What is a tensor? (Part II)

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goals of these tutorials

- very elementary introduction to tensors
- through the lens of linear algebra and numerical linear algebra
- highlight their roles in computations
- trace chronological development through three definitions
  ① a multi-indexed object that satisfies tensor transformation rules
  ② a multilinear map
  ③ an element of a tensor product of vector spaces
- last week: ① — trickiest one to discuss
- today: ② and ③ — much easier modern definitions

recap from part I

- take home idea: **a tensor is defined by its transformation rule**
- example: is this

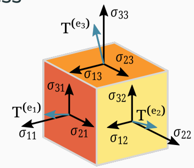
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

  a tensor?
- makes no sense
- definition ① requires a context

recap from part I

- if it comes from a measurement of stress

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



  then it is a contravariant 2-tensor

- if we are interested in its eigenvalues and eigenvectors

$$(XAX^{-1})Xv = \lambda Xv$$

  then it is a mixed 2-tensor

- if we are interested in its Hadamard product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix}$$

  then it is not a tensor

recap from part I

- indices tell us nothing
- $A \in \mathbb{R}^{m\times n}$ has two indices but if transformation rule is

$$A' = XA = [Xa_1, \dots, Xa_n] \quad \text{or} \quad A' = X^{-\mathsf{T}}A = [X^{-\mathsf{T}}a_1, \dots, X^{-\mathsf{T}}a_n]$$

  then it is covariant or contravariant 1-tensor respectively
- e.g., Householder QR algorithm, equivariant neural networks — all covariant 1-tensors no matter how many indices
- want coordinate-free definition that does not depend on indices

tensors via multilinear maps

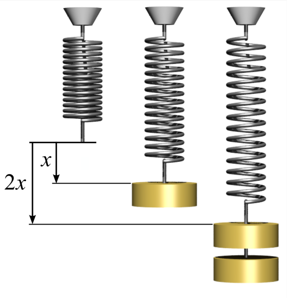
multilinear maps

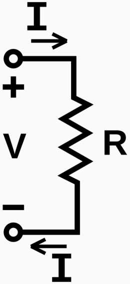
- $\mathbb{V}_1, \dots, \mathbb{V}_d$ and $\mathbb{W}$ vector spaces
- multilinear map or d-linear map is $\varphi : \mathbb{V}_1 \times \cdots \times \mathbb{V}_d \to \mathbb{W}$ with

$$\begin{aligned}\Phi(v, \dots, \lambda v_k + \lambda' v_k', \dots, v_d) \\ = \lambda \Phi(v_1, \dots, v_k, \dots, v_d) + \lambda' \Phi(v_1, \dots, v_k', \dots, v_d)\end{aligned}$$

  for $v_1 \in \mathbb{V}_1, \dots, v_k, v_k' \in \mathbb{V}_k, \dots, v_d \in \mathbb{V}_d$, $\lambda, \lambda' \in \mathbb{R}$
- write $\mathsf{M}^d(\mathbb{V}_1, \dots, \mathbb{V}_d; \mathbb{W})$ for set of all such maps
- write $\mathsf{M}^1(\mathbb{V}; \mathbb{W}) = \mathsf{L}(\mathbb{V}; \mathbb{W})$ for linear maps

why important





linearity principle: almost any natural process is linear in small amounts almost everywhere

multilinearity principle: if we keep all but one factors constant, the varying factor obeys principle of linearity

Hooke's and Ohm's laws both linear but not if we pass a current through the spring or stretch the resistor

$d = 1$

- $\mathscr{B} = \{v_1, \dots, v_m\}$ basis of $\mathbb{V}$, $\mathscr{B}^* = \{v_1^*, \dots, v_m^*\}$ dual basis
- any $v \in \mathbb{V}$ uniquely represented by $a \in \mathbb{R}^m$

$$\mathbb{V} \ni v = a_1 v_1 + \cdots + a_m v_m \quad \longleftrightarrow \quad [v]_{\mathscr{B}} := \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^m$$

- any $\varphi \in \mathbb{V}^*$ uniquely represented by $b \in \mathbb{R}^m$

$$\mathbb{V}^* \ni \varphi = b_1 v_1^* + \cdots + b_m v_m^* \quad \longleftrightarrow \quad [\varphi]_{\mathscr{B}^*} := \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

$d = 1$

- if $\mathscr{C} = \{v_1', \dots, v_m'\}$ another basis and $X \in \mathbb{R}^{m \times m}$ is

$$v_j' = \sum_{i=1}^{m} x_{ij} v_i$$

- change-of-basis theorem:

$$[v]_{\mathscr{C}} = X^{-1}[v]_{\mathscr{B}}, \qquad [\varphi]_{\mathscr{C}^*} = X^{\mathsf{T}}[\varphi]_{\mathscr{B}^*}$$

