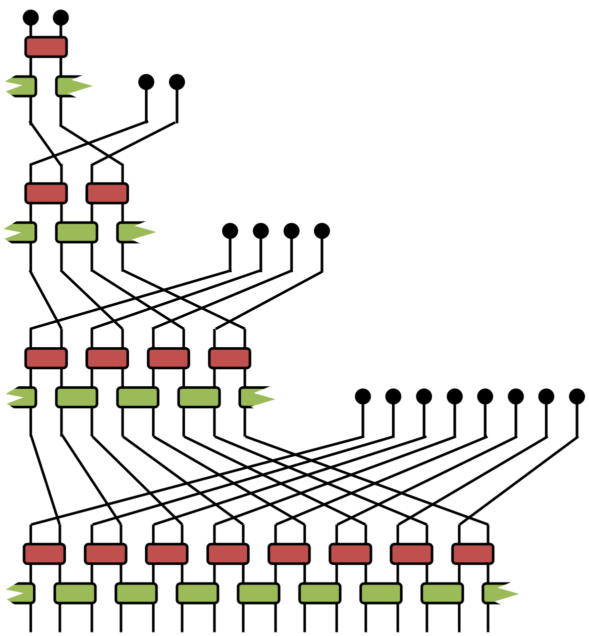
"Improved wavelet designs for image compression" J. C. McCord, **G.E.**, *in preparation*

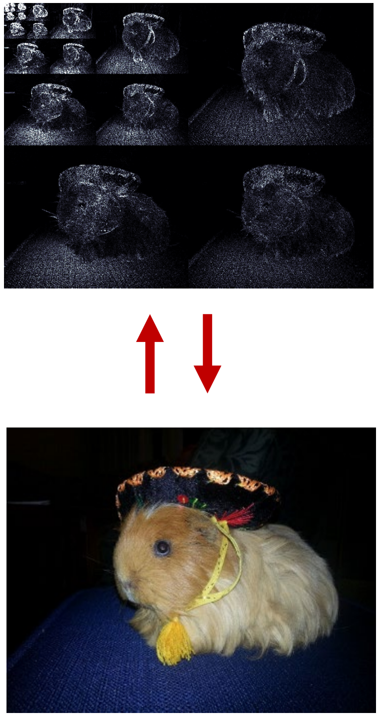
In many cases tensor networks can offer a new perspective for solving tasks in math / data science / engineering

The standard tools and methods developed for implementing tensor networks (e.g. TN optimization algorithms) can be superior to the established data science methods

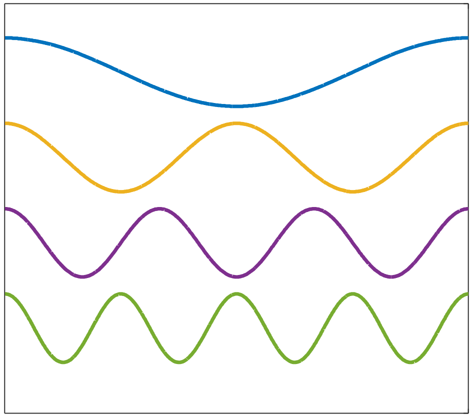
Overview

- What are wavelets? What are they useful for?
- How are wavelets related to tensor networks?
- How can tensor networks be use to construct improved wavelets for image compression?



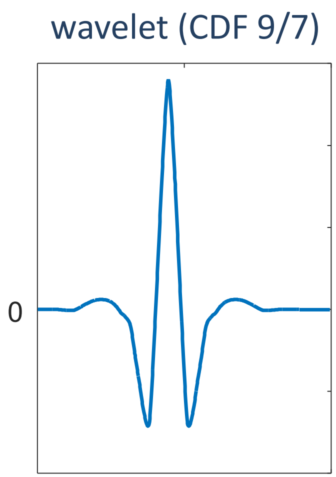


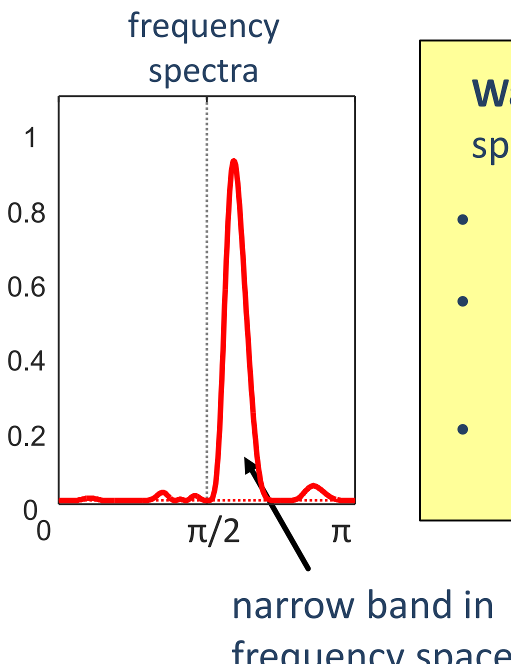
Introduction to Wavelets



Fourier expansions are ubiquitous in math, science and engineering

- many problems are simplified by expanding in Fourier modes
- smooth functions can be approximated by only a few non-zero Fourier coefficients

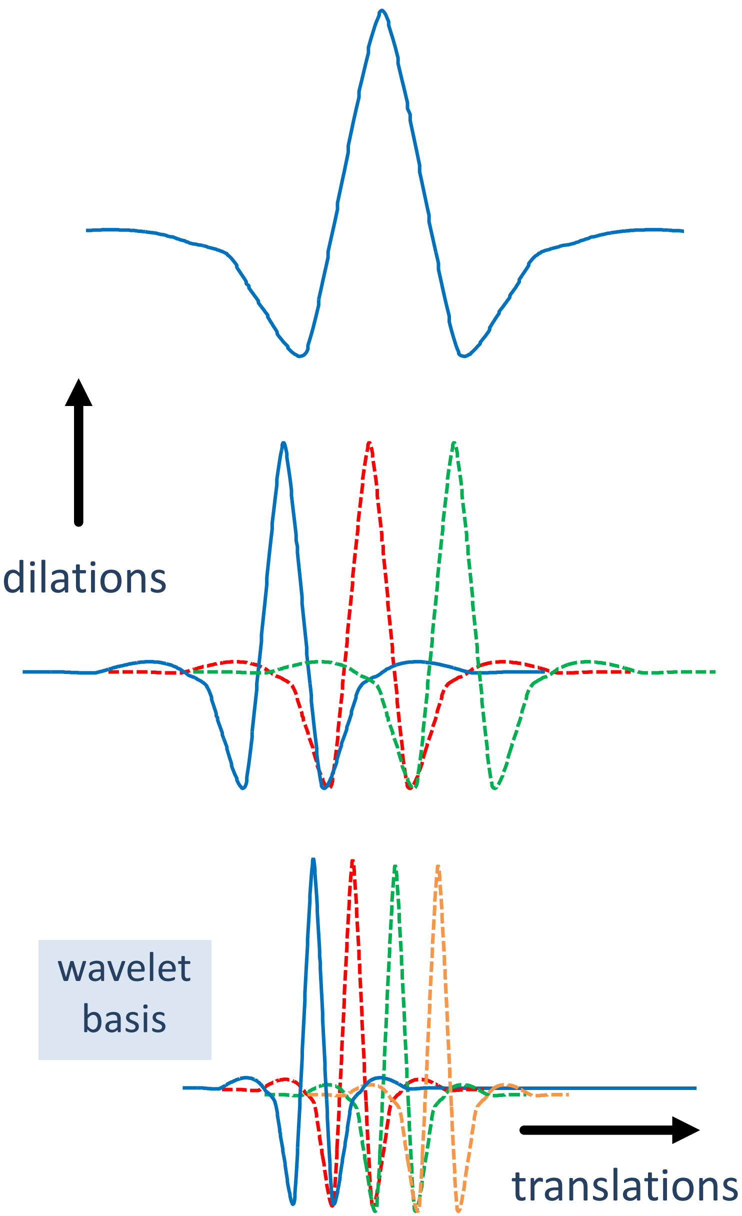




Wavelets are a good compromise between real-space and Fourier-space representations

- compact in real-space and in frequency-space
- developed by math and signal processing communities in late 80’s
- applications in signal and image processing, data compression

Introduction to Wavelets



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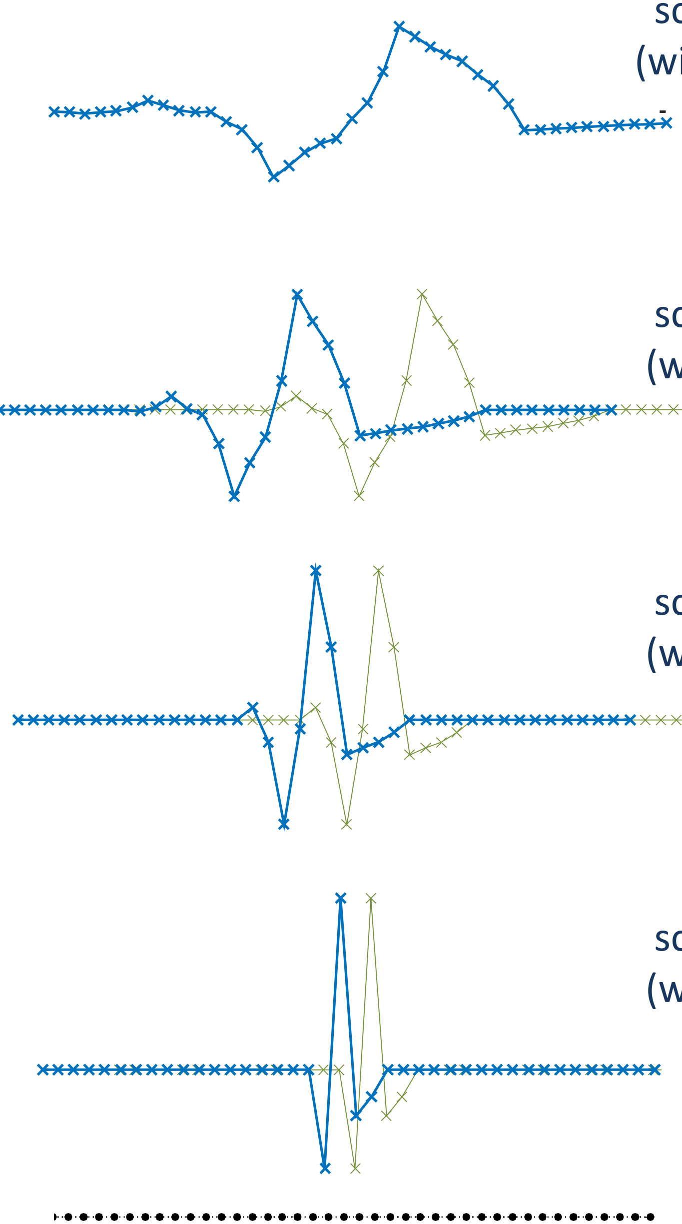
Daubechies Wavelets



scale z = 4 wavelet, $\boldsymbol{g_4}$
(with **N/16** translations)

scale z = 3 wavelet, $\boldsymbol{g_3}$
(with **N/8** translations)

scale z = 2 wavelet, $\boldsymbol{g_2}$
(with **N/4** translations)

scale z = 1 wavelet, $\boldsymbol{g_1}$
(with **N/2** translations)

