Geometry of configurations of points and symmetric rank
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Abstract: Decompositions of symmetric tensors can be viewed as sets of points in a projective space. The geometry of sets of this type is usually studied in terms of a resolution of the associated homogeneous ideal. I will illustrate how one can study problems like the minimality or the uniqueness of a given decomposition by means of algebraic invariants of the corresponding configuration of points.
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\[ T = \sum_{i=1}^{r} a_i T_i \]

\( T_i = L_i^d, \quad L_i \text{linear} \quad d = \deg(T), \quad r = \text{length}, \quad a_i \in \mathbb{C}. \]
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Coefficients \( a_i \)'s will become important later in the talk.
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- the expression is **non-redundant** if the $T_i$’s are independent and no $a_i$ is zero.
- the expression is **minimal** if there are no expressions of $T$ of smaller length. I.e. $r$ is the (symmetric) rank.
- the expression is **unique** if there are no other expressions of $T$ of length $\leq r$, except trivialities. In this case $T$ is **identifiable**.
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Introduction

Point of view

For symmetric tensors it is not hard to find \textbf{some} expression of $T$

$$T = \sum_{i=1}^{r} a_i T_i$$

It is hard to find a \textbf{minimal} decomposition, and/or to prove that it is \textbf{unique}.

- apolar equations;
- Strassen Additivity Problem for forms $T = T' \oplus T''$;
- matrix multiplication symmetrized.
The geometric (projective) setting

Most problems on tensors are invariant under rescaling (\(=\) multiplication by a non-zero scalar \(q\)).

\[
\text{rank}(T) = \text{rank}(qT) \quad \text{etc.}
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Advantages

- geometric insight on the problems;
- access to a huge set of tools from projective geometry.
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\[ T \Rightarrow [T] \in \mathbb{P}(\text{Sym}^d(\mathbb{C}^{n+1})). \]

Dictionary
The geometric (projective) setting

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**Dictionary**

- **linear form** \( L_i \)  \Rightarrow point \( P_i = [L_i] \in \mathbb{P}^n \)
- **power** \( L_i^d \)  \Rightarrow image of \( [L_i] \) in the Veronese map \( \nu_d : \mathbb{P}^n \to \mathbb{P}^N, N = \binom{n+d}{n} - 1 \)
- **expression** \( T = \sum_{i=1}^r a_i L_i^d \)  \Rightarrow \( T \in \text{linear span } \mathbb{P}^{r-1} \) of the \( \nu_d([L_i]) \)'s
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Definition A subset \( A = \{P_1, \ldots, P_r\} \subset \mathbb{P}^n \) is a (geometric) decomposition of \( T \) if

\[ [T] \in \langle v_d(P_1), \ldots, v_d(P_r) \rangle. \]
The geometric (projective) setting

\[ \mathbb{P}^n \to \mathbb{P}^N \]

Veronese map

Veronese variety

T
The geometric (projective) setting

**beware:** in general

\[
\left[ \sum_{i=1}^{r} a_i L_i^d \right] \neq \left[ \sum_{i=1}^{r} b_i L_i^d \right].
\]
Most celebrated criterion for uniqueness: **Kruskal’s criterion**.

**Kruskal’s rank of a matrix** $k(M) = \max \{ i : \text{all subsets of } i \text{ columns of } M \text{ are linearly independent} \}$.

**Kruskal's criterion (symmetric case)**

Let $T = \sum_{r_i=1} a_i L_i$. Let $M$ be the matrix whose $i$-th column is formed by the coefficients of $L_i$. If $r \leq dk(M) - d + 1$ then the expression is (minimal and) unique. (Original statement was for general 3-way tensors).
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Let $M$ be the matrix whose columns are given by the coefficients of the $L_i$’s, i.e. by projective coordinates for the points of $A = \{[L_1], \ldots, [L_r]\}$. 

Denote with $k_A$ the Kruskal’s rank of the matrix $M$ above.

$k_A = \max\{i : \text{all subsets of } A \text{ of cardinality } i \text{ are in Linear General Position (no three on a line, no four in a plane, ...)}\}$.

Certainly $k_A \leq n + 1$. 

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Geometry of configurations
The geometric (projective) setting

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**Bad.1** \( k_A \leq n + 1 \implies \) one cannot hope to apply the criterion when \( 2r > d(n + 1) - d + 1 \).
The geometric (projective) setting

Bad news for Kruskal’s criterion.