- recover transformation rules for contravariant and covariant 1-tensors

$$a' = X^{-1}a, \qquad b' = X^{\mathsf{T}}b$$

- vectors $\longleftrightarrow$ contravariant 1-tensors
- linear functionals $\longleftrightarrow$ covariant 1-tensors

$d = 2$

- bases $\mathscr{A} = \{u_1, \ldots, u_n\}$ on $\mathbb{U}$, $\mathscr{B} = \{v_1, \ldots, v_m\}$ on $\mathbb{V}$
- linear operator $\Phi : \mathbb{U} \to \mathbb{V}$ has matrix representation

$$[\Phi]_{\mathscr{A},\mathscr{B}} = A \in \mathbb{R}^{m \times n}$$

  where

$$\Phi(u_j) = \sum_{i=1}^{m} a_{ij} v_i$$

- new bases $\mathscr{A}'$ and $\mathscr{B}'$

$$[\Phi]_{\mathscr{A}',\mathscr{B}'} = A' \in \mathbb{R}^{m \times n}$$

- change-of-basis theorem: if $X \in \mathrm{GL}(m)$ change-of-basis matrix on $\mathbb{V}$, $Y \in \mathrm{GL}(n)$ change-of-basis matrix on $\mathbb{U}$, then

$$A' = X^{-1} A Y$$

$d = 2$

- special case $\mathbb{U} = \mathbb{V}$ with $\mathscr{A} = \mathscr{B}$ and $\mathscr{A}' = \mathscr{B}'$

$$A' = X^{-1} A X$$

- recover transformation rules for mixed 2-tensors
- bilinear functional $\beta : \mathbb{U} \times \mathbb{V} \to \mathbb{R}$ with

$$\beta(\lambda u + \lambda' u', v) = \lambda \beta(u, v) + \lambda' \beta(u', v),$$
$$\beta(u, \lambda v + \lambda' v') = \lambda \beta(u, v) + \lambda' \beta(u, v')$$

  for all $u, u' \in \mathbb{U}$, $v, v' \in \mathbb{V}$, $\lambda, \lambda' \in \mathbb{R}$
- matrix representation of β

$$[\beta]_{\mathscr{A}, \mathscr{B}} = A \in \mathbb{R}^{m \times n}$$

  given by

$$a_{ij} = \beta(u_i, v_j)$$

$d = 2$

- change-of-basis theorem: if

$$[\beta]_{\mathscr{A}',\mathscr{B}'} = A' \in \mathbb{R}^{m \times n},$$

  then

$$A' = X^{\mathsf{T}} A Y$$

- special case $\mathbb{U} = \mathbb{V}$, $\mathscr{A} = \mathscr{B}$, $\mathscr{A}' = \mathscr{B}'$

$$A' = X^{\mathsf{T}} A X$$

- recover transformation rules for covariant 2-tensors
- linear operators $\longleftrightarrow$ mixed 2-tensors
- bilinear functionals $\longleftrightarrow$ covariant 2-tensors

1- and 2-tensor transformation rules

contravariant 1-tensor:	$a' = X^{-1}a$	$a' = Xa$
covariant 1-tensor:	$a' = X^{\mathsf{T}}a$	$a' = X^{-\mathsf{T}}a$
covariant 2-tensor:	$A' = X^{\mathsf{T}}AX$	$A' = X^{-\mathsf{T}}AX^{-1}$
contravariant 2-tensor:	$A' = X^{-1}AX^{-\mathsf{T}}$	$A' = XAX^{\mathsf{T}}$
mixed 2-tensor:	$A' = X^{-1}AX$	$A' = XAX^{-1}$
contravariant 2-tensor:	$A' = X^{-1}AY^{-\mathsf{T}}$	$A' = XAY^{\mathsf{T}}$
covariant 2-tensor:	$A' = X^{\mathsf{T}}AY$	$A' = X^{-\mathsf{T}}AY^{-1}$
mixed 2-tensor:	$A' = X^{-1}AY$	$A' = XAY^{-1}$

$d = 3$

- bilinear operator $\mathrm{B} : \mathbb{U} \times \mathbb{V} \to \mathbb{W}$,

$$\mathrm{B}(\lambda u + \lambda' u', v) = \lambda \mathrm{B}(u, v) + \lambda' \mathrm{B}(u', v),$$
$$\mathrm{B}(u, \lambda v + \lambda' v') = \lambda \mathrm{B}(u, v) + \lambda' \mathrm{B}(u, v')$$

