On the geometry of tensor network states

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Based on

1. L; Qi, Yang; Ye, Ke On the geometry of tensor network states. Quantum Inf. Comput. 12 (2012), no. 3-4.

2. Gesmundo, Fulvio ; L; Walter, Michael Matrix product states and the quantum max-flow/min-cut conjectures. J. Math. Phys. 59 (2018), no. 10.

3. Work in progress with Thomas Barthel and Hang Huang

Notation

C_N cycle with N nodes

$TNS(C_N, m, d) \subset (\mathbb{C}^d)^{\otimes N}$ corresponding set of tensor network states where m bond dimension d the physical dimension,

Let $TNS : (\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^d)^{\oplus N} \to TNS(C_N, m, d)$ corresponding map.

Write $T_i = X_{i,1} \otimes e_1 + \cdots + X_{i,d} \otimes e_d \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^d$

$$TNS(T_1, \ldots, T_N) = \sum_{J \subset [d]^N} \text{trace}(X_{1,j_1} \cdots X_{N,j_N}) e_{j_1} \otimes \cdots \otimes e_{j_N}$$

Easy calculation dim image $\sim Ndm^2 - Nm^2$ in ambient space $\dim = d^N$.

Questions: What subset of $(\mathbb{C}^d)^{\otimes N}$ does it fill? Is image closed? .

Here closure under taking limits = Zariski closure

Cyclicly invariant tensors

$TNS^{\mathbb{Z}_N}(C_N, m, d) \subset [(\mathbb{C}^d)^{\otimes N}]^{\mathbb{Z}_N} \subset (\mathbb{C}^d)^{\otimes N}$ cyclicly invariant states obtained by placing the same tensor at each vertex.

Let $TNS^{\mathbb{Z}_N} : \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^d \to TNS^{\mathbb{Z}_N}(C_N, m, d)$ corresponding map.

Given $T = X_1 \otimes e_1 + \cdots + X_d \otimes e_d$ Then

$$TNS^{\mathbb{Z}_N}(T) = \sum_{J \subset [d]^N} \mathrm{trace}(X_{j_1} \cdots X_{j_N}) e_{j_1} \otimes \cdots \otimes e_{j_N}.$$

Easy calculation dim image $\sim dm^2 - m^2$ in space of dim $\sim \frac{d^N}{N}$

Questions:

What subset of $[(\mathbb{C}^d)^{\otimes N}]^{\mathbb{Z}_N}$ does it fill?

Does it lie in a proper linear subspace? Is image closed?

Results

Prop. (Gesmundo-L-Walter) $m \geq N$,
$Span(TNS^{\mathbb{Z}_N}(C_N, m, d)) = ((\mathbb{C}^d)^{\otimes N})^{\mathbb{Z}_N}$

Question: what about small m?

Thm. (L-Qi-Ye) $N \geq 3$, $d \geq m^2$ $TNS(C_N, m, d)$ is not closed.

Question: What about small d?

Thm. (B-H-L): $m = 2$, $N \geq 5$, $d \geq 2$. $TNS^{\mathbb{Z}_N}(C_N, m, d)$ is not closed.

Hope to prove soon: same result when $N \geq cm$ expect $c \leq 3$

Hope to prove: $TNS(C_N, m, d)$ not closed either when $N \geq F(m)$ some explicit $F(m)$

Space of cyclicly invariant tensors

Let $A = \mathbb{C}^d$.

Two groups act on $A^{\otimes N}$: $GL(A)$ and $\mathfrak{S}_N$.

$\mathbb{Z}_N \subset \mathfrak{S}_N$,

$(A^{\otimes 2})^{\mathbb{Z}_2} = S^2 A$

$(A^{\otimes 3})^{\mathbb{Z}_3} = S^3 A \oplus \Lambda^3 A$

$(A^{\otimes 4})^{\mathbb{Z}_4} = S^4 A \oplus S_{211} A \oplus S_{22} A = S^2(S^2 A) \oplus \Lambda^2(\Lambda^2 A)$

In general $A^{\otimes N} = \oplus_\pi S_\pi A \otimes [\pi]$ as $GL(A) \times \mathfrak{S}_N$-module

$(A^{\otimes N})^{\mathbb{Z}_N} = \oplus_\pi S_\pi A \otimes [\pi]^{\mathbb{Z}_N}$

Can describe explicitly as $GL(A)$-module.

Space of cyclicly invariant tensors

Natural bases where each basis vector v_k under $\sigma \in \mathbb{Z}_N$ satisfies $\sigma v_k = \omega v_k$ some ω with $\omega^N = 1$. Can insist moreover each v_k lives in an irred. $GL(A)$-module. $\rightsquigarrow$ basis vectors marked in two ways.

More naïvely: elements of form $\mathbb{Z}_N \cdot x_1 \otimes \cdots \otimes x_N$ with $x_j \in A$ span space.

Construct such via
$x_1 \otimes (e_1 \otimes e^N) + x_2 \otimes (e_2 \otimes e^1) + \cdots + x_N \otimes (e_N \otimes e^1)$.

can do if $m \geq N$, proving GLW Prop.

Do better with more refined basis?

Idea of proof of L-Qi-Ye Thm.

Consider iterated matrix multiplication tensor $IMM_m^N \in (\mathbb{C}^{m^2})^{\otimes N}$.