**Bad.1)** $k_A \leq n + 1 \Rightarrow$ one cannot hope to apply the criterion when $2r > d(n + 1) − d + 1$.

**Bad.2)** Kruskal’s criterion is sharp. One cannot hope to improve the previous range of application, by using $k_A$. (Derksen’s examples).
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What is so bad about the bad news?
Abstract secant variety (incidence variety):

$$A_{\sigma_r} = \{(T, A) : A \text{ is a decomposition for } T\}.$$ 

$$\dim(A_{\sigma_r}) = r(n + 1) - 1.$$
The geometric (projective) setting

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Consequence

No hope for uniqueness if \( r(n + 1) - 1 \geq N \), i.e. if

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The natural range for identifiability is $$r < r_g$$ (subgeneric rank)
The geometric (projective) setting

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**Theorem (Ottaviani-Vannieuwenhoven-C)**

If $r < r_g$ then a ‘generic Waring expression’ (in the Zariski sense) is minimal and unique, except for a finite, small list of values of $n, d$. 

Based on the works of Ciliberto-C, Ballico, Brambilla-Ottaviani, Ranestad-Voisin. 

The Kruskal’s range $r \leq (dn + 1)/2$ is much smaller than the range $r < r_g$ in which the uniqueness is expected (and known to hold generically).

Further analysis needed.
The geometric (projective) setting

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The geometry of Derksen’s example.

\[ \mathbb{P}^n \rightarrow \mathbb{P}^N \]

rational normal curve
The Game

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**Definition**

For a finite set $A \subset \mathcal{P}_n$ set:

- **Hilbert** $i$-th rank $h_A(i)$ = (affine) dimension of the linear span of $v_i(A)$;
- **Kruskal** $i$-th rank $k_A(i)$ = the Kruskal’s rank of $v_i(A)$.

**Reshaped Kruskal’s Theorem**

A decomposition $A$ of length $r$ is minimal and unique when $r \leq k_A(d_1) + k_A(d_2) + k_A(d_3) - 2$. 

*Not sharp!*

**Bad.** Even if the $k_A(d_i)$'s are maximal (= $d_i + n$), the range of application remains much lower than the range $r < r_g$.
The Game

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**Bad.3** Even if the \( k_A(d_i) \)'s are maximal \((= \binom{d_i+n}{n})\), the range of application remains much lower than the range \( r < r_g \).
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Definition

For a finite set $A \subset \mathbb{P}^n$ define

- **Hilbert function**: $i \mapsto h_A(i)$;

- **Kruskal function**: $i \mapsto k_A(i)$:

$(h_A(0) = k_A(0) = 1)$.

- **Difference Hilbert function**: $Dh_A(i) = h_A(i) - h_A(i - 1)$;

- **Difference Kruskal function**: $Dk_A(i) = k_A(i) - k_A(i - 1)$. 
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Difference Kruskal function: $Dk_A(i) = k_A(i) - k_A(i - 1)$.

If $A$ is general enough $h_A(i) = k_A(i)$.  

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The Game

Many properties are known for the Hilbert function $h_A$.

- $h_A$ is increasing from 1 to the maximum $r = \#A$,
- $\sum_{i=1}^{\infty} Dh_A(i) = r$;
- if $Dh_A(i) \leq i$ then $Dh_A(i+1) \leq Dh_A(i)$;
- Cayley-Bacharach properties $CB(i)$;
- ...
Most important properties.

Let $A, B$ be two decompositions of $T$. Set $Z = A \cup B$. 

- $h_Z(d) < \#Z$;
- $D_Z(d + 1) > 0$;
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Let $A, B$ be two decompositions of $T$. Set $Z = A \cup B$.

- $h_Z(d) < \#Z$;
- $D_Z(d + 1) > 0$;
- if $A, B$ are disjoint, then

$$\dim(\langle v_d(A) \rangle \cap \langle v_d(B) \rangle) = \#Z - h_Z(d) = \sum_{i=d+1}^{\infty} Dh_Z(i).$$

By means of properties of the Hilbert function and the Kruskal function of $A$, one can improve the Reshaped Kruskal’s Theorem.
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**Player 1** (You) provides a beautiful set \( A \) of \( r \) points, decomposition of \( T \);
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**Player 1** (You) provides a beautiful set $A$ of $r$ points, decomposition of $T$;

**Player 2** (the Rival) determines a set $B$ of cardinality $\leq r$, which gives for $Z = A \cup B$ the inequality $Dh_Z(d + 1) > 0$
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If the Rival cannot find $B$, you win (i.e. $A$ is minimal, unique, ...)

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Geometry of configurations

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A decomposition $A$ of cardinality $r$ of a **ternary form** $T$ of degree $d$ is minimal and unique if:

- $d = 2m$ is even, $k_A(m - 1) = \min\{(\frac{m+1}{2}), r\}$, $h_A(m) = r \leq \left(\frac{m+2}{2}\right) - 2$;
- $d = 2m + 1$ is odd, $k_A(m) = \min\{(\frac{m+2}{2}), r\}$, $h_A(m + 1) = r \leq \left(\frac{m+2}{2}\right) + \lfloor\frac{m}{2}\rfloor$.

The Game

Theorem

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There is a (non-sharp) version for forms in any number of variables.