- bases $\mathscr{A} = \{u_1, \dots, u_n\}$, $\mathscr{B} = \{v_1, \dots, v_m\}$, $\mathscr{C} = \{w_1, \dots, w_p\}$,

$$\mathrm{B}(u_i, v_j) = \sum_{i=1}^{p} a_{ijk} w_k$$

- change-of-basis theorem: if

$$[\mathrm{B}]_{\mathscr{A},\mathscr{B},\mathscr{C}} = A \quad \text{and} \quad [\mathrm{B}]_{\mathscr{A}',\mathscr{B}',\mathscr{C}'} = A' \in \mathbb{R}^{m \times n \times p},$$

then

$$A' = (X^{\mathsf{T}}, Y^{\mathsf{T}}, Z^{-1}) \cdot A$$

$d = 3$

- trilinear functional $\tau : \mathbb{U} \times \mathbb{V} \times \mathbb{W} \to \mathbb{R}$,

$$\tau(\lambda u + \lambda' u', v, w) = \lambda\tau(u, v, w) + \lambda'\tau(u', v, w),$$
$$\tau(u, \lambda v + \lambda' v', w) = \lambda\tau(u, v, w) + \lambda'\tau(u, v', w),$$
$$\tau(u, v, \lambda w + \lambda' w') = \lambda\tau(u, v, w) + \lambda'\tau(u, v, w')$$

- bases $\mathscr{A} = \{u_1, \dots, u_n\}$, $\mathscr{B} = \{v_1, \dots, v_m\}$, $\mathscr{C} = \{w_1, \dots, w_p\}$,

$$\tau(u_i, v_j, w_k) = a_{ijk}$$

- change-of-basis theorem: if

$$[\tau]_{\mathscr{A},\mathscr{B},\mathscr{C}} = A \quad \text{and} \quad [\tau]_{\mathscr{A}',\mathscr{B}',\mathscr{C}'} = A' \in \mathbb{R}^{m\times n\times p},$$

then

$$A' = (X^{\mathsf{T}}, Y^{\mathsf{T}}, Z^{\mathsf{T}}) \cdot A$$

extends to arbitrary order

- bilinear operators $\longleftrightarrow$ mixed 3-tensor of covariant order 2 contravariant order 1
- trilinear functionals $\longleftrightarrow$ covariant 3-tensor
- recover all transformation rules in definition ①
  - covariant d-tensor:

$$A' = (X_1^{\mathsf{T}}, X_2^{\mathsf{T}}, \dots, X_d^{\mathsf{T}}) \cdot A$$

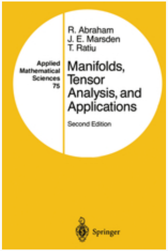
  - contravariant d-tensor:

$$A' = (X_1^{-1}, X_2^{-1}, \dots, X_d^{-1}) \cdot A$$

  - mixed d-tensor:

$$A' = (X_1^{-1}, \dots, X_p^{-1}, X_{p+1}^{\mathsf{T}}, \dots, X_d^{\mathsf{T}}) \cdot A$$

- definition ② is the easiest definition of a tensor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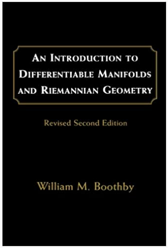
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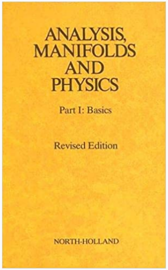
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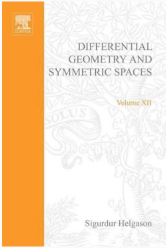
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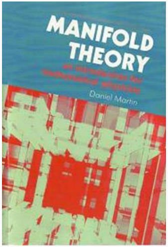
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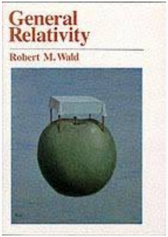
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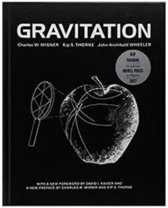
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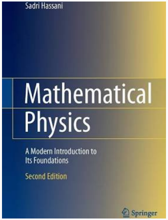
Sadri Hassani
Mathematical Physics
A Modern Introduction to Its Foundations
Second Edition
Springer

issue

- many more multilinear maps than there are types of tensors
- $d = 2$:
  - linear operators

$$\Phi : \mathbb{U}^* \to \mathbb{V}, \quad \Phi : \mathbb{U} \to \mathbb{V}^*, \quad \Phi : \mathbb{U}^* \to \mathbb{V}^*$$

  - bilinear functionals

$$\beta : \mathbb{U}^* \times \mathbb{V} \to \mathbb{R}, \quad \beta : \mathbb{U} \times \mathbb{V}^* \to \mathbb{R}, \quad \beta : \mathbb{U}^* \times \mathbb{V}^* \to \mathbb{R}$$

- $d = 3$:
  - bilinear operators

$$\mathrm{B} : \mathbb{U}^* \times \mathbb{V} \to \mathbb{W},\ \mathrm{B} : \mathbb{U} \times \mathbb{V}^* \to \mathbb{W}, \dots, \mathrm{B} : \mathbb{U}^* \times \mathbb{V}^* \to \mathbb{W}^*$$

  - trilinear functionals

$$\tau : \mathbb{U}^* \times \mathbb{V} \times \mathbb{W} \to \mathbb{R},\ \tau : \mathbb{U} \times \mathbb{V}^* \times \mathbb{W} \to \mathbb{R}, \dots, \tau : \mathbb{U}^* \times \mathbb{V}^* \times \mathbb{W}^* \to \mathbb{R}$$

  - more complicated maps

$$\Phi_1 : \mathbb{U} \to \mathsf{L}(\mathbb{V}; \mathbb{W}), \quad \Phi_2 : \mathsf{L}(\mathbb{U}; \mathbb{V}) \to \mathbb{W},$$
$$\beta_1 : \mathbb{U} \times \mathsf{L}(\mathbb{V}; \mathbb{W}) \to \mathbb{R}, \quad \beta_2 : \mathsf{L}(\mathbb{U}; \mathbb{V}) \times \mathbb{W} \to \mathbb{R}$$

issue

- possibilities increase exponentially with order d
- ought to be only as many as types of transformation rules

$$
\begin{aligned}
&\text{covariant 2-tensor:} && \Phi : \mathbb{U} \to \mathbb{V}^*, && \beta : \mathbb{U} \times \mathbb{V} \to \mathbb{R} \\
&\text{contravariant 2-tensor:} && \Phi : \mathbb{U}^* \to \mathbb{V}, && \beta : \mathbb{U}^* \times \mathbb{V}^* \to \mathbb{R} \\
&\text{mixed 2-tensor:} && \Phi : \mathbb{U} \to \mathbb{V}, && \beta : \mathbb{U} \times \mathbb{V}^* \to \mathbb{R}, \\
& && \Phi : \mathbb{U}^* \to \mathbb{V}^*, && \beta : \mathbb{U}^* \times \mathbb{V} \to \mathbb{R}
\end{aligned}
$$

- definition ③ accomplishes this without reference to the transformation rules

imperfect fix

- only allow $\mathbb{W} = \mathbb{R}$
- d-tensor of contravariant order p and covariant order $d - p$ is multilinear functional

$$\varphi : \mathbb{V}_1^* \times \cdots \times \mathbb{V}_p^* \times \mathbb{V}_{p+1} \times \cdots \times \mathbb{V}_d \to \mathbb{R}$$

- excludes vectors, by far the most common 1-tensor
- excludes linear operators, by far the most common 2-tensor
- excludes bilinear operators, by far the most common 3-tensor
- e.g., instead of talking about $v \in \mathbb{V}$, need to talk about linear functionals $f : \mathbb{V}^* \to \mathbb{R}$

multilinear operators are useful

- favorite example: higher derivatives of multivariate functions
- but need a norm on $\mathsf{M}^d(\mathbb{V}_1, \ldots, \mathbb{V}_d; \mathbb{W})$
- $\mathsf{M}^d(\mathbb{V}_1, \ldots, \mathbb{V}_d; \mathbb{W})$ is itself a vector space
- if $\mathbb{V}_1, \ldots, \mathbb{V}_d$ and $\mathbb{W}$ endowed with norms, then

$$\|\Phi\|_\sigma := \sup_{v_1, \ldots, v_d \neq 0} \frac{\|\Phi(v_1, \ldots, v_d)\|}{\|v_1\| \cdots \|v_d\|}$$

  defines a norm on $\mathsf{M}^d(\mathbb{V}_1, \ldots, \mathbb{V}_d; \mathbb{W})$
- slightly abused notation: same $\|\cdot\|$ denote norms on different spaces

higher-order derivatives

- $\mathbb{V}$, $\mathbb{W}$ normed spaces; $\Omega \subseteq \mathbb{V}$ open
- derivative of $f : \Omega \to \mathbb{W}$ at $v \in \Omega$ is linear operator $Df(v) : \mathbb{V} \to \mathbb{W}$,

$$\lim_{h \to 0} \frac{\|f(v+h) - f(v) - [Df(v)](h)\|}{\|h\|} = 0$$

- since $Df(v) \in \mathsf{L}(\mathbb{V}; \mathbb{W})$, apply same definition to $Df : \Omega \to \mathsf{L}(\mathbb{V}; \mathbb{W})$
- get $D^2 f(v) : \mathbb{V} \to \mathsf{L}(\mathbb{V}; \mathbb{W})$ as $D(Df)$,

$$\lim_{h \to 0} \frac{\|Df(v+h) - Df(v) - [D^2 f(v)](h)\|}{\|h\|} = 0$$

- apply recursively to get derivatives of arbitrary order

$$\begin{aligned} Df(v) &\in \mathsf{L}(\mathbb{V}; \mathbb{W}), & D^2 f(v) &\in \mathsf{L}\big(\mathbb{V}; \mathsf{L}(\mathbb{V}; \mathbb{W})\big), \\ D^3 f(v) &\in \mathsf{L}\big(\mathbb{V}; \mathsf{L}(\mathbb{V}; \mathsf{L}(\mathbb{V}; \mathbb{W}))\big), & D^4 f(v) &\in \mathsf{L}\Big(\mathbb{V}; \mathsf{L}\big(\mathbb{V}; \mathsf{L}(\mathbb{V}; \mathsf{L}(\mathbb{V}; \mathbb{W}))\big)\Big) \end{aligned}$$

higher-order derivatives

- how to avoid nested spaces of linear maps?
- use multilinear maps

$$\mathsf{L}\big(\mathbb{V};\mathsf{M}^{d-1}(\mathbb{V},\ldots,\mathbb{V};\mathbb{W})\big) = \mathsf{M}^{d}(\mathbb{V},\ldots,\mathbb{V};\mathbb{W})$$

- if $\Phi : \mathbb{V} \to \mathsf{M}^{d}(\mathbb{V},\ldots,\mathbb{V};\mathbb{W})$ linear, then

$$[\Phi(h)](h_1,\ldots,h_d)$$

  linear in h for fixed $h_1,\ldots,h_d$, d-linear in $h_1,\ldots,h_d$ for fixed h
- $D^d f(v) : \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{W}$ may be regarded as multilinear operator
- Taylor's theorem

$$f(v+h) = f(v) + [Df(v)](h) + \frac{1}{2}[D^2 f(v)](h,h) + \cdots$$
$$\cdots + \frac{1}{d!}[D^d f(v)](h,\ldots,h) + R(h)$$

  remainder $\|R(h)\|/\|h\|^d \to 0$ as $h \to 0$

why not hypermatrices

- why not choose bases and just treat them as hypermatrices $A \in \mathbb{R}^{n_1 \times \cdots \times n_d}$?
- may not be computable: writing down a hypermatrix given bases in general #P-hard
- may not be possible: multilinear maps extend to modules, which may not have bases
- may not be useful: even for $d = 1, 2$, writing down matrix unhelpful when vector spaces have special structures
- may not be meaningful: ‘indices’ may be continuous

writing down hypermatrix is #P-hard

- $0 \le d_1 \le d_2 \le \cdots \le d_n$, generalized Vandermonde matrix

$$V_{(d_1,\dots,d_n)}(x) := \begin{bmatrix} x_1^{d_1} & x_2^{d_1} & \dots & x_n^{d_1} \\ x_1^{d_2} & x_2^{d_2} & \dots & x_n^{d_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{d_{n-1}} & x_2^{d_{n-1}} & \dots & x_n^{d_{n-1}} \\ x_1^{d_n} & x_2^{d_n} & \dots & x_n^{d_n} \end{bmatrix}$$

- usual Vandermonde matrix

$$V_{(0,1,\dots,n-1)}(x) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}$$

- if $d_i \ge i$, then $\det V_{(d_1,d_2,\dots,d_n)}(x)$ divisible by $\det V_{(0,1,\dots,n-1)}(x)$

writing down hypermatrix is #P-hard

- for any integers $0 \le p_1 \le p_2 \le \cdots \le p_n$,

$$s_{(p_1,p_2,\ldots,p_n)}(x) := \frac{\det V_{(p_1,p_2+1,\ldots,p_n+n-1)}(x)}{\det V_{(0,1,\ldots,n-1)}(x)}$$

  is a symmetric polynomial in the variables $x_1,\ldots,x_n$

$$s(x_1,x_2,\ldots,x_n) = s\bigl(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}\bigr)$$

- $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$ vector spaces of symmetric polynomials of degrees d, d', and $d+d'$ respectively
- $\mathrm{B} : \mathbb{U} \times \mathbb{V} \to \mathbb{W}$ bilinear operator given by

$$\mathrm{B}\bigl(s(x),t(x)\bigr) = s(x)t(x)$$

  for $s(x)$ of degree d and $t(x)$ of degree d'

writing down hypermatrix is #P-hard

- $\mathscr{A}$ be basis of $\mathbb{U}$ given by

$$\left\{s_{(p_1,p_2,\dots,p_n)}(x) \in \mathbb{U} : p_1 \le p_2 \le \cdots \le p_n \text{ integer partition of } d\right\}$$

- $\mathscr{B}$, $\mathscr{C}$ similar bases for $\mathbb{V}$, $\mathbb{W}$
- B may in principle be written down as 3-dimensional hypermatrix

$$[\mathrm{B}]_{\mathscr{A},\mathscr{B},\mathscr{C}} = A \in \mathbb{R}^{d \times d' \times (d+d')}$$

with

$$\mathrm{B}(u_i, v_j) = \sum_{i=1}^{p} a_{ijk} w_k$$

- a_{ijk}'s are Littlewood–Richardson coefficients, well-known to be #P-complete [Narayanan, 2006]
- linear algebra significance: Horn's conjecture on eigenvalues of sums of Hermitian matrices [Klyachko 1998; Knutson–Tao, 1999]

multilinear maps in computations

higher derivatives of log det

- barrier function for positive definite cone $\mathbb{S}^n_{++}$

$$f : \mathbb{S}^n_{++} \to \mathbb{R}, \qquad f(X) = -\log\det X$$

- gradient is

$$\nabla f : \mathbb{S}^n_{++} \to \mathbb{S}^n, \qquad \nabla f(X) = -X^{-1}$$

- Hessian, i.e., $\nabla^2 f \coloneqq D(\nabla f)$, at any $X \in \mathbb{S}^n_{++}$ is linear operator

$$\nabla^2 f(X) : \mathbb{S}^n \to \mathbb{S}^n, \qquad H \mapsto X^{-1} H X^{-1}$$

- standard formulas useless

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}, \qquad \nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial f}{\partial x_n^2} \end{bmatrix}$$

higher derivatives of log det

- writing down (hyper)matrix representations of (multi)linear maps may not be useful even when $d = 1, 2$
- in SDP, vector space is $\mathbb{S}^n$, $n \times n$ real symmetric matrices
- bad idea to identify it with $\mathbb{R}^{n(n+1)/2}$
- what about higher derivatives of $f(X) = -\log\det X$?
- formulas like

$$\nabla^d f = \left[\frac{\partial^d f}{\partial x_i \partial x_j \cdots \partial x_k} \right]_{i,j,\dots,k=1}^n$$

  even less illuminating
- need to view them as multilinear maps

higher derivatives of log det

- write $F = \nabla f$, i.e., $F(X) = -X^{-1}$
- then $DF(X) = \nabla^2 f(X)$ and now we want $D^2 F(X)$
- by earlier discussion, this is bilinear operator

$$D^2 F(X) : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n$$

- not hard to show that it is given by

$$(H_1, H_2) \mapsto -X^{-1} H_1 X^{-1} H_2 X^{-1} - X^{-1} H_2 X^{-1} H_1 X^{-1}.$$

- dth derivative is d-linear operator

$$D^d F(X) : \mathbb{S}^n \times \cdots \times \mathbb{S}^n \to \mathbb{S}^n$$

  that sends $(H_1, H_2, \ldots, H_d)$ to

$$(-1)^{d+1} \sum_{\sigma \in \mathfrak{S}_d} X^{-1} H_{\sigma(1)} X^{-1} H_{\sigma(2)} X^{-1} \cdots X^{-1} H_{\sigma(d)} X^{-1}$$

self-concordance

- need third derivative to check self-concordance
- convex $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ self-concordant at $x \in \Omega$ if

$$\left|\nabla^3 f(x)(h, h, h)\right| \leq 2\sigma \left|\nabla^2 f(x)(h, h)\right|^{3/2}$$

  for all $h \in \mathbb{R}^n$ [Nesterov–Nemirovskii, 1994]
- convex programming problem may be solved to ε-accuracy in polynomial time if it has self-concordant barrier functions,
- e.g., LP, QP, SOCP, SDP, GP

- not useful when vector space is not $\mathbb{R}^n$:

$$\nabla^2 f(x)(h,h) = \sum_{i,j=1}^{n} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} h_i h_j,$$
$$\nabla^3 f(x)(h,h,h) = \sum_{i,j,k=1}^{n} \frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k$$

- to check that $f(X) = -\log\det(X)$ is self-concordant, need to show

$$\left|\operatorname{tr}\left(H^\intercal[\nabla^3 f(X)](H,H)\right)\right| \le 2\sigma \left|\operatorname{tr}\left(H^\intercal[\nabla^2 f(X)](H)\right)\right|^{3/2}$$

- easy with multilinear map formulas

$$[\nabla^2 f(X)](H) = X^{-1} H X^{-1},$$
$$[\nabla^3 f(X)](H,H) = -2X^{-1} H X^{-1} H X^{-1}$$

bilinear complexity

- given $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$, how to construct a bilinear operator

$$\mathrm{B} : \mathbb{U} \times \mathbb{V} \to \mathbb{W}?$$

- take linear functional $\varphi : \mathbb{U} \to \mathbb{R}$, linear functional $\psi : \mathbb{V} \to \mathbb{R}$, vector $w \in \mathbb{W}$, define

$$\mathrm{B}(u, v) = \varphi(u)\psi(v)w$$

for any $u \in \mathbb{U}$, $v \in \mathbb{V}$

- evaluating B requires exactly one multiplication of variables

bilinear complexity

- for example $\mathbb{U} = \mathbb{V} = \mathbb{W} = \mathbb{R}^3$ with

$$\varphi(u) = u_1 + 2u_2 + 3u_3,$$
$$\psi(v) = 2v_1 + 3v_2 + 4v_3,$$
$$w = (3, 4, 5)$$

  then

$$\mathrm{B}(u, v) = \begin{bmatrix} 3(u_1 + 2u_2 + 3u_3)(2v_1 + 3v_2 + 4v_3) \\ 4(u_1 + 2u_2 + 3u_3)(2v_1 + 3v_2 + 4v_3) \\ 5(u_1 + 2u_2 + 3u_3)(2v_1 + 3v_2 + 4v_3) \end{bmatrix}$$

- multiplications like $2u_2$ or $4v_3$ are all scalar multiplications, i.e., one of the factors is a constant
- only variable multiplication like $(u_1 + 2u_2 + 3u_3)(2v_1 + 3v_2 + 4v_3)$ counts

bilinear complexity

- this is the notion of bilinear complexity [Strassen, 1987]
- once we fixed φ, ψ, w, evaluation of these can be hardwired or hardcoded
- e.g., discrete Fourier transform

$$\begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \\ \vdots \\ x'_{n-1} \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

- use FFT to evaluate $\varphi(x) = \sum_{j=0}^{n-1} \omega^{jk} x_j$
- bilinear complexity of FFT is zero

bilinear complexity

- may often bound number of additions and scalar multiplications in terms of number of variable multiplications
- e.g., if an algorithm takes n^p variable multiplications, may show that it takes at most $10n^p$ additions and scalar multiplications
- so algorithm still $O(n^p)$ even if we count all arithmetic operations
- most importantly, **bilinear complexity = tensor rank**

$$\operatorname{rank}(\mathrm{B}) = \min\left\{ r : \mathrm{B}(u,v) = \sum_{i=1}^{r} \varphi_i(u)\psi_i(v)w_i \right\}$$

- if only need $\mathrm{B}(u,v)$ up to ε-accuracy, **border rank**

$$\overline{\operatorname{rank}}(\mathrm{B}) = \min\left\{ r : \mathrm{B}(u,v) = \lim_{\varepsilon \to 0^+} \sum_{i=1}^{r} \varphi_i^\varepsilon(u)\psi_i^\varepsilon(v)w_i^\varepsilon \right\}$$

- due to [Strassen, 1973] and [Bini–Lotti–Romani, 1980] respectively

Gauss's algorithm

- complex multiplication with three real multiplications

$$
\begin{aligned}
(a+bi)(c+di) &= (ac-bd)+i(bc+ad) \\
&= (ac-bd)+i[(a+b)(c+d)-ac-bd]
\end{aligned}
$$

- $\mathrm{B} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $(z, w) \mapsto zw$ is $\mathbb{R}$-bilinear

$$
\mathrm{B} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathrm{B}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) = \begin{bmatrix} ac-bd \\ bc+ad \end{bmatrix}
$$

- usual:

$$
\mathrm{B}(z,w) = [e_1^*(z)e_1^*(w) - e_2^*(z)e_2^*(w)]e_1 + [e_1^*(z)e_2^*(w) + e_2^*(z)e_1^*(w)]e_2
$$

- Gauss:

$$
\begin{aligned}
\mathrm{B}(z,w) = {} & [(e_1^* + e_2^*)(z)(e_1^* + e_2^*)(w)]e_2 \\
& + [e_1^*(z)e_1^*(w)](e_1 - e_2) - [e_2^*(z)e_2^*(w)](e_1 + e_2)
\end{aligned}
$$

Gauss's algorithm

- Gauss optimal in both exact and approximate sense:

$$\operatorname{rank}(\mathrm{B}) = 3 = \overline{\operatorname{rank}}(\mathrm{B})$$

- why useful?
- complex matrix multiplication:

$$(A + iB)(C + iD) = (AC - BD) + i[(A + B)(C + D) - AC - BD]$$

for $A + iB, C + iD \in \mathbb{C}^{n\times n}$ with $A, B, C, D \in \mathbb{R}^{n\times n}$

- which is why we should allow for modules
    - $\mathbb{C}$ two-dimensional vector space over $\mathbb{R}$
    - $\mathbb{C}^{n\times n}$ two-dimensional free module over $\mathbb{R}^{n\times n}$

other simple example?

