Tropical tensor networks for ground states of spin glasses

Lei Wang (王磊)
wanglei@iphy.ac.cn
Institute of Physics, CAS

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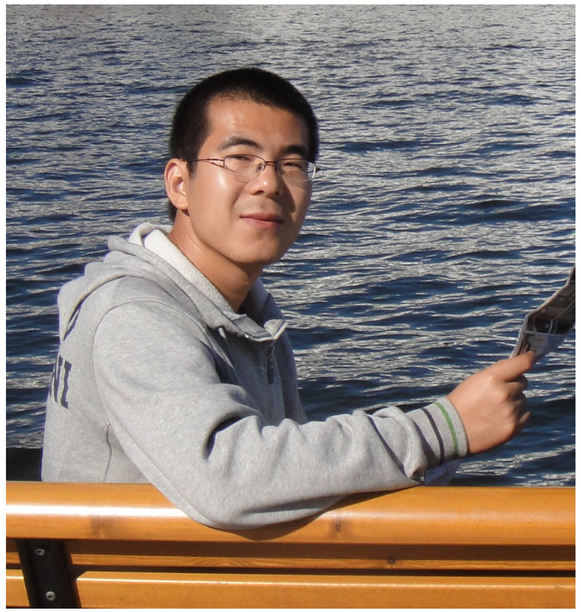
https://github.com/TensorBFS/TropicalTensors.jl

Thanks to my collaborators

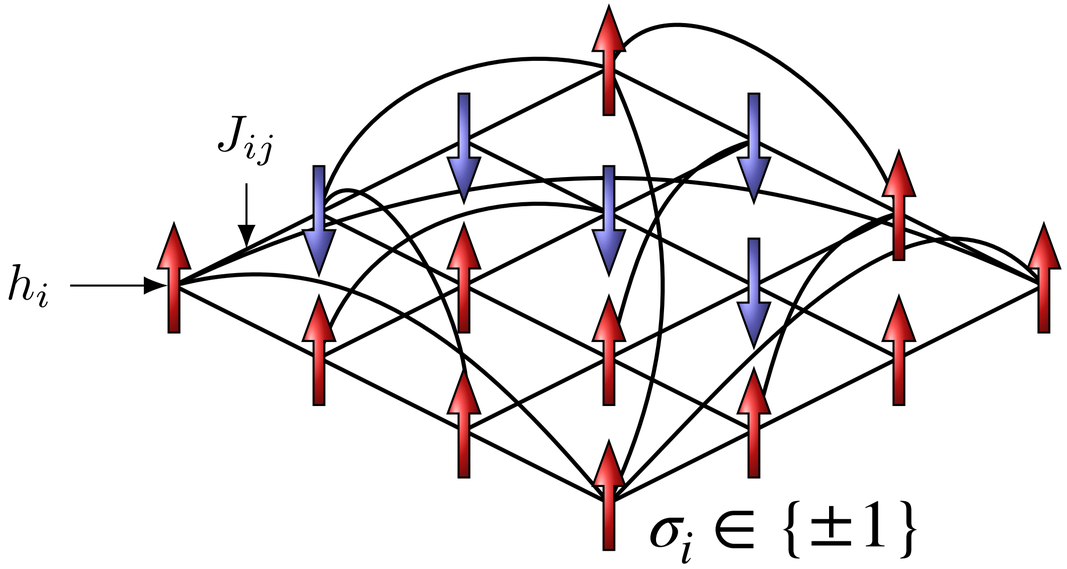
Jin-Guo Liu
Harvard & QuEra Computing



Pan Zhang
ITP CAS

tutorial notebook https://giggleliu.github.io/notebooks/tropical/tropicaltensornetwork.html

Spin glasses



$$E(\{\sigma\}) = \sum_{i<j} J_{ij}\sigma_i\sigma_j + \sum_i h_i\sigma_i$$

$$E^* = \min_{\sigma} E(\{\sigma\}) \quad ???$$

- Same as QUBO, has broad applications in statistical physics, machine learning, coding theory…
- Finding ground state is NP-hard: Barahona 1982
- Ising formulation for many NP problems: Lucas 1302.5834
  (including Karp's 21 NP-complete problems)

Example: frustrated Ising model on a football

https://github.com/QuantumBFS/SSSS/blob/master/Challenge.md

What is the ground state ?

- Energy
- Spin configuration

Optimization problem

How many are they ?

“residual entropy”

Counting problem

Tropical algebra

$$x \oplus y = \min(x, y)$$

$$x \odot y = x + y$$

Tropical algebra

Addition table

$\oplus$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	2	3	3	3	3	3
4	1	2	3	4	4	4	4
5	1	2	3	4	5	5	5
6	1	2	3	4	5	6	6
7	1	2	3	4	5	6	7

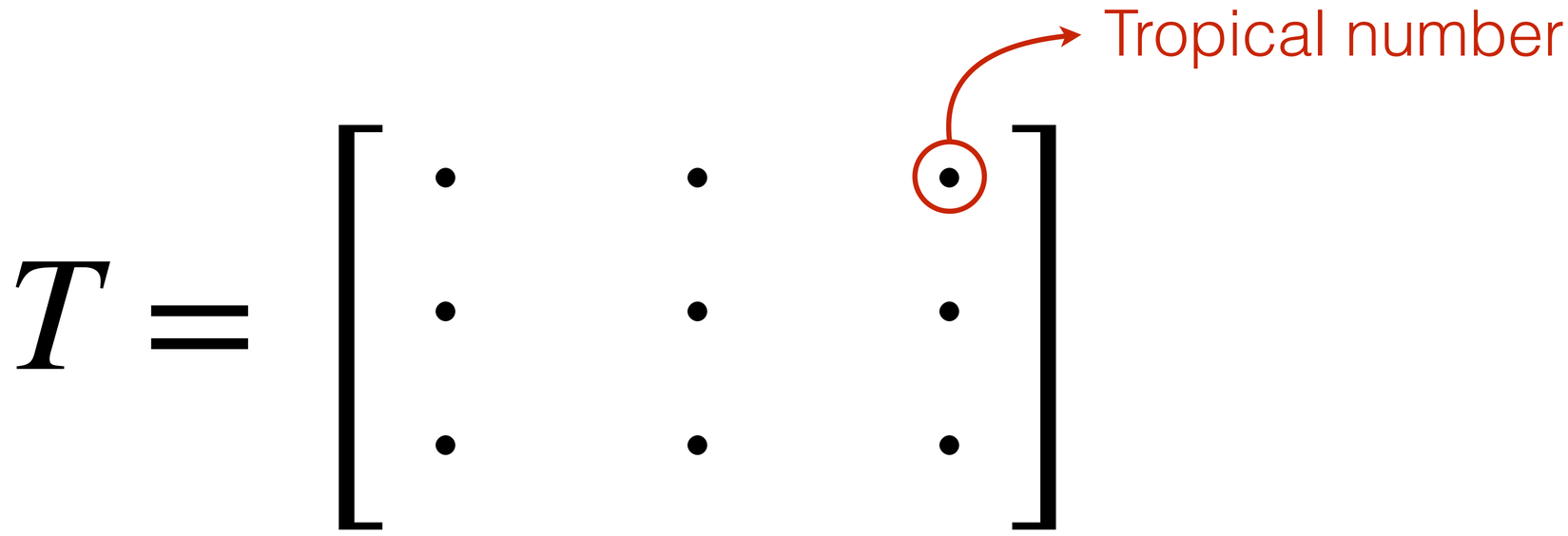
Multiplication table

$\odot$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14

$$\infty \oplus x = x$$

$$\infty \odot x = \infty$$

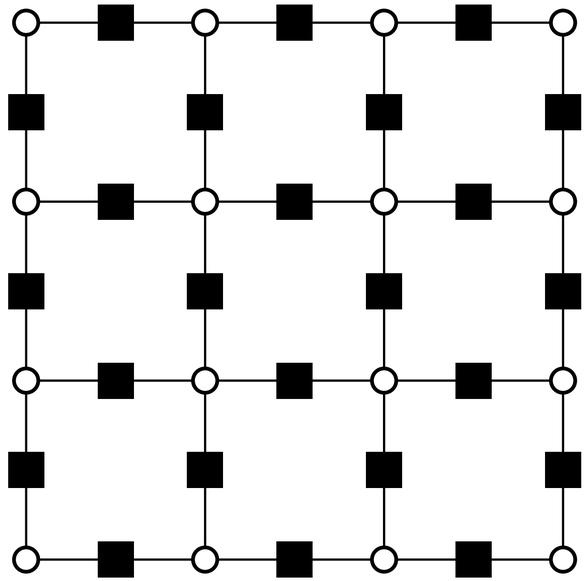
Tropical tensor networks



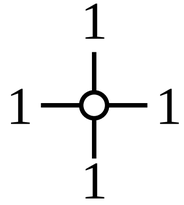
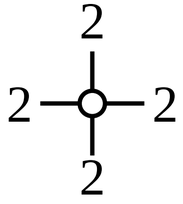
$(\oplus, \odot)$ semiring is sufficient to define tensor network contraction

Tropical tensor networks for Ising spin glasses

$$E(\{\sigma\}) = \sum_{i<j} J_{ij}\sigma_i\sigma_j + \sum_i h_i\sigma_i$$

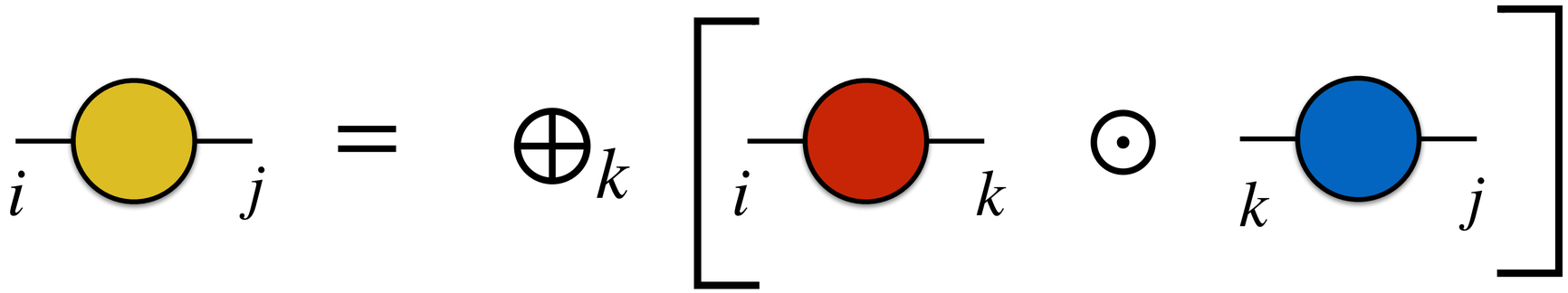
$E^* =$ 

 $= \begin{pmatrix} J_{ij} & -J_{ij} \\ -J_{ij} & J_{ij} \end{pmatrix}$

 $= h_i$  $= -h_i$

all other elements are ∞

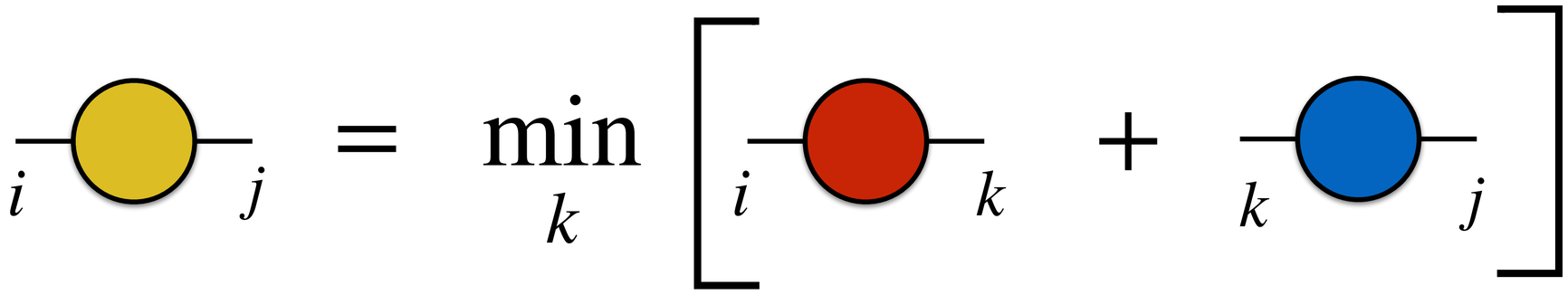
Tropical tensor network contraction
→ ground state energy **(value problem)**



Tropical tensor stores min-marginals of the energy

*also known as algebraic dynamic programming

Tropical tensor network contraction
→ ground state energy **(value problem)**



Tropical tensor stores min-marginals of the energy

*also known as algebraic dynamic programming

Physical understanding of the tropical algebra

zero temperature limit of the partition function

$$Z = \sum_{\{\sigma\}} e^{-\beta E} \qquad \text{consider} \quad E^* = \lim_{\beta\to\infty} \frac{-1}{\beta} \ln Z$$

a tensor network with ordinary algebra

$$\lim_{\beta\to\infty} \frac{-1}{\beta} \ln(e^{-\beta x} + e^{-\beta y}) = x \oplus y$$

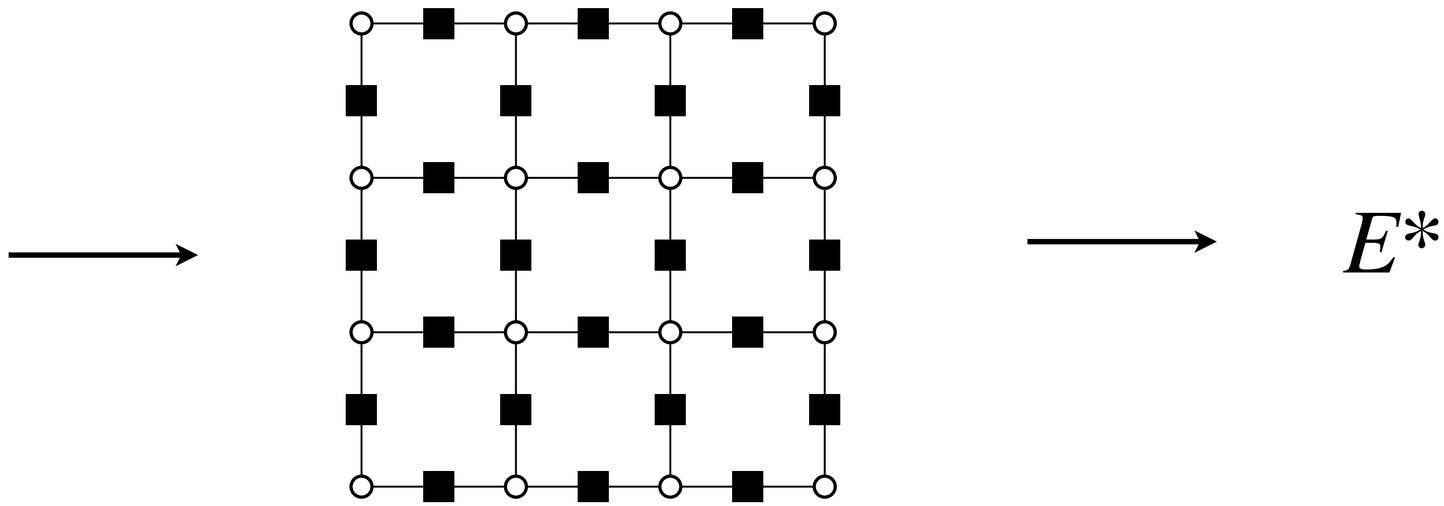
$$\lim_{\beta\to\infty} \frac{-1}{\beta} \ln(e^{-\beta x} \cdot e^{-\beta y}) = x \odot y$$

*also corresponds to log-number system, which avoids numerical issue at low temperature 1903.07721

Gradient with respect to the field → ground state configuration **(optimization problem)**

h_i

J_{ij}



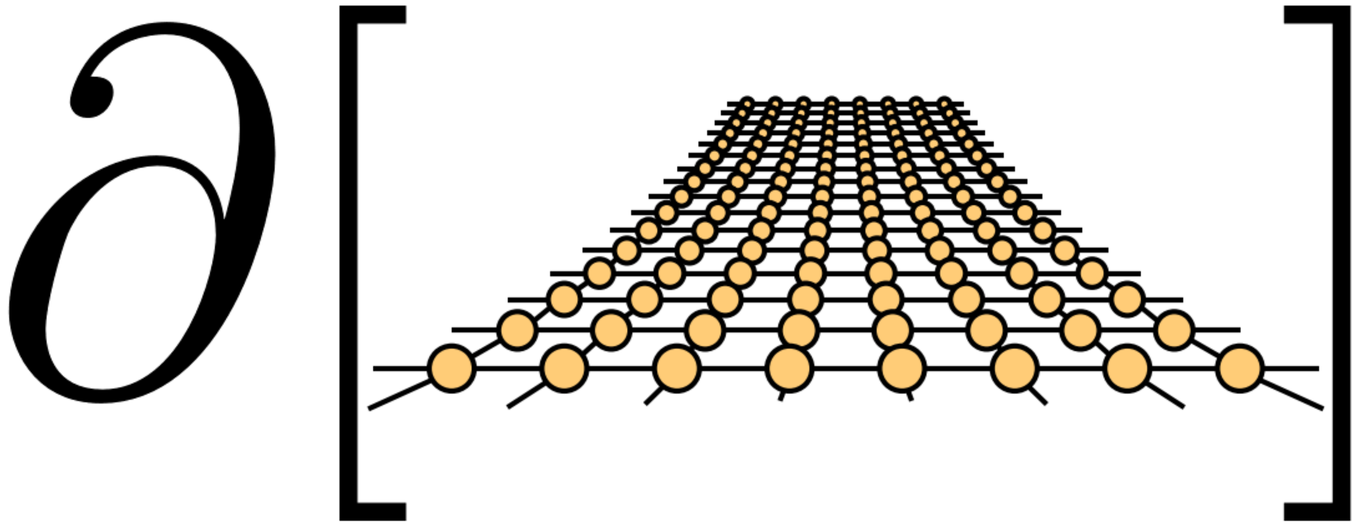
E^*

$$\sigma_i^* = \frac{\partial E^*}{\partial h_i}$$

This can be done with automatic differentiation through contraction

Interclude: Differentiable programming tensor networks

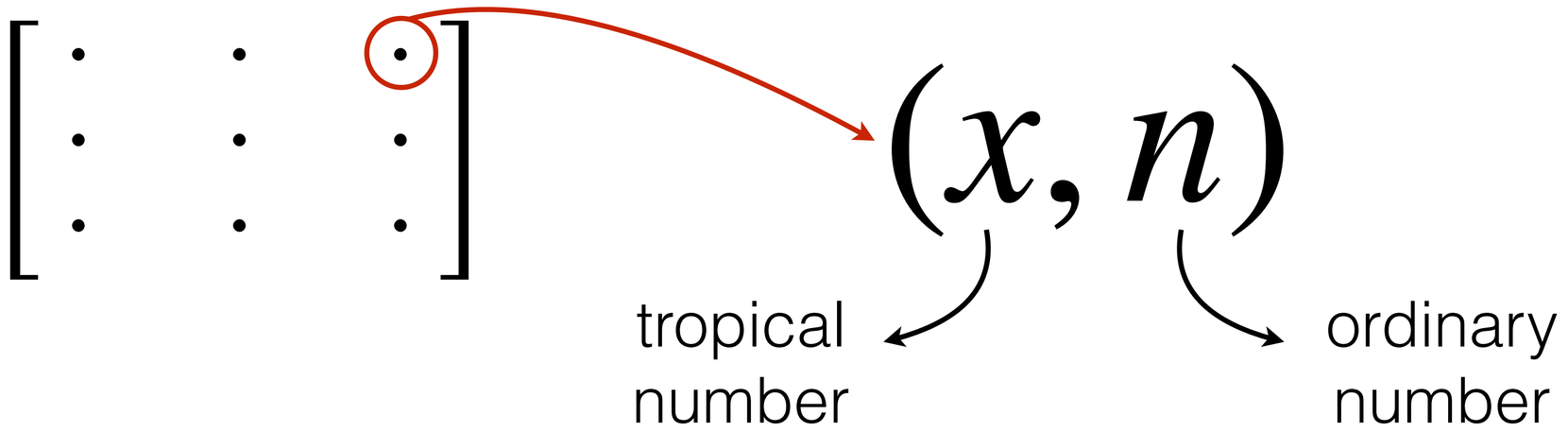
"training tensor networks as if they were neural networks"



Variational optimization Liao et al PRX '19, Hasik et al, SciPost '20

Extracting physical quantities (response function, scaling dimension,…) Tu et al, 2101.03935
Lyu et al, 2102.08136

Mix tropical with ordinary algebra →
ground state degeneracy **(counting problem)**

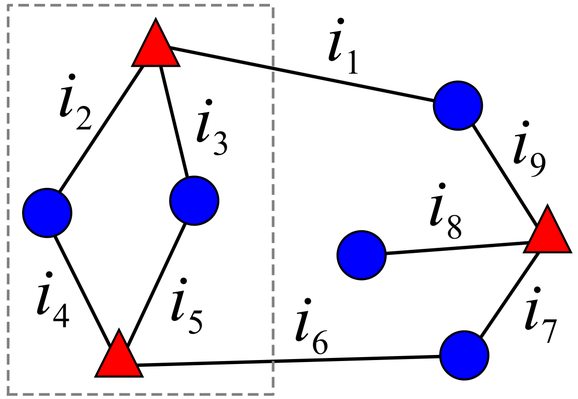


$$(x,n) \odot (y,m) = (?,?)$$
$$(x,n) \oplus (y,m) = (?,?)$$

The second field counts the minimal energy configurations

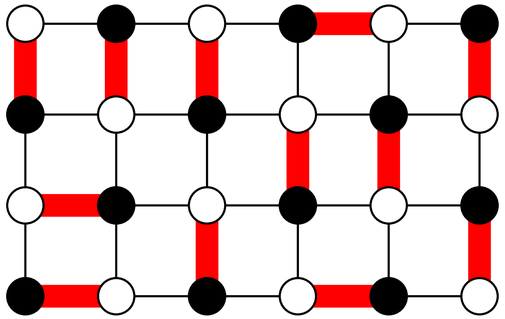
Counting with tensor networks

#SAT problems



García-Sáez et al, ‘12
Biamonte et al, ‘15
Kourtis et al ‘19

Residual entropy density
of infinite periodic system

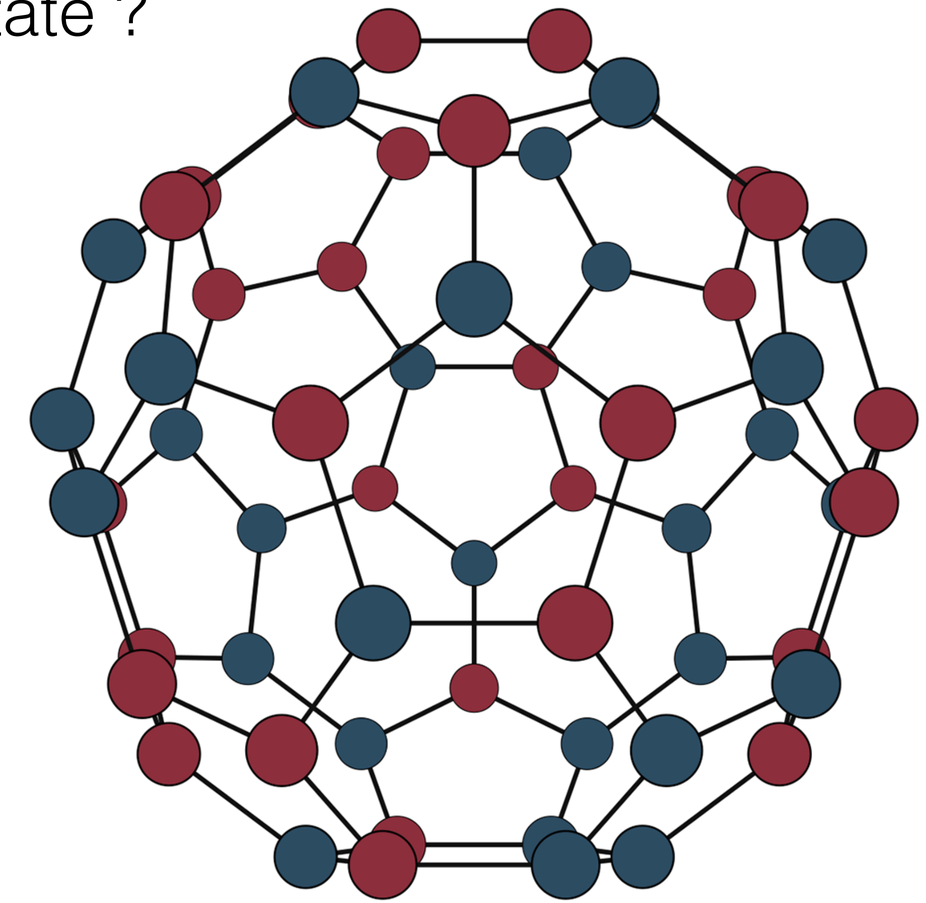


Vanderstraeten et al ’18
Vanhecke et al ‘21

Construct 0-1 tensor for # constraint satisfaction problems

Frustrated Ising model on a football

What is the ground state ?



How many are they ?

16,000

Tropical Tensor Networks

It is not conceptually new (tensor network reformulation of dynamic programming inference method in graphical models)

Exact tensor network contraction is still difficult in general (#P hard)

Nevertheless, it nicely leverages the engine of deep learning: differentiable tensors on GPU/TPU/…

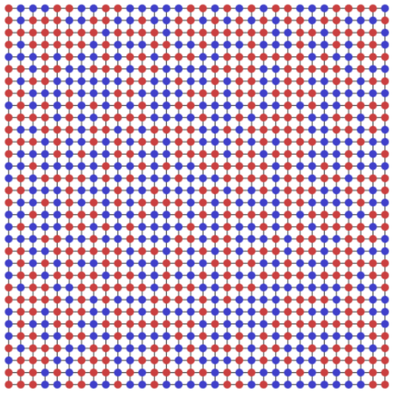




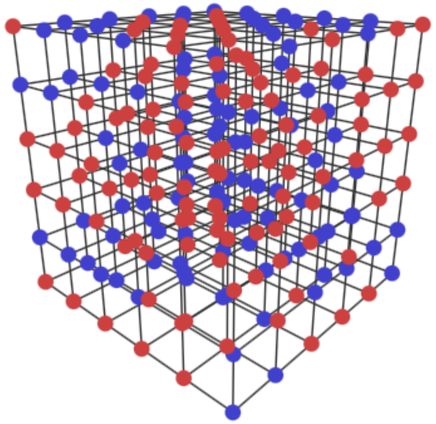
“Tropical BLAS”
TensorBFS/TropicalGEMM.jl