Ottaviani-Vannieuwenoven-C, Angelini-C-Mazzon, Ballico, Mourrain-Oneto.
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Bad news.

**Bad.4** In any case we are still far (half-way, for ternary forms) from the expected range $r < r_g$. 
Theorem

A decomposition $A$ of cardinality $r$ of a ternary form $T$ of degree $d$ is minimal and unique if:

- $d = 2m$ is even, $k_A(m-1) = \min\{(m+1)/2, r\}$, $h_A(m) = r \leq (m+2)/2 - 2$;
- $d = 2m + 1$ is odd, $k_A(m) = \min\{(m+2)/2, r\}$, $h_A(m+1) = r \leq (m+2)/2 + \lfloor(m/2)\rfloor$.

Bad news.

**Bad.4** In any case we are still far (half-way, for ternary forms) from the expected range $r < r_g$.

(Even if the version for many variables is not sharp).
The fine tuning

The intrinsic weakness, that makes hopeless to cover the whole range with a similar analysis: The analysis only takes care of properties of the decomposition $A$. 

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The fine tuning

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Example

Forms of degree $d = 9$ in $n + 1 = 3$ variables, $r = 18(= r_g - 1)$. 

Veronese map $v_9$
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Veronese map $v_9$
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Forms of degree $d = 9$ in $n + 1 = 3$ variables, $r = 18(= r_g - 1)$. A general.
The fine tuning

Problem
Find a strategy to determine whether $T$ lies in
- the bad locus $W$ in which $A$ is not unique; or
- the bad locus $W'$ in which $A$ is not minimal.
The fine tuning

The strategy of fine tuning

- Step 1. Determine parameters of all possible alternative decompositions $B$ for \textbf{forms in the span} of $\nu_d(A)$. 

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The strategy of fine tuning

- Step 1. Determine parameters of all possible alternative decompositions $B$ for \textbf{forms in the span} of $v_d(A)$.
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Parametric expressions for alternative $B$ can be found by analyzing:

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The procedure turns out to be (reasonably) **easy** for ternary forms, much more complicated for forms in $n + 1 \geq 4$ variables.

**Challenge** for algebraic geometers: parametrize sets $B$ such that

$$Dh_{A \cup B}(d + 1) > 0.$$
Step 2: the fine tuning

Once we have the ideals of all $B$’s that decompose some forms in $\langle v_d(A) \rangle$, then compute if the fixed $T$ in the span can be obtained by some alternative decomposition $B$. 
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Theorem

$T$ belongs to the intersection of $\langle v_d(A) \rangle$ and $\langle v_d(B) \rangle$ iff

$$(l_A)_d + (l_B)_d \subseteq T^\vee$$
Examples

$T$ ternary form of degree 9, $r = 18$. 

Even if $A$ is a general set of 18 points in $P^2$, yet the span of $v_9(A)$ contains forms $T$ having (exactly one) second decomposition $B$, disjoint from $A$, of cardinality 17.

Thus the rank of $T$ is 17, and the two decompositions $A, B$ are separate.
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The shape of $Dh_{A∪B}$ implies that:

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Thus the admissible \( B \)'s are parametrized by the choice of a quintic and a septic form in \( I_A \).
Moreover \( I_B \) can be easily recovered from \( I_A \) and the two forms that determine the linkage. The fine tuning analysis is effective.

Concrete case: $T =$

\[
= [9666x_0^9 + 13004x_0^8x_1 + 12463x_0^7x_1^2 - 13235x_0^6x_1^3 - 15442x_0^5x_1^4 + 15509x_0^4x_1^5 + -6311x_0^3x_1^6 + \\
-2390x_0^2x_1^7 + 547x_0x_1^8 - 119x_1^9 - 14916x_0^8x_2 + 1822x_0^7x_1x_2 - 8022x_0^6x_1^2x_2 - 9386x_0^5x_1^3x_2 + \\
-2742x_0^4x_1^4x_2 + 10541x_0^3x_1^5x_2 + 1156x_0^2x_1^6x_2 - 12023x_0^7x_1^2x_2 + 4417x_1^8x_2 - 11823x_0^7x_2^2 - 737x_0^6x_1^2x_2 + \\
-7616x_0^5x_1^3x_2 + 11293x_0^4x_1^4x_2 - 8260x_0^3x_1^5x_2 - 9332x_0^2x_1^6x_2 + 7078x_0^6x_1^2x_2 - 4553x_1^7x_2^2 - 15941x_0^6x_2^3 + \\
+4339x_0^5x_1^3x_2^2 - 4251x_0^4x_1^4x_2^2 + 9854x_0^3x_1^5x_2^2 - 22x_0^2x_1^6x_2^2 + 8408x_0^5x_1^2x_2^3 + 11858x_1^6x_2^3 + \\
-9161x_0^5x_1^3x_2^3 - 9854x_0^4x_1^4x_2^3 - 13165x_0^3x_1^5x_2^3 - 2105x_0^2x_1^6x_2^3 - 8715x_0^4x_1^4x_2^4 + 390x_1^5x_2^4 - 9955x_0^4x_2^5 + \\
-11013x_0^3x_1^5x_2^5 - 10651x_0^2x_1^6x_2^5 - 3850x_0^3x_1^5x_2^5 + 4029x_1^4x_2^5 - 11735x_0^3x_2^6 - 12427x_0^2x_1^6x_2^6 + 12255x_0^2x_1^6x_2^6 + \\
-3686x_1^3x_2^6 - 2271x_0^2x_1^7 + 5939x_0x_1^8x_2 - 3402x_1^2x_2^7 + 13298x_0^2x_2^8 + 6455x_1^8x_2^8 + x_2^9].
\]
Examples