N-dim vector space

Example:

Daubechies D4 wavelets

- complete, orthonormal basis
- have 2 vanishing moments (orthogonal to constant + linear functions)
- useful for resolving information at different scales

large scale wavelets encode long-ranged information

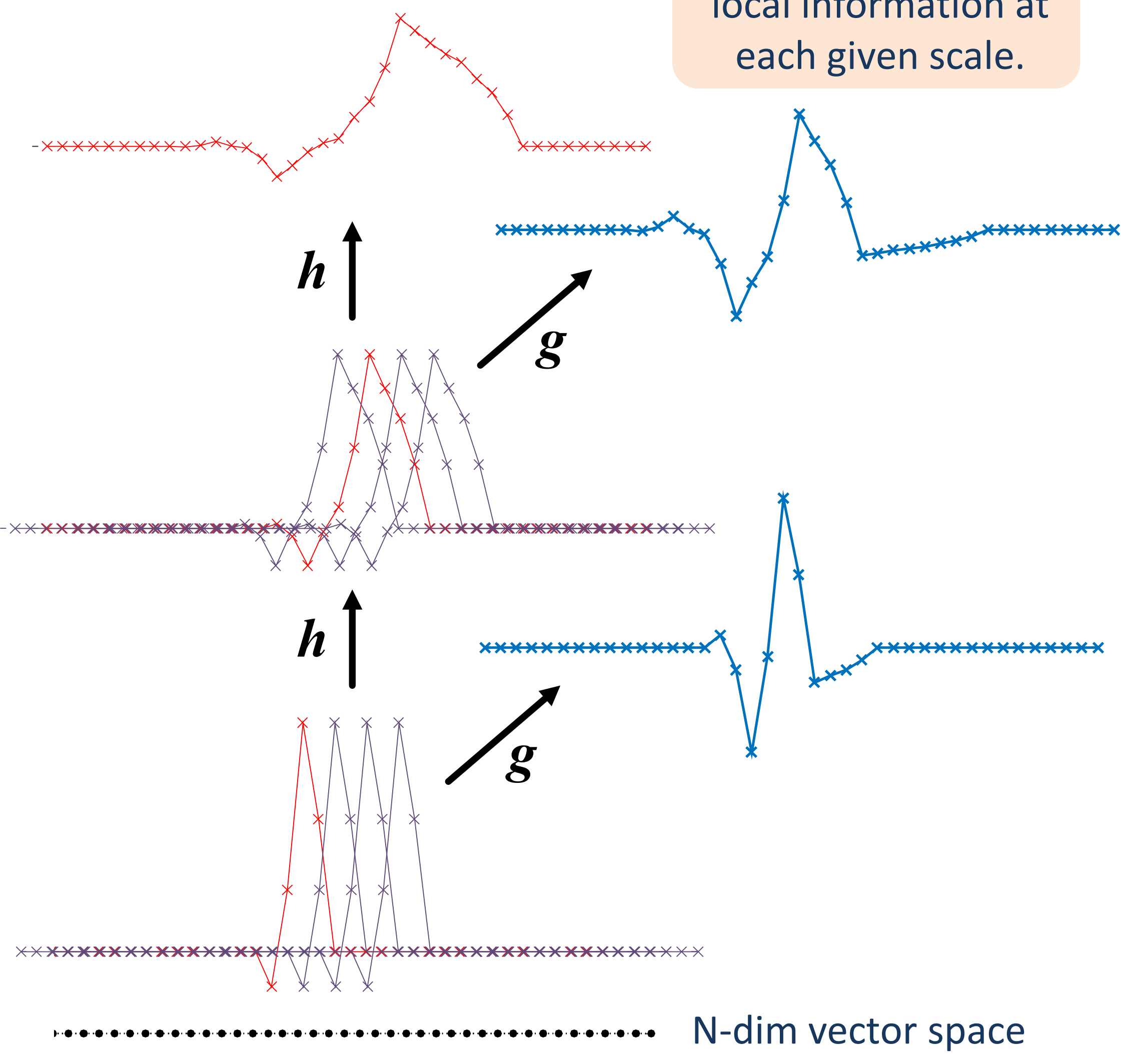
↑

small scale wavelets encode short-ranged information

Daubechies Wavelets

Scaling functions: transform data between different length scales

Wavelets: encode the local information at each given scale.



How can we construct wavelets?

- first construct scaling function (allows recursive construction of functions at different scales)

D4 scaling sequence

$$\boldsymbol{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} -0.1294 \\ 0.2241 \\ 0.8365 \\ 0.4830 \end{bmatrix}$$

- wavelets then defined from scaling functions using wavelet sequence

D4 wavelet sequence

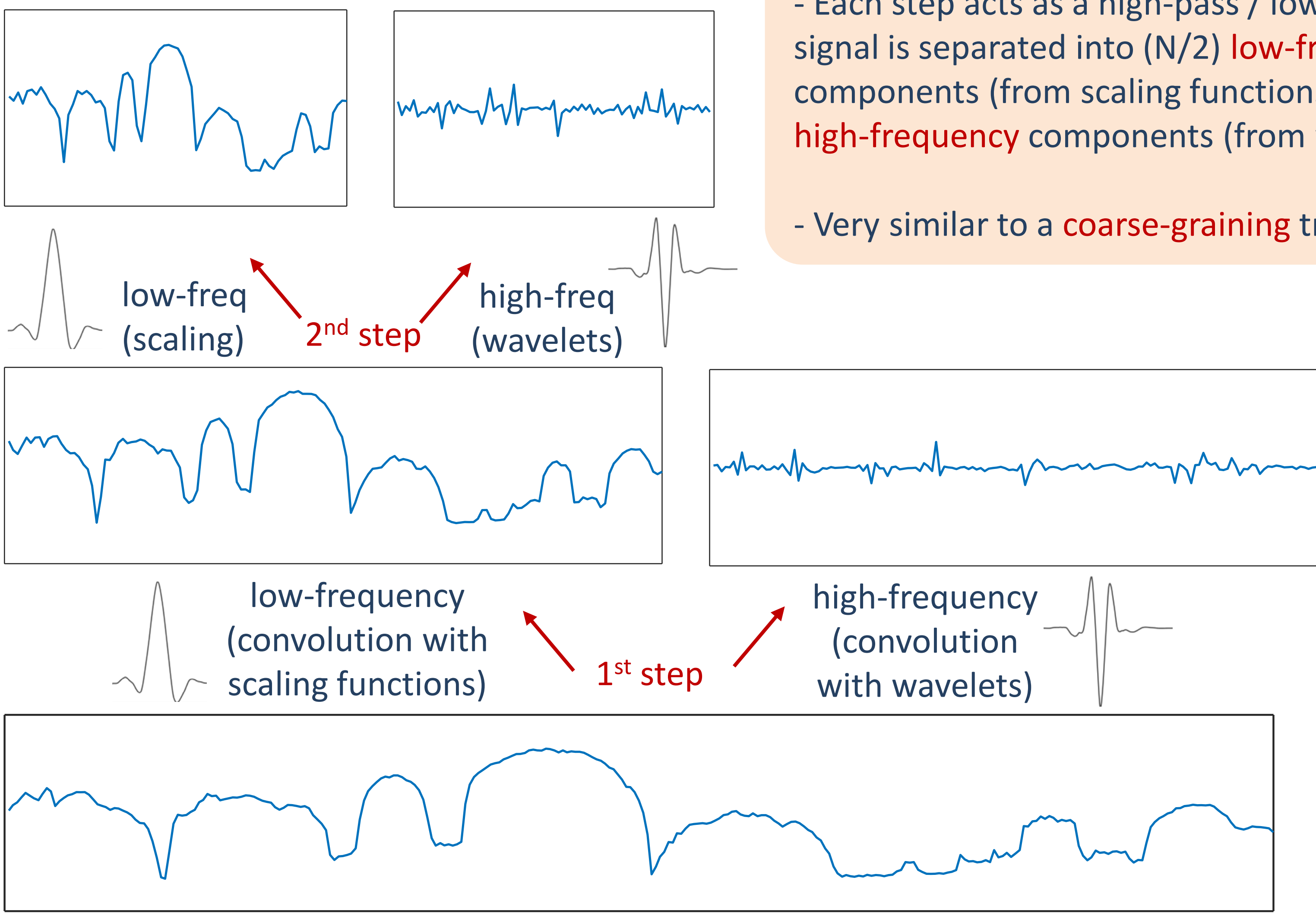
$$\boldsymbol{g} = \begin{bmatrix} -h_4 \\ h_3 \\ -h_2 \\ h_1 \end{bmatrix} = \begin{bmatrix} -0.4830 \\ 0.8365 \\ -0.2241 \\ -0.1294 \end{bmatrix}$$

Introduction to Wavelets

- A discrete wavelet transform (DWT) is a type of multi-resolution analysis (MRA)

- Each step acts as a high-pass / low-pass filter: the signal is separated into (N/2) low-frequency components (from scaling functions) and (N/2) high-frequency components (from wavelets)

- Very similar to a coarse-graining transformation!



