HOMEWORK FROM CLAY LECTURE 3

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(1) Write $F_{110} \subset A^*\otimes B^*$ for a candidate for I_{110} in the border apolarity method. Recall that the (210)-test is that the symmetrization map $\phi^{210}_{F_{110}} : F_{110}\otimes A^* \to S^2A^*\otimes B^*$ has image of codimension at least r. Show that this is equivalent to the map $\psi : F_{110}^{\perp}\otimes A \to \Lambda^2 A\otimes B$ having at least an r dimensional kernel. Hints: First consider the transpose of $\phi^{210}_{F_{110}}$, and then recall that $A\otimes A = S^2A\otimes\Lambda^2 A$. Remark: Using this one reduces the calculation of the (210) test for $M_{\langle 2\rangle}$ from computing the rank of a 40×24 matrix to computing the rank of a 24×24 matrix.

(2) Show that when $r = m$ (minimal border rank), the (210)-test becomes a familiar space of polynomials in the ideal of $\sigma_m(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1})$.

(3) Let $U = V = W = \mathbb{C}^2$. Show that we have the following decompositions as $SL(U)\times SL(V)$-modules

$$\Lambda^2(U^*\otimes V)\otimes V^* = S^2U^*\otimes V \oplus \Lambda^2U^*\otimes V \oplus \Lambda^2U^*\otimes S^3V$$
$$S^2(U^*\otimes V)\otimes V^* = S^2U^*\otimes S^3V \oplus \Lambda^2U^*\otimes V \oplus S^2U^*\otimes V$$

(4) Write $F_{110}^{\perp} = M_{\langle 2\rangle}(C^*)\oplus E'_{110}$. Using the previous two exercises, show that the dimension of the kernel of ψ for $M_{\langle 2\rangle}$ equals the dimension of the kernel of the map

$$E'_{110}\otimes A \to \Lambda^2U^*\otimes S^3V\otimes W.$$

Note that this amounts to calculating the rank of a 4×6 matrix.

(5) Prove $\underline{\mathbf{R}}(M_{\langle 2\rangle}) > 6$ (and thus equals 7).

(6) Show that for $T \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m = A\otimes B\otimes C$ that if $\underline{\mathbf{R}}(T) \leq m$, then $\mathbb{P}T(A^*) \cap Seg(\mathbb{P}B \times \mathbb{P}C) \neq \emptyset$ and similarly for the other factors.

(7) Show more generally that for $T \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m = A\otimes B\otimes C$ that if $\underline{\mathbf{R}}(T) \leq m$, then there exists a flag $F_1 \subset F_2 \subset \cdots \subset F_m = T(A^*)$, such that $\mathbb{P}F_j \subset \sigma_j(Seg(\mathbb{P}B \times \mathbb{P}C))$ for $1 \leq j \leq m$ and similarly for the other factors. (This is called the "flag condition".)

(8) Show that an analog of the flag condition holds for border apolarity. Namely show that if we have a winning ideal, then $F_{110}^{\perp} \subset A\otimes B$ admits a flag $F_1 \subset \cdots \subset F_r = F_{110}^{\perp}$ such that for $j < m$, $\mathbb{P}F_j \subset \sigma_j(Seg(\mathbb{P}B \times \mathbb{P}C))$. Remark: this was the new development that enabled the previously unattainable lower bound on the border rank of perm_3, because in that case the relevant weight spaces have large multiplicities.

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