Exact computation on 1 Nvidia V100 GPU

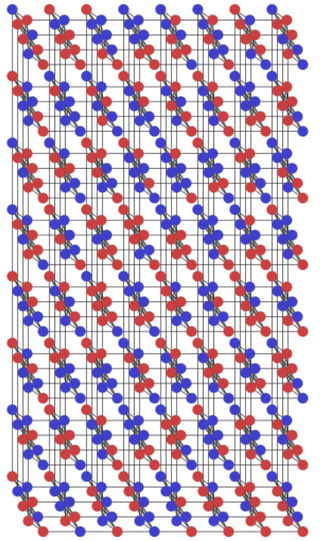
Ising spin glasses



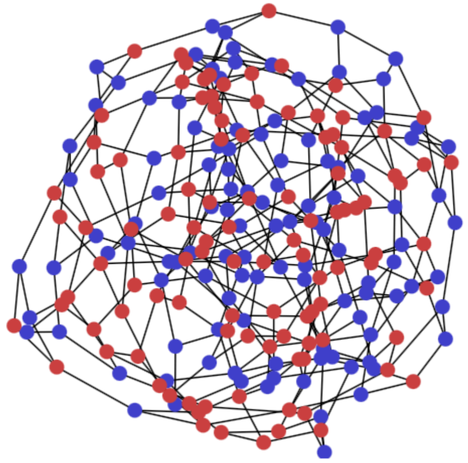
32^2 square



6^3 cube



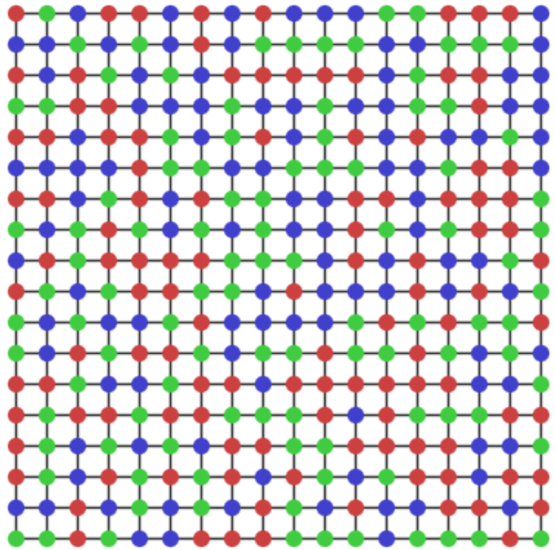
$8*8^2$ chimera graph



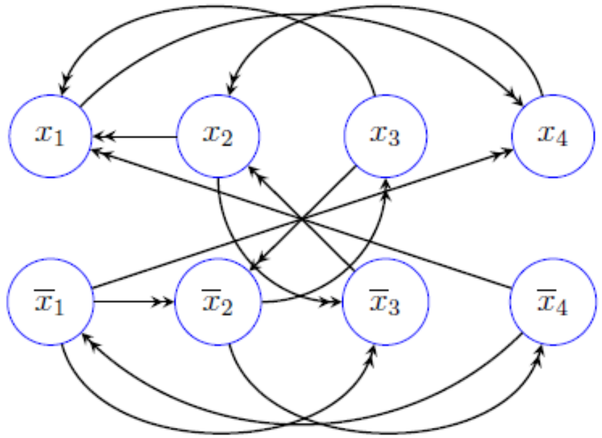
220 spin random 3-regular graph

More combinatorial optimization & counting problems

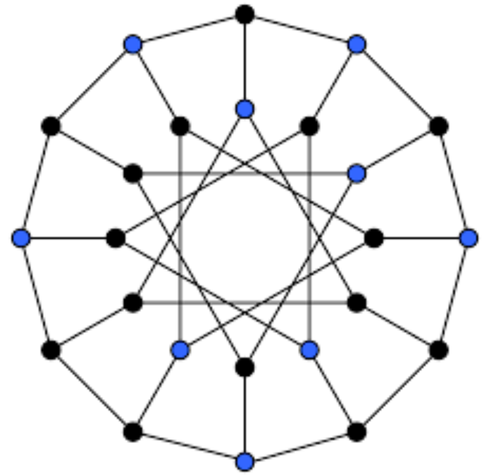
Potts glass



Max 2-SAT



Maximal independent set



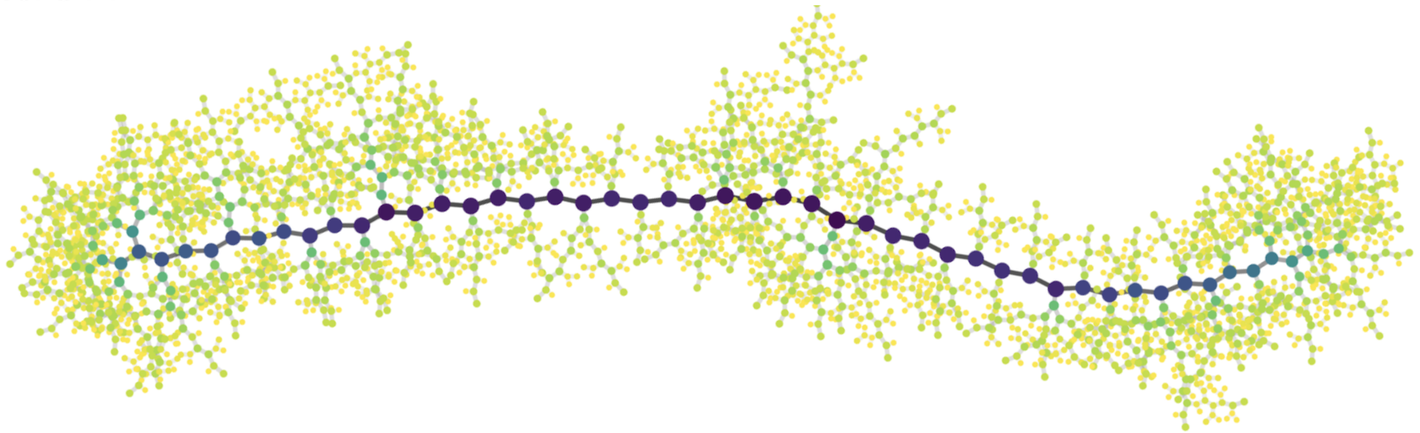
Construct local tropical tensors for the energy function, then contract

Tensor network contraction order

- Optimal contraction order is NP hard
- Many heuristics (talk by Johnnie Gray on Thursday April 1)

  …Gray et al 2002.01935, Huang et al 2005.06787, Pan el al 2103.03074…

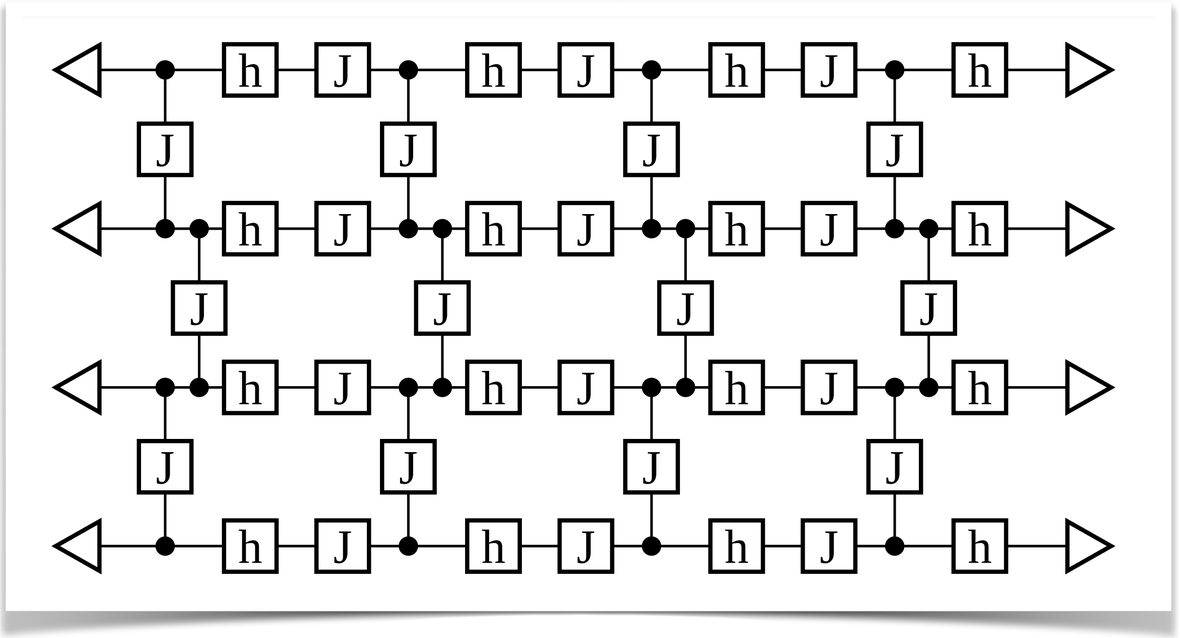
- Relevant to graphical model inference and quantum circuit simulation



Repurpose high-performance tensor network contraction engine for optimization

overload $\oplus$ and $\odot$ operations

Solve spin glass with a quantum circuit simulator




```
Warning: Input type of `ArrayReg` is not Complex, got Tropical{Float64}
```