original signal on N-lattice points

Introduction to Wavelets

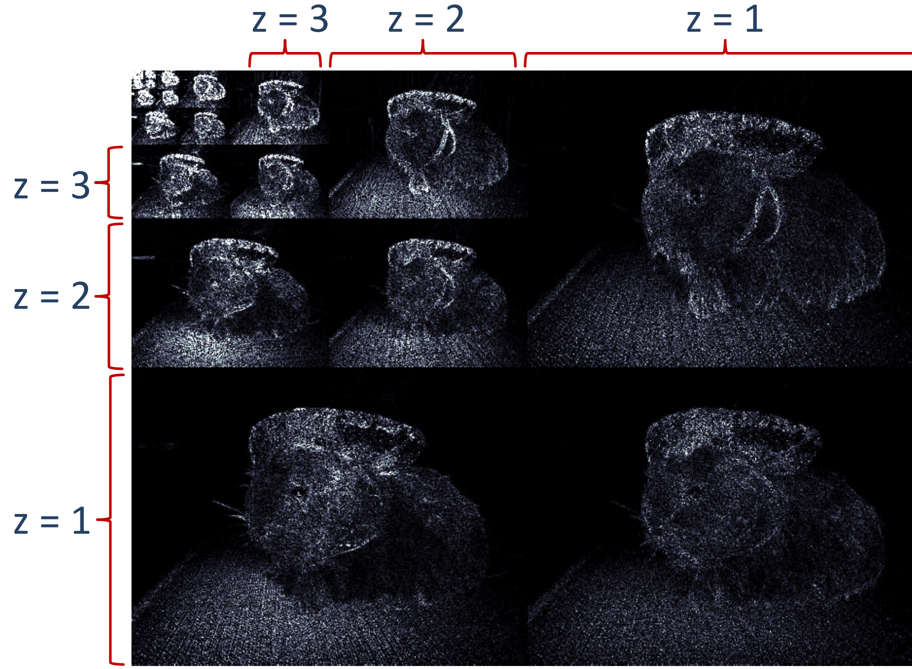
- the discrete wavelet transform decomposes the image into the information at different scales `z`

- bright pixels in transformed image represent large high-freq components (i.e. sharp changes in the image)

- transformed image still contains all of the information of the original image (we have just made a change of basis!)

- wavelets have myriad uses in signal / image processing

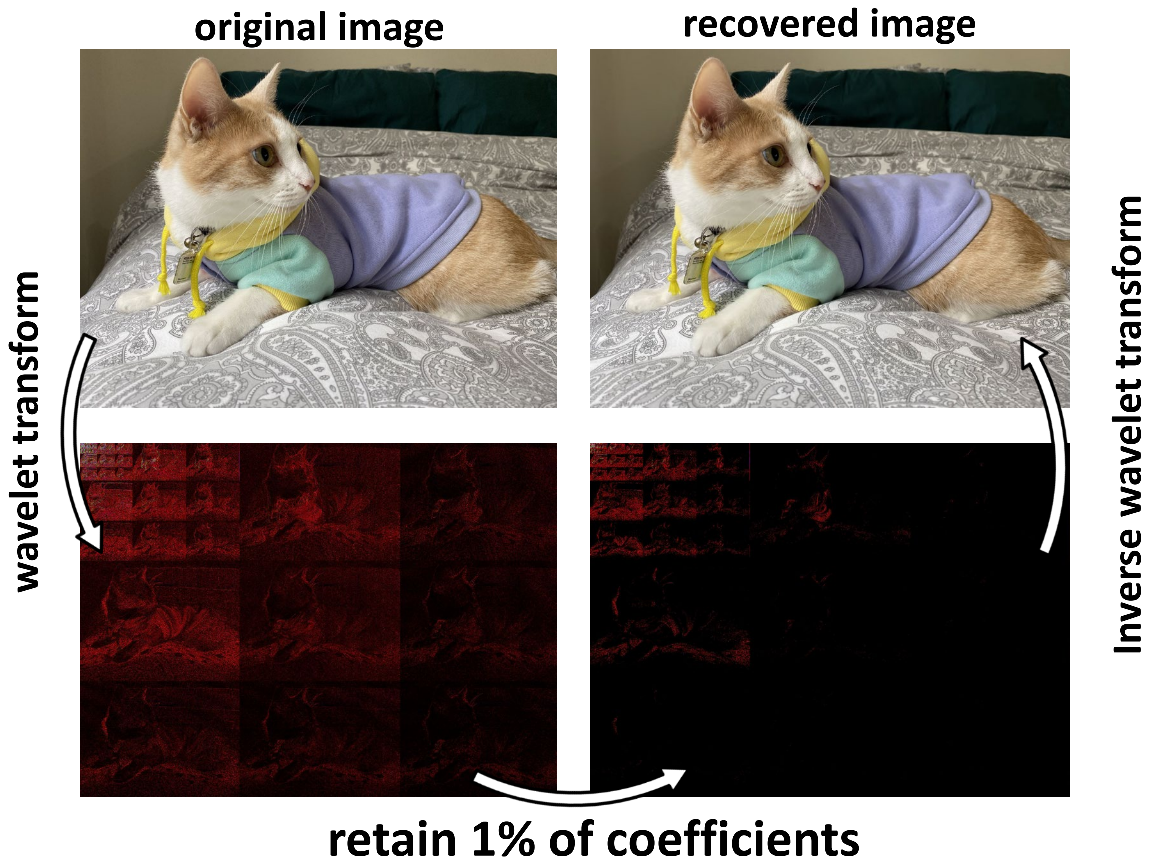
- an important application is image / video compression **(JPEG2000, MPEG, AVC, H.264, H.265)**



↑ 2D discrete wavelet transform

Original Image (“bubbles” the guinea pig)

Wavelet Application to Image Compression



- Most coefficients of the transformed image are close to zero (as wavelets are orthogonal to smooth functions)
- The transformed image is thresholded as to store only the largest wavelet coefficients (and discard the rest).
- This is the key part of JPEG2000 format, and many other standards for image, audio and video compression

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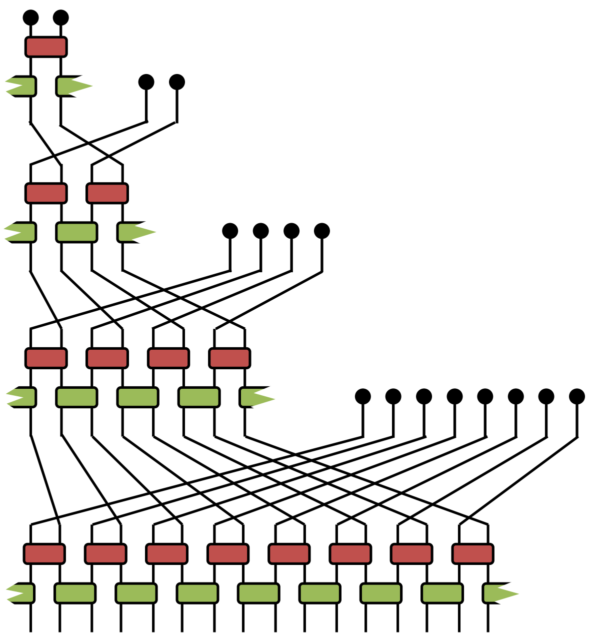
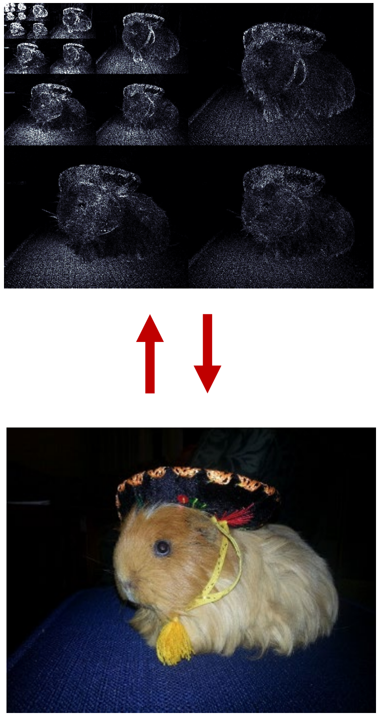
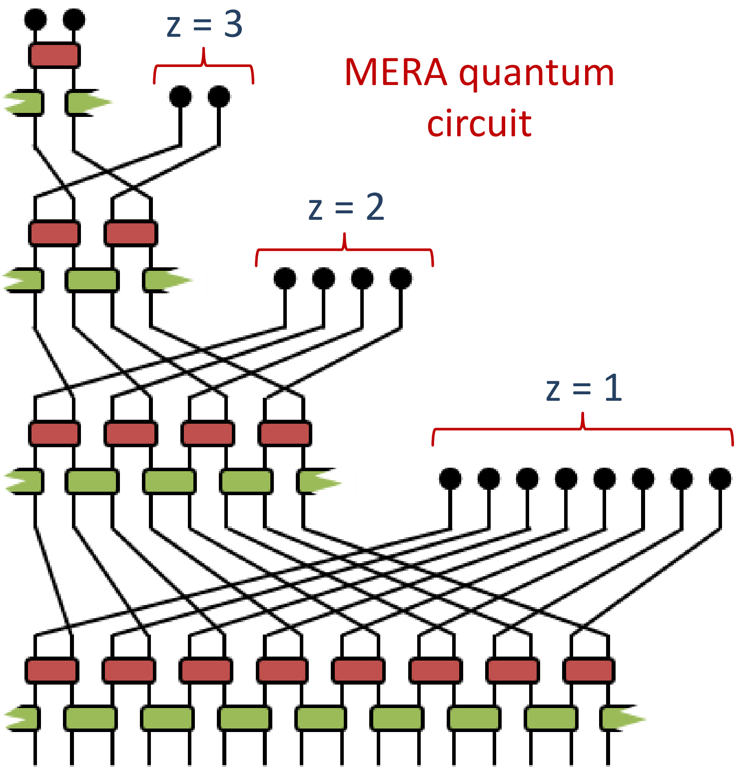
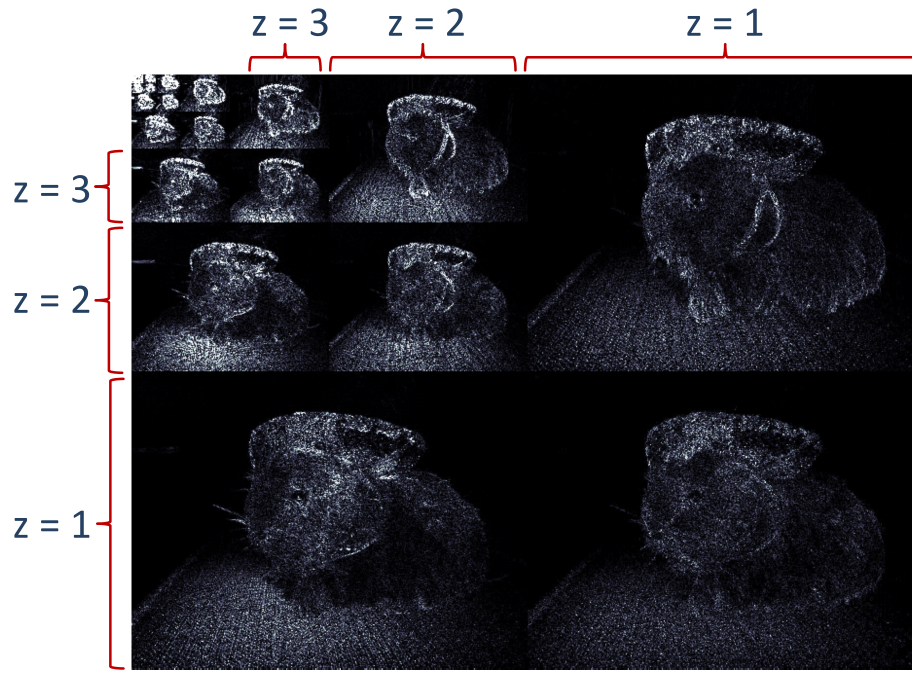




Introduction to Wavelets



The discrete wavelet transform displays many similarities to coarse-graining and the MERA. Are these similarities superficial or something deeper?



2D discrete wavelet transform

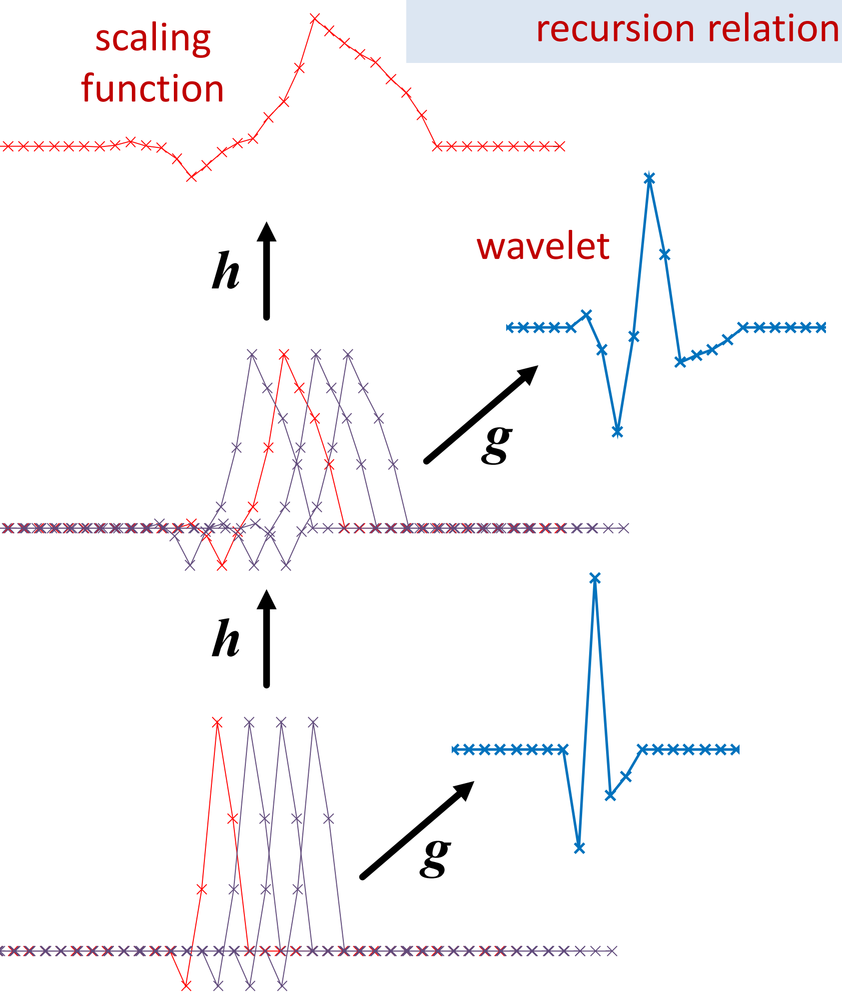
Original Image (“bubbles” the guinea pig)

Circuit representation of wavelets

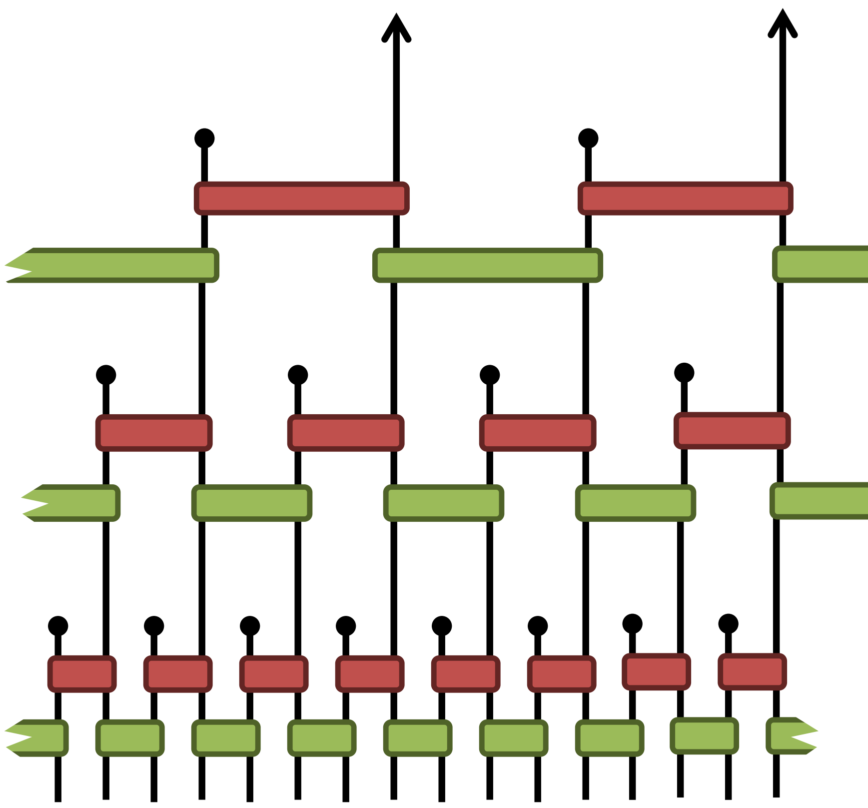
"Entanglement Renormalization and Wavelets"
G.E., Steven. R. White, **Phys. Rev. Lett 116, 140403** (2016)

Discrete wavelet transforms are precisely equivalent to (Gaussian) MERA tensor networks!

Wavelet transform described by recursion relation:



Recursion relation can be encoded as a (direct-sum) unitary circuit:



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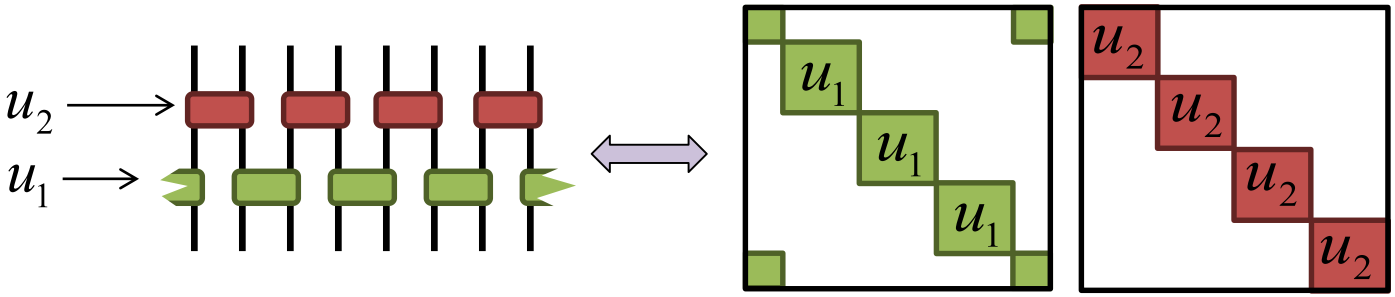


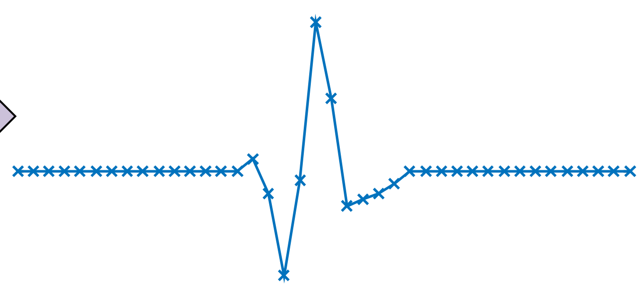
Circuit diagram here represents **direct sum** of matrices (not **tensor product!**)

Define 2x2 unitary matrix

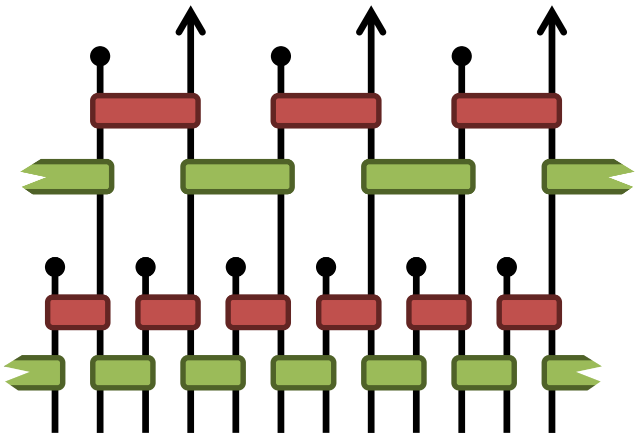
$$u(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

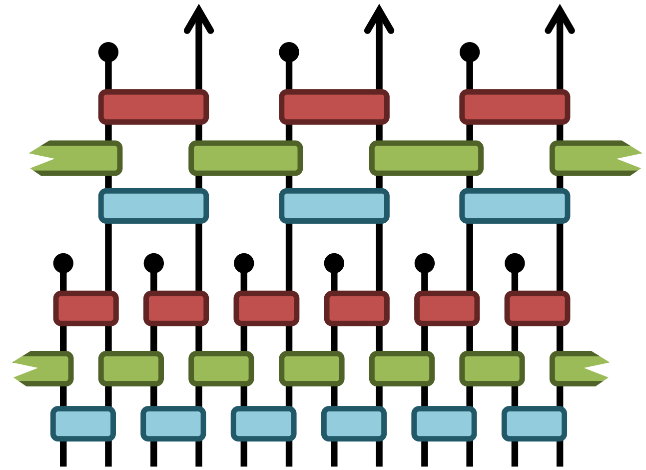
Set rotation angles: $\theta_1 = \pi/12$, $\theta_2 = -\pi/6$

The circuit then encodes the Daubechies D4 wavelets



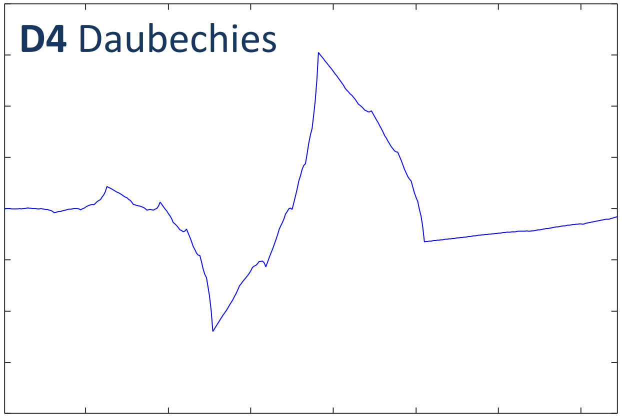
Circuit representation of wavelets



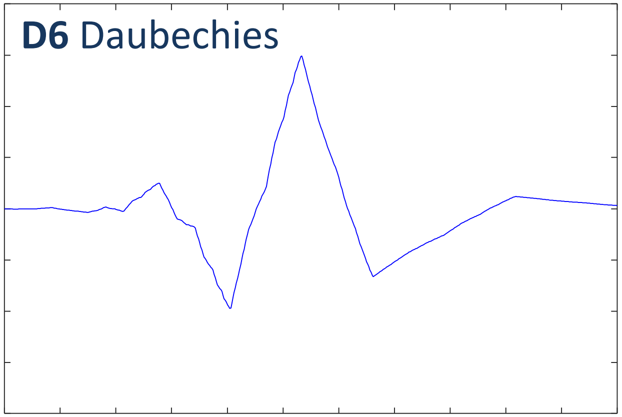


Daubechies D4 wavelets can be represented as unitary circuits with exactly the same structure as MERA (but direct-sum rather than tensor-product).

Higher-order Daubechies wavelets (or other wavelet types, such as symlets or coiflets) can also be represented as MERA-like circuits.



orthogonal to **constant** + **linear** functions



orthogonal to **constant** + **linear** + **quadratic** functions

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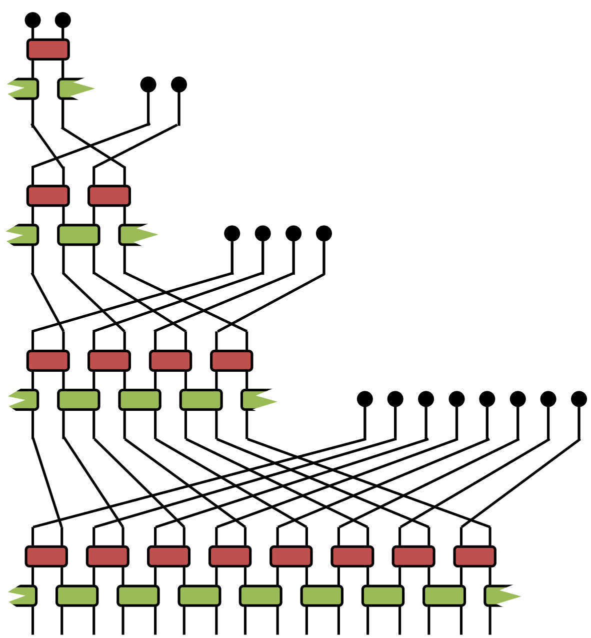
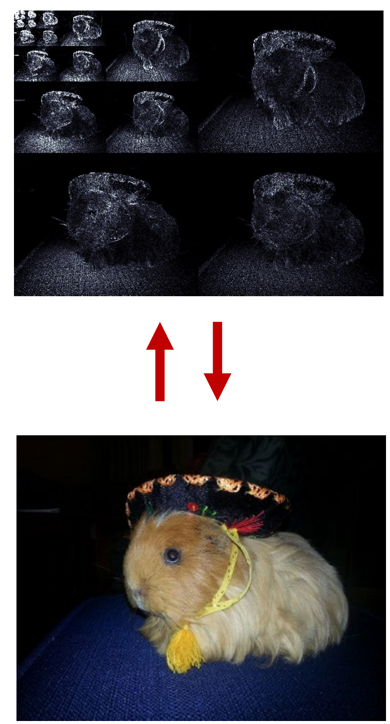


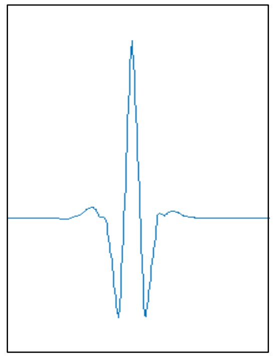


Wavelets for Image Compression

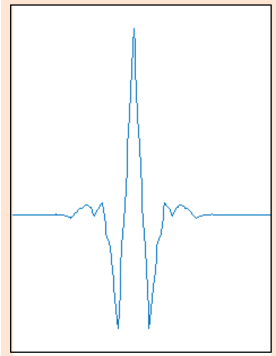
What wavelets are good for image compression?

Daubechies

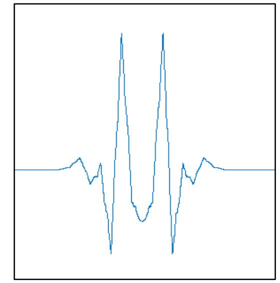
Coiflets

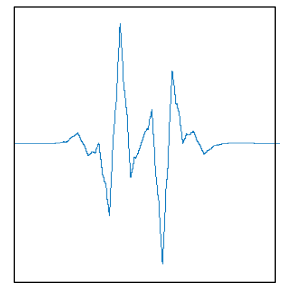


CDF wavelets



Scale-3 symmetric





desirable properties	Daubechies	Coiflets	CDF wavelets	Scale-3 symmetric
orthogonality?	**yes**	**yes**	**near orthogonal**	**yes**
symmetric?	**no**	**near symmetric**	**yes**	**yes**
compression ratio?	**okay**	**good**	**good**	**bad**

JPEG2000, MPEG, AVC, H.264, H.265

Wavelet Design using Tensor Networks

"Representation and design of wavelets using unitary circuits"
G.E., Steven. R. White, **Phys. Rev. A 97,** 052314 (2018)

Tensor networks offer a radically different way to construct wavelets than the standard approaches!

The approach we follow:

(1) Choose a network structure: defines size of wavelets and their translational/scaling properties

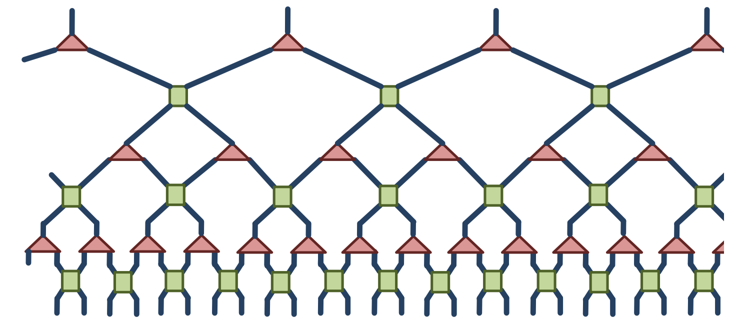
(2) Impose symmetries: global symmetries imposed by local constraints on tensors

(3) Optimise free parameters: minimize the chosen loss function.

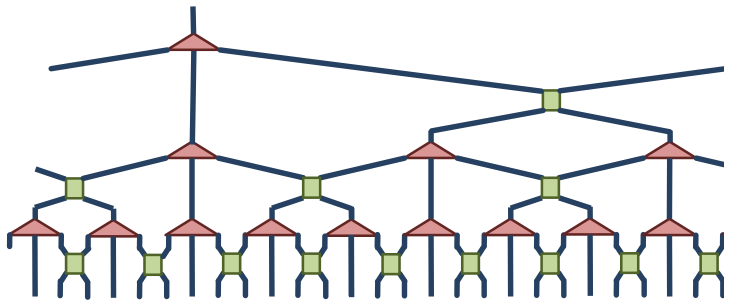
We know that there are many different ways to construct multi-scale tensor networks:

- different rescaling factors
- different organization of blocks
- different causal structure

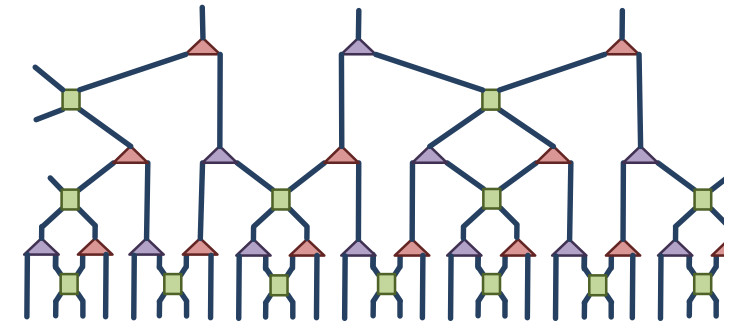
(i) Binary MERA



(ii) Ternary MERA:



(iii) Modified Binary MERA:



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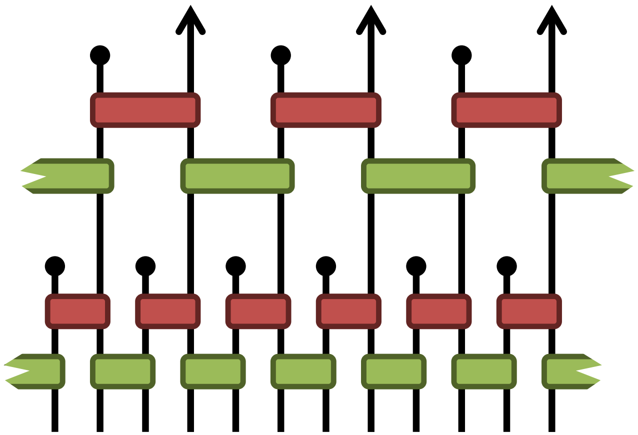
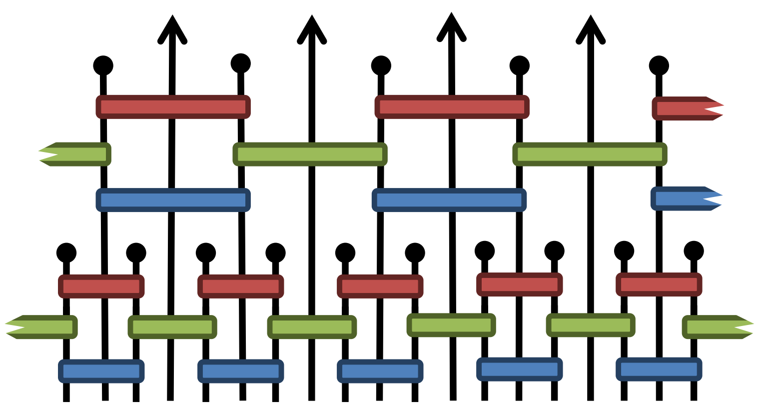
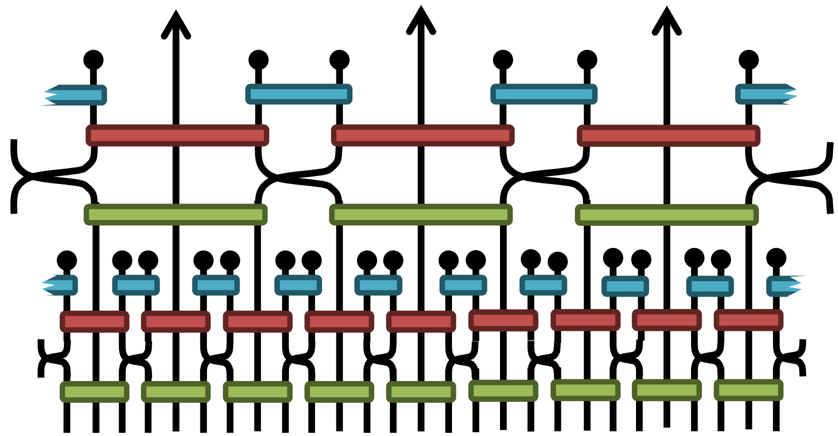
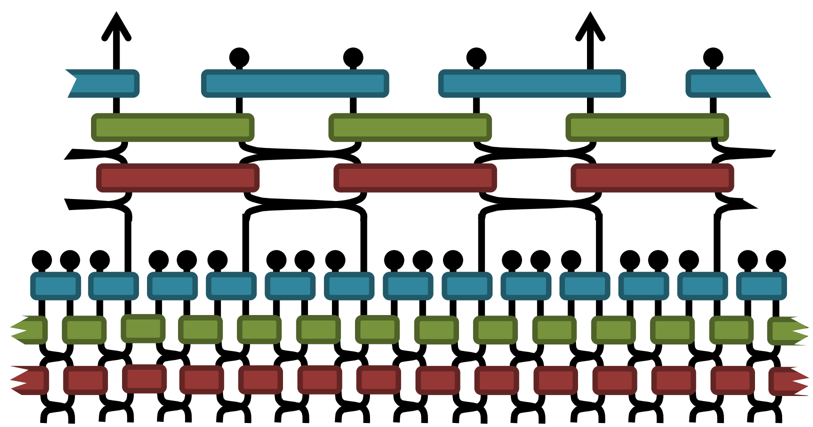
Many wavelets (Daubechies, Symmlets, Coiflets) correspond to binary circuits:





However the circuit formalism allows us to easily formulate more general wavelets, many of which were previously unknown:





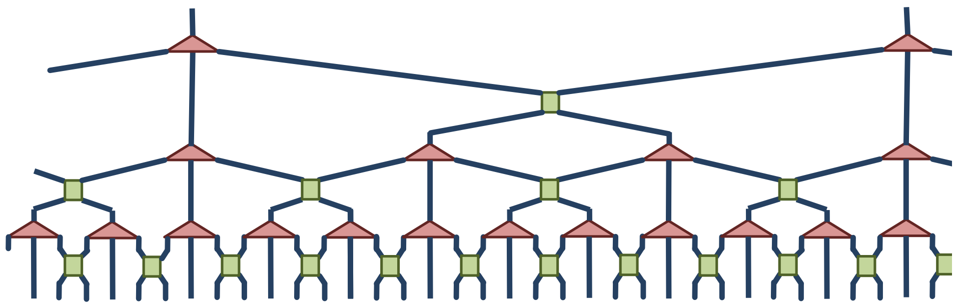
Each circuit corresponds to a different class of wavelets
(different support, translational and rescaling properties)

Wavelet Design using Tensor Networks

"Representation and design of wavelets using unitary circuits"
G.E., Steven. R. White, **Phys. Rev. A 97,** 052314 (2018)

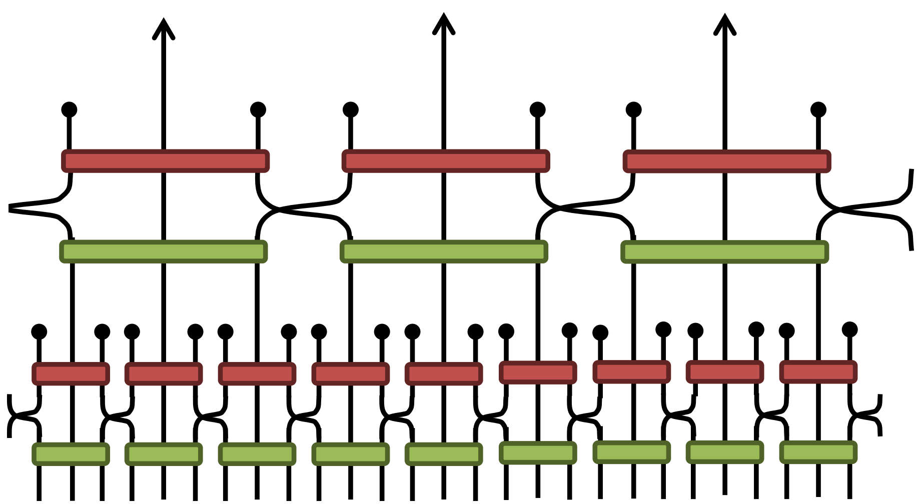
We tested many possibilities but today we focus on ternary circuits (3-to-1 rescaling):

Ternary MERA:



generalization

Ternary unitary circuit:



Wavelet Design using Tensor Networks

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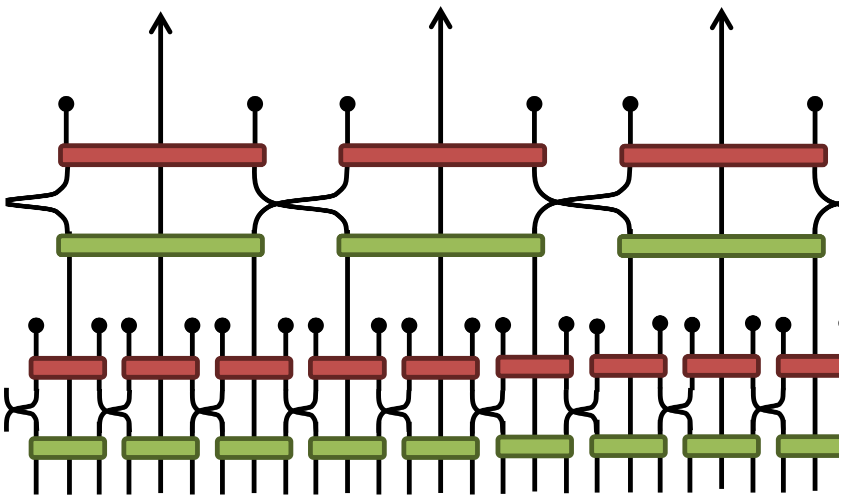
Ternary unitary circuit:



Q: How to we incorporate a global symmetry on a tensor network?

A: We impose a symmetry constraint on each individual tensor!

Orthogonality of wavelets ⇒ every tensor should be unitary

Reflection symmetry wavelets ⇒ every tensor should be reflection symmetric

Wavelet Design using Tensor Networks

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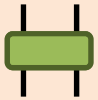
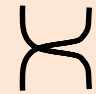
2x2 matrices:

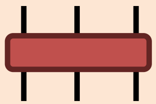


only a single unique matrix: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

corresponds to a swap gate:



3x3 matrices:



1-parameter family of matrices:

$$\frac{1}{2}\begin{bmatrix} \cos(\theta)+1 & \sqrt{2}\sin(\theta) & \cos(\theta)-1 \\ -\sqrt{2}\sin(\theta) & 2\cos(\theta) & -\sqrt{2}\sin(\theta) \\ \cos(\theta)-1 & \sqrt{2}\sin(\theta) & \cos(\theta)+1 \end{bmatrix}$$

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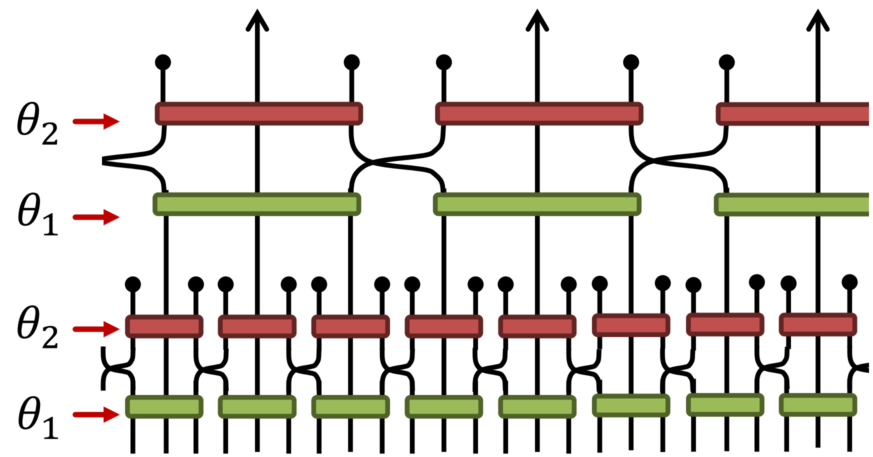
Ternary unitary circuit:



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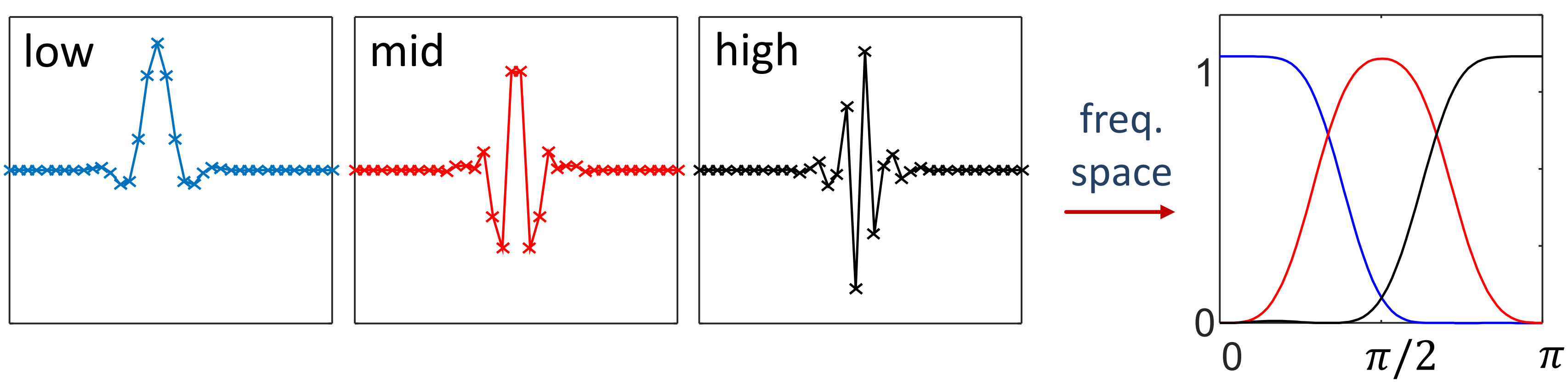
(3) Optimise free parameters: minimize the chosen loss function.

Q: How do we set the free parameters $\{\theta_1, \theta_2, \ldots\}$ of the network?

A: We numerically optimise according to some specified loss function; this is chosen to enforce that each layer separates into well defined low/mid/high frequencies.



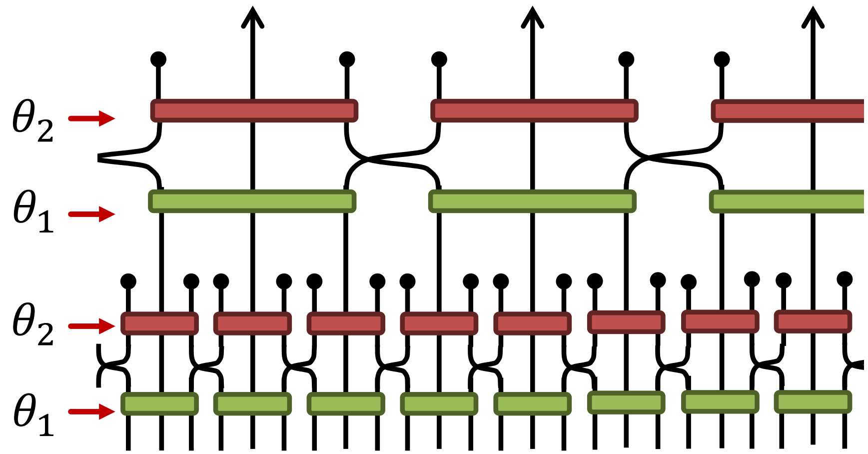
Wavelet Design using Tensor Networks

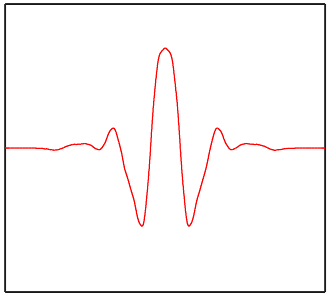
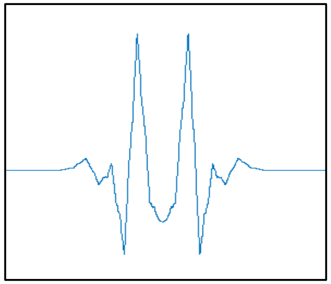
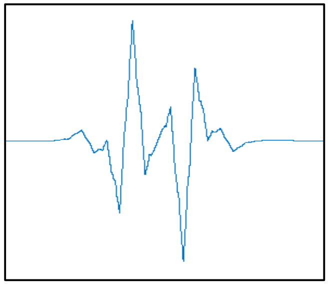
"Representation and design of wavelets using unitary circuits"
G.E., Steven. R. White, **Phys. Rev. A 97,** 052314 (2018)

Result: we construct new families of (anti-) symmetric wavelets with dilation factor 3

These are not the first known examples of symmetric wavelets of dilation factor 3, but have greatly improved properties over previous examples!

Ternary unitary circuit:



	symmetric wavelet	antisymmetric wavelet	
New orthogonal, symmetric wavelets: smooth with good compactness.			Very handsome!
Previous orthogonal, symmetric wavelets: jagged and wide support.			Butt ugly!

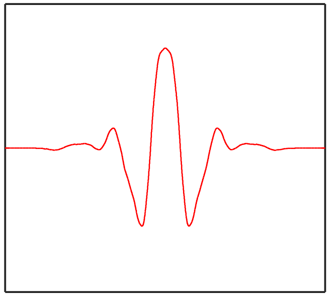
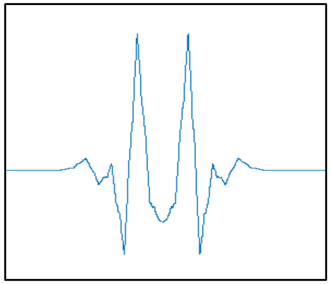
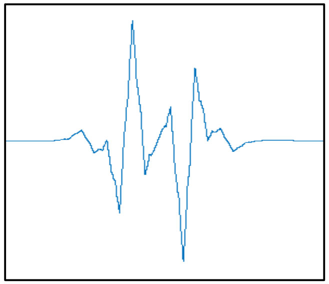
Wavelet Design using Tensor Networks

How well do these new wavelets work in practice?

We have been doing some large scale testing and experimentation of new wavelet designs for image compression.

James McCord

"Improved wavelet designs for image compression"
J. C. McCord, **G.E.**, ***in preparation***

	symmetric wavelet	antisymmetric wavelet	
New orthogonal, symmetric wavelets: smooth with good compactness.			Very handsome!
Previous orthogonal, symmetric wavelets: jagged and wide support.			Butt ugly!

Wavelet Design using Tensor Networks

sample code at:
www.tensors.net/research

CDF 9/7 wavelets:

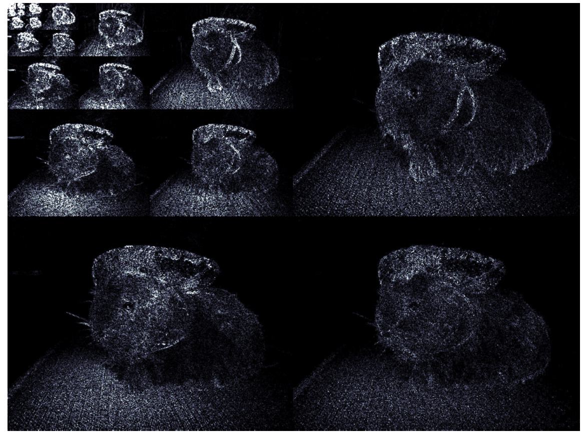
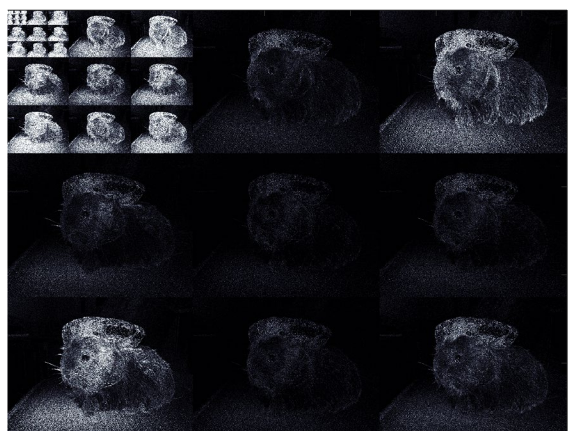


image → transform to wavelet basis → **truncate** (keep only largest 2% of coefficients) → inverse transform → compressed image

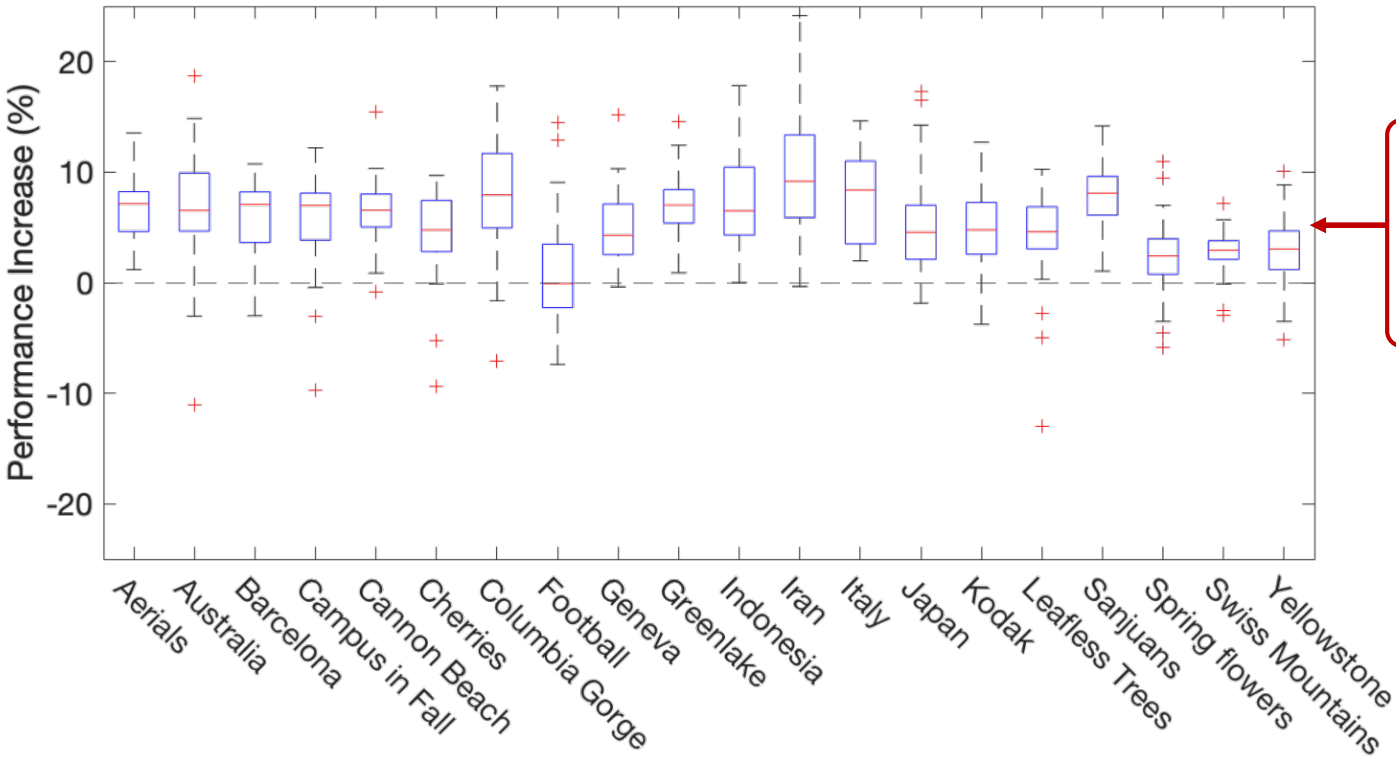
New scale-3 wavelets:



Wavelet Design using Tensor Networks

- We compare the new wavelets against the CDF 9/7 wavelets, which are regarded as the best known wavelets for practical applications.
- We compare compressed images based on multi-scale structural similarity (SSIM), which is a good measure of perceived image quality.
- Calculate the minimum number of coefficients that to be retained (in the transformed image) in order to achieve fixed quality: SSIM > 0.9.

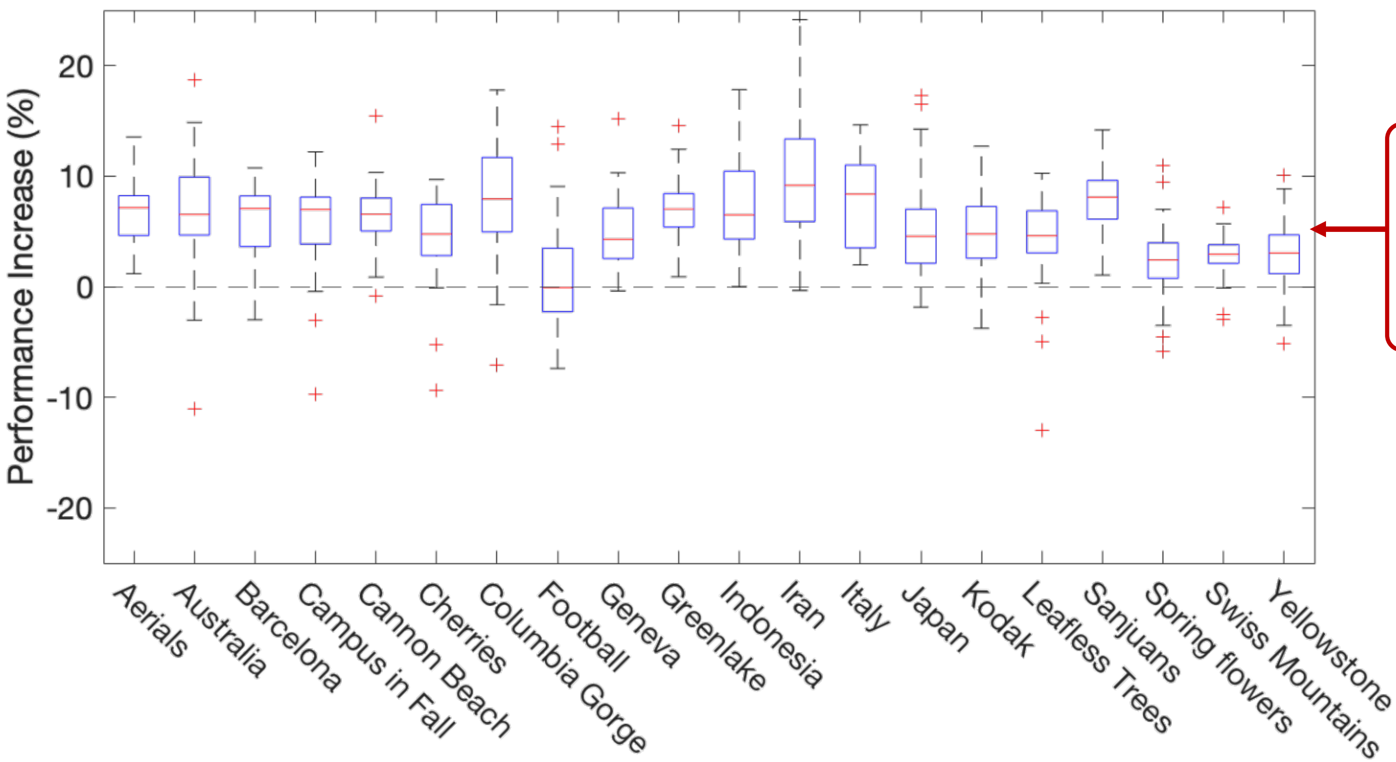
Test database: 1000+ colour photographs.



On average, the new wavelets give 7% more efficient compression!

Wavelet Design using Tensor Networks

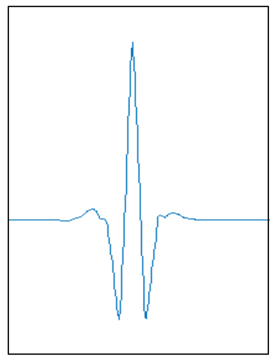
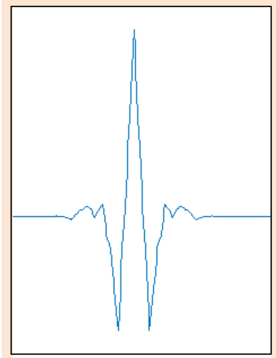
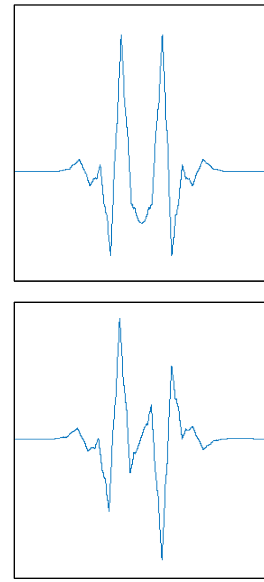
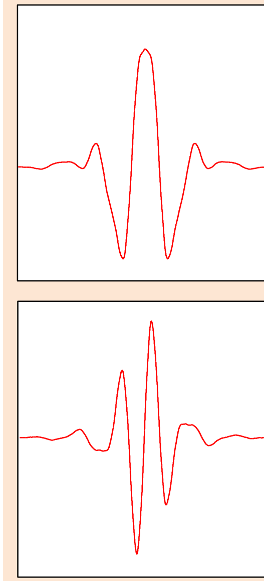
- the improvement in efficiency becomes larger at higher quality factors.
- this improvement in efficiency could be especially significant in certain settings, such as medical imaging.
- similar improvements are expected over previous wavelet based video compression routines (JPEG2000, MPEG, AVC, H.264, H.265), which could reduce the bandwidth required to stream video.



On average, the new wavelets give 7% more efficient compression!

Wavelets for Image Compression

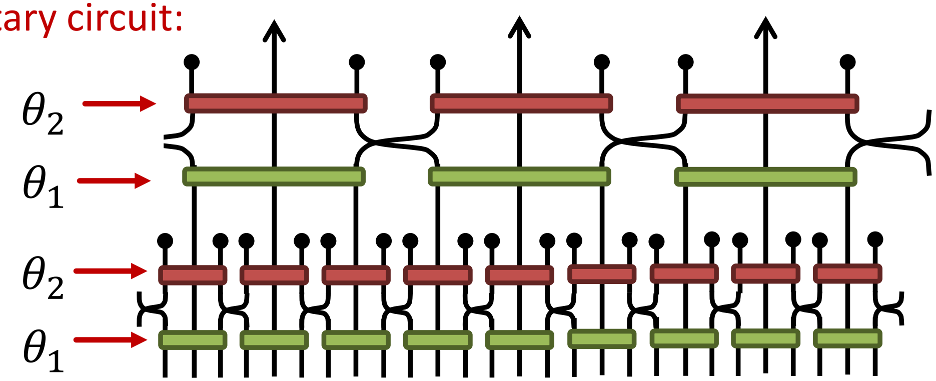
The new wavelets seems to be a clear improvement over previous best wavelets…

	Daubechies	Coiflets	CDF wavelets	Scale-3 symmetric	New scale-3 symmetric
					
desirable properties					
orthogonality?	**yes**	**yes**	**near orthogonal**	**yes**	**yes**
symmetric?	**no**	**near symmetric**	**yes**	**yes**	**yes**
compression ratio?	**okay**	**good**	**good**	**bad**	**superior**

JPEG2000, MPEG, AVC, H.264, H.265 (CDF wavelets)

Wavelet Design using Tensor Networks

Ternary unitary circuit:



In progress: **finely tuned wavelets**

Can we tune the free angles $\{\theta_1, \theta_2, \ldots\}$ as to construct optimally efficient wavelets for certain data-types? (e.g. fingerprints, medical images)

This a nice example to demonstrate the utility of tensor networks outside of physics!

standard tensor network ideas and methods

new setting

reformulation of wavelets, with application to image/video compression using wavelets

profit???

More generally: tensor networks have many connections between existing ideas in data science; there are myriad potential applications!

Thanks!