- Gauss essentially the only one in two dimensions
- parallel evaluation of standard inner product and standard symplectic form on $\mathbb{R}^2$

$$g(x, y) = x_1 y_1 + x_2 y_2 \quad \text{and} \quad \omega(x, y) = x_1 y_2 - x_2 y_1.$$

- algorithm similar to Gauss's gives result with $\operatorname{rank}(\mathrm{B}) = 3 = \overline{\operatorname{rank}}(\mathrm{B})$
- three dimensions: skew-symmetric matrix-vector product

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ay + bz \\ -ax + cz \\ -bx - cy \end{bmatrix}$$

- in this case[1] $\operatorname{rank}(\mathrm{B}) = 5 = \overline{\operatorname{rank}}(\mathrm{B})$

[1] thanks to J. M. Landsberg (for $\mathbb{C}$) and Visu Makam (for $\mathbb{R}$)

Strassen's algorithm

- 2×2 matrix multiplication with seven multiplications

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 b_1 + a_2 b_2 & \beta + \gamma + (a_1 + a_2 - a_3 - a_4) b_4 \\ \alpha + \gamma + a_4 (b_2 + b_3 - b_1 - b_4) & \alpha + \beta + \gamma \end{bmatrix}$$

with

$$\alpha = (a_3 - a_1)(b_3 - b_4),\ \beta = (a_3 + a_4)(b_3 - b_1),\ \gamma = a_1 b_1 + (a_3 + a_4 - a_1)(b_1 + b_4 - b_3)$$

- consequence: inverting $n \times n$ matrix in $5.64 n^{\log_2 7}$ arithmetic operations (both additions and multiplications) [Strassen, 1969]
- huge surprise as there were results showing $n^3/3$ required by Gaussian elimination cannot be improved
- such results assume row and column operations, Strassen used block operations
- $\operatorname{rank}(\mathrm{B}) = 7 = \overline{\operatorname{rank}}(\mathrm{B})$ [Landsberg, 2006]

matrix multiplication tensor

- bilinear operator

$$\mu_{m,n,p} : \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times p} \to \mathbb{R}^{n\times p}, \quad (A, B) \mapsto AB$$

  called matrix multiplication tensor

- exponent of matrix multiplication is

$$\omega := \inf\{p \in \mathbb{R} : \operatorname{rank}(\mu_{n,n,n}) = O(n^p)\}$$

- current bound $\omega < 2.3728596$ [Alman–Vassilevska Williams, 2021]
- ω underlies nearly every problem in numerical linear algebra

exponent of matrix multiplication

- inversion: given $A \in \mathrm{GL}(n)$, find $A^{-1} \in \mathrm{GL}(n)$
- determinant: given $A \in \mathrm{GL}(n)$, find $\det(A) \in \mathbb{R}$
- null basis: given $A \in \mathbb{R}^{n \times n}$, find a basis $v_1, \ldots, v_m \in \mathbb{R}^n$ of $\ker(A)$
- linear system: given $A \in \mathrm{GL}(n)$ and $b \in \mathbb{R}^n$, find $v \in \mathbb{R}^n$ so that $Av = b$
- LU decomposition: given $A \in \mathbb{R}^{m \times n}$ of full rank, find permutation P, unit lower triangular $L \in \mathbb{R}^{m \times m}$, upper triangular $U \in \mathbb{R}^{m \times n}$ so that $PA = LU$
- QR decomposition: given $A \in \mathbb{R}^{n \times n}$, find orthogonal $Q \in \mathrm{O}(n)$, upper triangular $U \in \mathbb{R}^{n \times n}$ so that $A = QR$

exponent of matrix multiplication

- eigenvalue decomposition: given $A \in \mathbb{S}^n$, find $Q \in \mathrm{O}(n)$ and diagonal $\Lambda \in \mathbb{R}^{n \times n}$ so that $A = Q\Lambda Q^{\mathsf{T}}$
- Hessenberg decomposition: given $A \in \mathbb{R}^{n \times n}$, find $Q \in \mathrm{O}(n)$ and upper Hessenberg $H \in \mathbb{R}^{n \times n}$ so that $A = QHQ^{\mathsf{T}}$
- characteristic polynomial: given $A \in \mathbb{R}^{n \times n}$, find $(a_0, \dots, a_{n-1}) \in \mathbb{R}^n$ so that $\det(xI - A) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
- sparsification: given $A \in \mathbb{R}^{n \times n}$ and $c \in [1, \infty)$, find $X, Y \in \mathrm{GL}(n)$ so that $\mathrm{nnz}(XAY^{-1}) \le cn$

exponent of nearly all matrix computations

any $\varepsilon > 0$, there is an algorithm for each of these problems in $O(n^{\omega+\varepsilon})$ arithmetic operations (including additions and scalar multiplications)

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