\[ T = \sum_{i=1}^{18} a_i L_i^9 \]

\[ (a_i) = (10308, -9437, -13956, -12270, 2135, -4854, -2213, 1755, -13629, 7308, -8496, 2940, 11348, -12437, -6712, 4086, -823, -2818) \]

\[ A \leftrightarrow \begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 2 & 4 & 1 & 5 & 6 & 1 & 1 & 6 & -7 & 3 & 2 & 6 & -7 \\ 1 & 1 & 2 & 2 & -2 & 1 & 2 & 5 & 2 & 2 & 7 & 7 & 5 & 2 & 7 & -5 & 3 & 6 \\ 1 & 2 & 1 & 3 & 0 & 4 & -3 & 1 & 3 & 3 & 7 & 3 & 4 & 3 & 4 & 1 & -4 & 6 \end{pmatrix} \]
Examples

\[ T = \sum_{i=1}^{18} a_i L_i^9 = \sum_{i=1}^{17} M_i^9 \]

\[ B \leftrightarrow \]

\[
\begin{pmatrix}
1 & 62.6659 & 29.7378 \\
1 & 13.368 + 38.1825 i & -19.099 + 7.53788 i \\
1 & 13.368 - 38.1825 i & -19.099 - 7.53788 i \\
1 & 35.333 & 40.797 \\
1 & 14.7061 & 27.8538 \\
1 & 10.7119 & 4.95399 \\
1 & -0.796312 & 2.23381 \\
1 & 1.06064 + 0.13583 i & 1.62951 - 0.563286 i \\
1 & 1.06064 - 0.13583 i & 1.62951 + 0.563286 i \\
1 & 0.737271 & -0.0631582 \\
1 & -0.245331 & -0.76262 \\
1 & -0.187307 & 0.100519 \\
1 & -0.0870499 & -0.126324 \\
1 & 0.00104432 & 0.00164595 \\
1 & 0.306581 + 0.0182712 i & -0.877193 - 0.031211 i \\
1 & 0.306581 - 0.0182712 i & -0.877193 + 0.031211 i \\
1 & 0.390447 & 0.585521
\end{pmatrix}
\]
Examples

\[ T = \sum_{i=1}^{18} a_i L_i^9 = \sum_{i=1}^{17} M_i^9 \]

From the **wrong** (= too long) decomposition \( A = ([L_1], \ldots, [L_{18}]) \) one finds the **minimal** decomposition \( B = ([M_1], \ldots, [M_{17}]) \).
Examples

Sextic ternary forms of rank $r = 9$ ($< r_g = 10$).
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Problem

Given one decomposition $T = \sum_{i=1}^{9} L_i^9$, find the second, different decomposition $T = \sum_{i=1}^{9} M_i^9$. 
Examples

\[ B \] is linked to \[ A \] by a complete intersection of type \((3, 6)\).

Fine tuning: Take in \( I_A \) a cubic form (unique) and a sextic form (depends on 9 parameters) and compute \( I_B \).

\[
\begin{align*}
M & \rightarrow \mathbb{R}^3(-5) \\
& \rightarrow \mathbb{R}^3(-4) \oplus \mathbb{R}(-3) \\
& \rightarrow I_A \\
& \rightarrow 0
\end{align*}
\]

\[
M = \begin{pmatrix}
2 & 1 & 1 & 1 \\
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\end{pmatrix}
\]

\[
\Rightarrow \begin{pmatrix}
0 \\
x \\
\ell \\
q
\end{pmatrix} = M'
\]

\( \ell \) = general linear form, \( q \) = general quadratic form (9 parameters).

\( I_B \) is generated by the maximal minors of \( M' \).

Then solve the fine tuning equation

\[
T^\vee = (I_A)^6 + (I_B)^6
\]

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<tr>
<th>( \sigma_9 )</th>
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<td>( \dim )</td>
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THANK YOU FOR YOUR ATTENTION