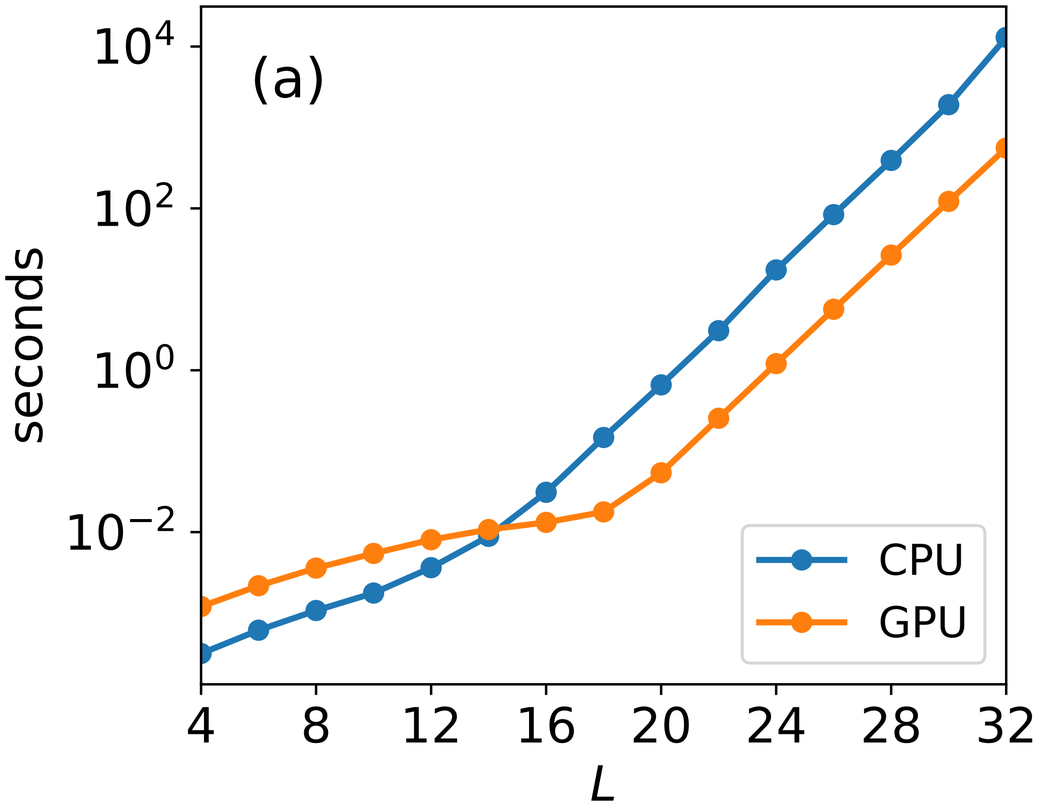
Contracting tropical tensor networks via simulating non-unitary circuits with unexpected algebra

https://yaoquantum.org/

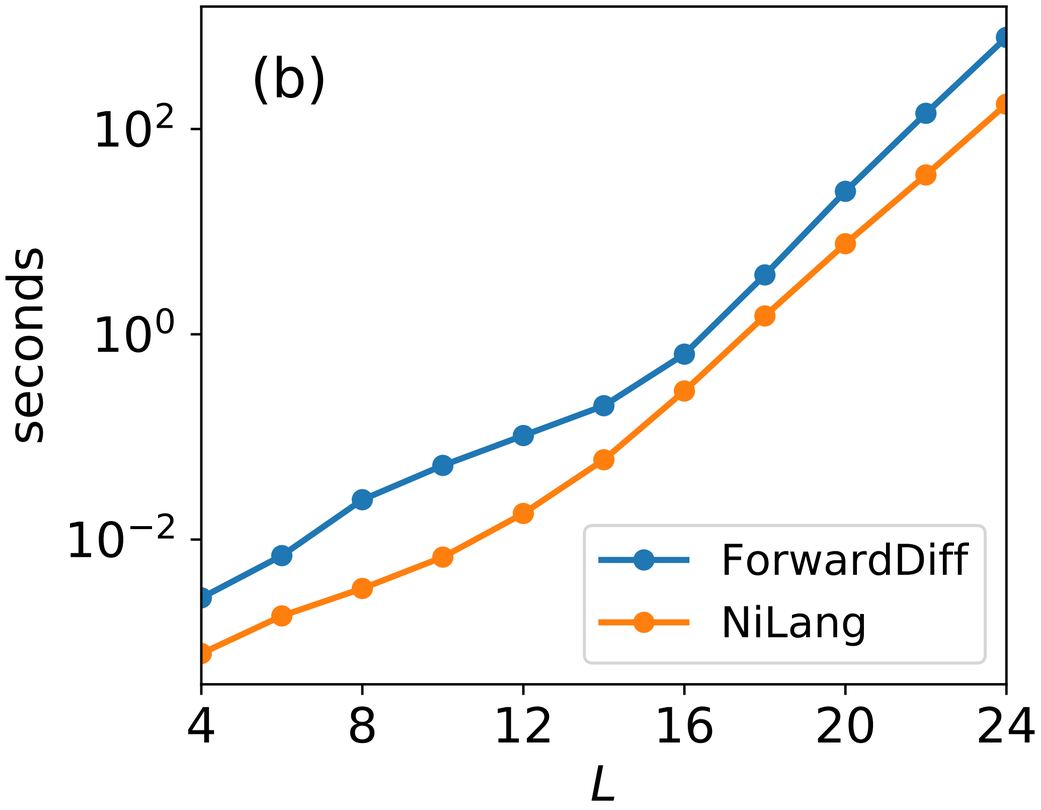
Luo, Liu, Zhang, LW, 1912.10877

Square lattice spin glasses

Time for ground state energy

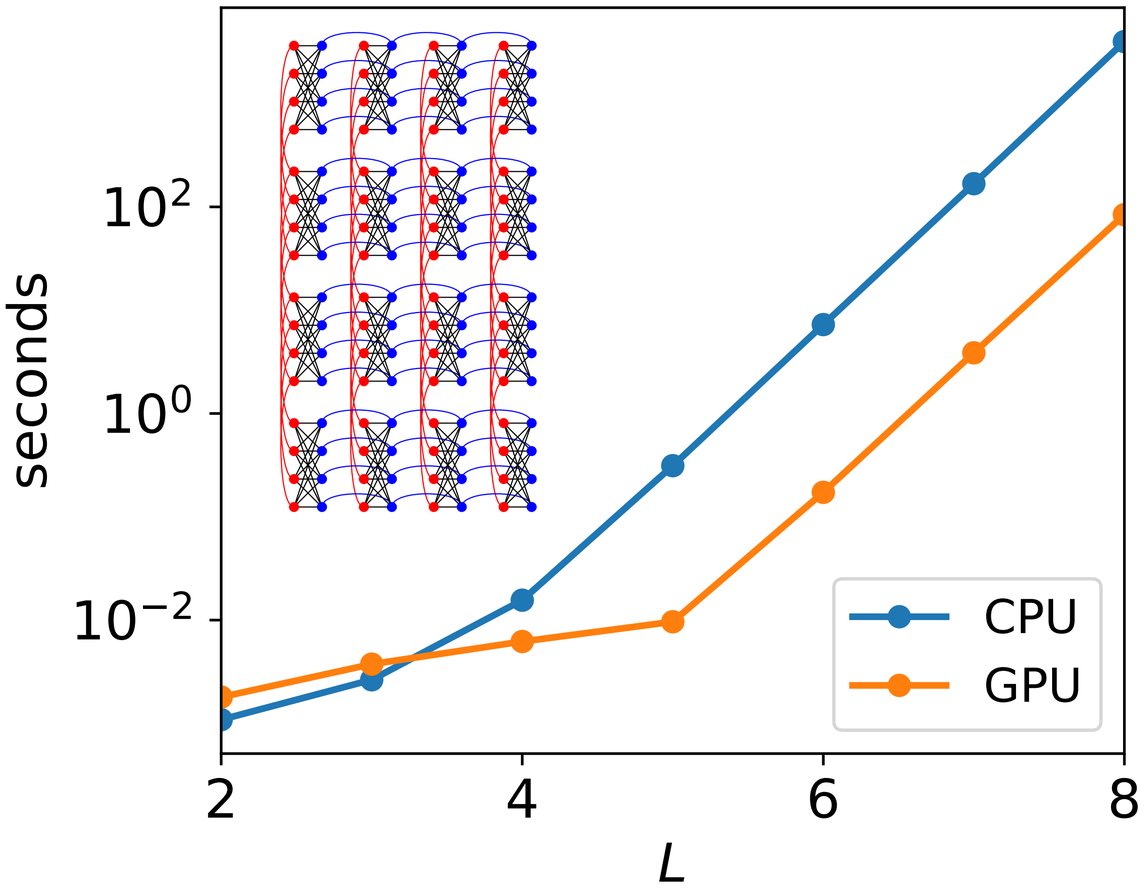


Time for ground state configuration

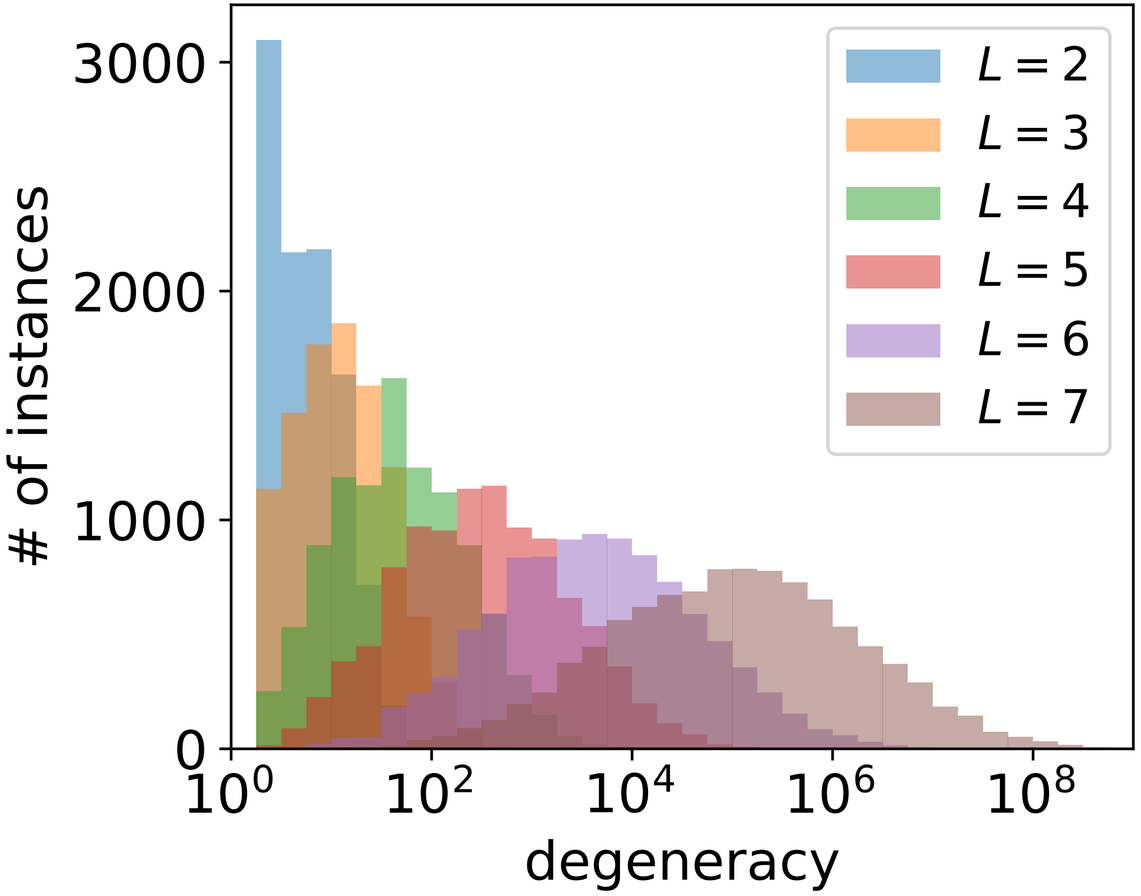


Chimera graph Ising spin glass

Time for ground state energy



Histogram of ground state degeneracy



Summary

Tensor network contraction for combinatorial optimization

Differentiable programming as a unified way to optimal solution

Generic design utilizes computing power for unexpected applications

WANTED: Approximated contraction under tropical algebra ?

Thank You!