$IMM_m^N(Z_1, \ldots, Z_N) \mapsto \mathrm{trace}(Z_1 \cdots Z_N)$

Prop. $IMM_m^N \in TNS^{\mathbb{Z}_N}(C_N, m, m^2)$.

Proof: Write $A = A_j = \mathbb{C}^d = E^* \otimes E$, $E = \mathbb{C}^m$. Take
$T = \mathrm{Id}_{E^* \otimes E} \in (E^* \otimes E) \otimes (E \otimes E^*) = \mathrm{End}(E^* \otimes E)$.

Prop. $TNS(C_N, m, m^2) = \mathrm{End}(A_1) \times \cdots \times \mathrm{End}(A_N) \cdot IMM_m^N$.

and $TNS^{\mathbb{Z}_N}(C_N, m, m^2) = \mathrm{End}(A) \cdot IMM_m^N$

Note
$\mathrm{End}(A_1) \times \cdots \times \mathrm{End}(A_N) \cdot IMM_m^N \subseteq \overline{GL(A_1) \times \cdots \times GL(A_N) \cdot IMM_m^N}$

An orbit closure!

Detour: Orbits and their closures

In general $G \subset GL(V)$, (our case $V = A^{\otimes N}$ or $V = A_1 \otimes \cdots \otimes A_N$, $G = GL(A)$ or $GL(A_1) \times \cdots \times GL(A_N)$)

given $v \in V$, let $G_v := \{g \in G \mid g \cdot v = v\}$. (our case $v = IMM_m^N$)

Then, given $w \in \overline{G \cdot v}$, $\dim G_w \geq \dim G_v$. with $=$ iff $w \in G \cdot v$.

$w \in \mathrm{End}(V) \cdot v$, either $w \in GL(V) \cdot v$ or $w = X \cdot v$, $\det(X) = 0$.

Our case: if $T \in \mathrm{End}(A_1) \times \cdots \times \mathrm{End}(A_N) \cdot IMM_m^N$, then either $T \in GL(A_1) \times \cdots \times GL(A_N) \cdot IMM_m^N$ or T is not concise (lives in smaller space, i.e., $\mathbb{C}^{d-1} \otimes (\mathbb{C}^d)^{\otimes N-1}$ or $\sigma \cdot \mathbb{C}^{d-1} \otimes (\mathbb{C}^d)^{\otimes N-1}$ $\sigma \in \mathbb{Z}_N$ permutation of factors.)

Idea of proof cont'd

To show $TNS(C_N, m, m^2) = \mathrm{End}(A)^{\times N} \cdot IMM_m^N$ is not closed

find $T \in \overline{GL(A)^{\times N} \cdot IMM_m^N}$ such that

i. $\dim G_T > \dim G_{IMM_m^N}$,

ii. T is concise

Example

$N = 3$, $d = m^2$. View $A = E^* \otimes E$

Let
$g(t) = (X_0 + tX_1, Y_0 + tY_1, Z_0 + tZ_1) \in GL(A) \times GL(A) \times GL(A)$
where

X_0, Y_0, Z_1 projection onto diagonal matrices.

X_1, Y_1, Z_0 projection onto off diagonal matrices

Explicitly $m = 2$ $(X_0 + tX_1) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & tb \\ tc & d \end{pmatrix}$

$\dim G_{IMM_m^3} = 3m^2 - 1$.

Let $[T] = \lim_{t \to 0} g(t) \cdot [IMM_m^3]$

$\dim G_T = 4m^2 - 2m$.

More factors, just take other components of $g(t)$ to be Id.

Case $d > m^2$ reduce to $d = m^2$ via vector bundle construction

Work in Progress

$d < m^2$, $TNS(C_N, m, d)$, $TNS^{\mathbb{Z}_N}(C_N, m, d)$ no longer orbit closures.

Idea: work directly.

$v^{\otimes N} + w^{\otimes N} \in TNS^{\mathbb{Z}_N}(C_N, m, d) \subset TNS(C_N, m, d)\ \forall v, w \in A$.

$\mathbb{Z}_N \cdot (e_1^{\otimes N-1} \otimes e_2) = \lim_{t \to 0} \frac{1}{t}[(e_1 + te_2)^{\otimes N} - e_1^{\otimes N}]$

so $\mathbb{Z}_N \cdot (e_1^{\otimes N-1} \otimes e_2) \in \overline{TNS^{\mathbb{Z}_N}(C_N, m, d)}$.

Goal: show $\mathbb{Z}_N \cdot (e_1^{\otimes N-1} \otimes e_2) \notin TNS^{\mathbb{Z}_N}(C_N, m, d)$

Work in Progress cont'd

To show $\mathbb{Z}_N \cdot (e_1^{\otimes N-1} \otimes e_2) \notin TNS^{\mathbb{Z}_N}(C_N, m, d)$

Recall, for $m \times m$ matrix X, for $k \geq m$, can express X^k in terms of $\mathrm{Id}, X, X^2, \ldots, X^{m-1}$ (characteristic polynomial).

$\Rightarrow$

trace$(X_1^i X_2^j) = 0$, $i + j = N$ in terms of trace$(X_1^s X_2^t)$, $s, t \leq m - 1$.

Lots of identities

Finally, show if holds $\forall\ (i, j) \neq (N - 1, 1)$ also $= 0$
$(i, j) = (N - 1, 1)$.

Can refine looking at more general words in X_1, X_2 (up to cyclic perm.)

Thank you for your attention

For more on **tensors**, their geometry and applications, resp. **geometry and complexity**, resp. **recent developments**: