

Border rank lower bounds part 1:

Representation theory: the systematic utilization of symmetry in (multi-)linear algebra

J.M. Landsberg

Texas A&M University

Supported by NSF grant CCF-1814254 and Clay foundation
(Clay senior scholar)

Goals

- Exhibit the utility of representation theory in the study of tensors.
- Explain border rank lower bounds for matrix multiplication.

Tensors

V complex vector space $\dim(V) = \mathbf{v}$.

$V^{\otimes d} := \{T : V^* \times \cdots \times V^* \rightarrow \mathbb{C} \text{ multilinear}\}$

$\mathfrak{S}_d$: permutation group acts on $V^{\otimes d}$:

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_d) := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$$

Special subspaces:

$$S^d V := \{T \in V^{\otimes d} \mid \sigma \cdot T = T \ \forall \sigma \in \mathfrak{S}_d\}$$

symmetric tensors, homogeneous degree d polynomials on V^* .

Write $v_1 v_2 \cdots v_d := \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \in S^d V$.

$$\Lambda^d V := \{T \in V^{\otimes d} \mid \sigma \cdot T = \text{sgn}(\sigma) T \ \forall \sigma \in \mathfrak{S}_d\}$$

skew-symmetric tensors. Write

$$v_1 \wedge \cdots \wedge v_d := \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \in \Lambda^d V.$$

What else in $V^{\otimes d}$?

Projective space, projective varieties

Recall: $\mathbf{Q}, \dots, \underline{\mathbf{R}}, \mathbf{R}$ notions of rank: all *invariant* under rescaling $T \mapsto \lambda T, \lambda \in \mathbb{C}^*$.

Recall: want to take limits

$\leadsto$ convenient to work in projective space $\mathbb{P}V := V \setminus \{0\} / \mathbb{C}^*$,
 $T \in V$, write $[T] \in \mathbb{P}V$, $X \subset \mathbb{P}V$, write $\hat{X} \subset V$.

$X \subset \mathbb{P}V$ zero set of collection of homogeneous polynomials called *algebraic variety* (or *projective variety*)

Examples of projective varieties

Ex. $X = \text{Seg}(\mathbb{P}A \times \mathbb{P}B) \subset \mathbb{P}(A \otimes B)$ *Segre variety* of rank one matrices. Zero set of $\Lambda^2 A^* \otimes \Lambda^2 B^* \subset S^2(A^* \otimes B^*)$.

Ex. $X = \text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A \otimes B \otimes C)$ *Segre variety* of rank one tensors. Zero set of $\Lambda^2 A^* \otimes \Lambda^2(B^* \otimes C^*) + \Lambda^2 B^* \otimes \Lambda^2(A^* \otimes C^*) + \Lambda^2 C^* \otimes \Lambda^2(A^* \otimes B^*) \subset S^2(A^* \otimes B^* \otimes C^*)$.

Ex. $X = v_d(\mathbb{P}V) \subset \mathbb{P}S^d V$ *Veronese variety* of rank one deg d polynomials, i.e. $v_d(\mathbb{P}V) = \{[v^d] \mid v \in V\}$ also $= GL(V) \cdot [w^d]$

Ex. $X = G(d, V) = \{[v_1 \wedge \cdots \wedge v_d] \mid v_j \in V\} \subset \mathbb{P}\Lambda^d V$ *Grassmann variety* d -planes through the origin in V :

$$[v_1 \wedge \cdots \wedge v_d] = [w_1 \wedge \cdots \wedge w_d] \text{ iff } \langle v_1, \dots, v_d \rangle = \langle w_1, \dots, w_d \rangle$$

$X \subset \mathbb{P}V$ any variety, can consider its secant varieties:

$$\sigma_r(X) := \overline{\{[v] \in \mathbb{P}V \mid v = x_1 + \cdots + x_r, x_j \in \hat{X}\}}$$

$\sigma_r(\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$: tensors of border rank $\leq r$.

Varieties and group actions

Say $X \subset \mathbb{P}V$ is a G -variety for some $G \subset GL(V)$, if $g \cdot x \in X$ for all $x \in X$ and all $g \in G$.

$Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is a
 $G = GL(A) \times GL(B) \times GL(C) \subset GL(A \otimes B \otimes C)$ variety.

X is a G -variety $\Rightarrow \sigma_r(X)$ is a G -variety.

Essential observation: $Z \subset \mathbb{P}V$ is a G -variety $\Rightarrow I_Z \subset Sym(V^*)$
is a G -module

Representation Theory

A *representation* of a group G is a group homomorphism $\mu : G \rightarrow GL(V)$. Say V is a G -module.

$W \subset V$ is a *submodule* if $\forall g \in G, w \in W, \mu(g)w \in W$.

V is *irreducible* if it has no proper submodules.

Non-Example: action of $\mathfrak{S}_d$ on $\mathbb{C}^d$ is not irreducible.

Examples: $\Lambda^d V, S^d V \subset V^{\otimes d}$ are irreducible $GL(V)$ -modules.

Lemma (Schur's Lemma)

Let V, W be irreducible G -modules and $f : V \rightarrow W$ a G -module map. Then either $f = 0$ or f is an isomorphism. If furthermore $V = W$, then $f = \lambda \text{Id}$ for some constant λ .

Decomposition of $V^{\otimes 3}$ as a $GL(V)$ -module

Saw $S^3 V \oplus \Lambda^3 V \subset V^{\otimes 3}$ What else?

What about partially symmetric? $S^2 V \otimes V \subset V^{\otimes 3}$ not irred.,
 $S^2 V \otimes V \supset S^3 V$. Consider kernel of symmetrization map
 $S^2 V \otimes V \rightarrow S^3 V$. A $GL(V)$ -submodule!

Ex. $v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_1 \otimes v_3 \otimes v_2 - v_3 \otimes v_1 \otimes v_2$ is in kernel.

Representations of $GL(V)$ in $V^{\otimes d}$

Let π : partition of d . Have associated *Young diagram*

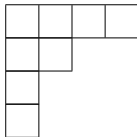


Figure: Young diagram for $\pi = (4, 2, 1, 1)$

Label boxes $\{1, \dots, d\}$, called a *Young tableau without repetitions*.
For example

3	7	2	5
1	4		
6			
8			

Representations of $GL(V)$ in $V^{\otimes d}$

default Young tableau

1	2	3	4
5	6		
7			
8			

Given any Young tableau without repetitions, write

$$V^{\otimes d} = V_1 \otimes V_2 \otimes \cdots \otimes V_d,$$

for each row symmetrize the corresponding copies of V ,

e.g., for $\pi = (4, 2, 1, 1)$ after the symmetrization, one obtains an element of $S^4 V \otimes S^2 V \otimes V \otimes V \subset V^{\otimes 8}$. Next, skew-symmetrize along the columns.

Example, default Young tableau for $\pi = (2, 1)$, get

$$v_i \otimes v_j \otimes v_k \mapsto v_i \otimes v_j \otimes v_k + v_j \otimes v_i \otimes v_k \mapsto$$

$$v_i \otimes v_j \otimes v_k - v_k \otimes v_j \otimes v_i + v_j \otimes v_i \otimes v_k - v_k \otimes v_i \otimes v_j.$$

Representations of the permutation group and $GL(V)$

For default Young tableau, write

$$\text{proj}_{\pi-def} : V^{\otimes d} \rightarrow V^{\otimes d}$$

$$S_{\pi-def} V := \text{proj}_{\pi-def}(V^{\otimes d}) : \text{image.}$$

$S_{\pi-def} V$ is a $GL(V)$ -module.

Write $\ell(\pi)$ for number of parts of π (number of cols. of YD)

Let $\{e_j\}$ basis V and $\ell(\pi) \leq \dim V$. Set

$$v_{\pi-def} := \text{proj}_{\pi-def}(e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \otimes \cdots \otimes e_\ell^{\otimes p_\ell}).$$

Fact: $\mathfrak{S}_d$ -modules indexed by partitions of d . Write $[\pi]$ for corresponding module. The span of the $\mathfrak{S}_d$ -orbit of $v_{\pi-def}$ is an irreducible $\mathfrak{S}_d$ -module isomorphic to $[\pi]$.

Representations of the permutation group and $GL(V)$

Let $S_\pi V$ denote the isomorphism class of $S_{\pi-def} V$.

Theorem (Schur-Weyl duality)

As a $GL(V) \times \mathfrak{S}_d$ -module,

$$V^{\otimes d} = \bigoplus_{|\pi|=d} S_\pi V \otimes [\pi].$$

W, M : G -modules with M irreducible, the *isotypic component* of M in W is the largest submodule of W isomorphic to $M^{\oplus m} = M \otimes \mathbb{C}^m$ for some m (multiplicity). Isotypic component of $S_\pi V$ in $V^{\otimes d}$ is $S_\pi V \otimes [\pi]$ Multiplicity of $S_\pi V$ in $V^{\otimes d}$ is $\dim[\pi]$. Multiplicity of $[\pi]$ in $V^{\otimes d}$ is $\dim S_\pi V$.

Other $GL(V)$ modules: Define $\det^{-1} : GL(V) \rightarrow GL(\mathbb{C}^1)$,
 $g \mapsto \det(g)^{-1}$. All other $GL(V)$ -modules are $S_\pi V \otimes (\det^{-1})^{\otimes k}$.

Dimensions

Write $x \in \pi$ for a box in the Young diagram of π .

hook length of x = number of boxes to the right of it in its row, plus the number of boxes below it in its column, plus one,

hook lengths for $(4, 2, 1, 1)$:

7	4	2	1
4	1		
2			
1			

content $c(x)$ of x is zero if x is on the main diagonal, j , if it is on the j -th diagonal above the main diagonal, and $-j$ if it is on the j -th diagonal below the main diagonal.

Dimensions cont'd

$$\dim S_\pi \mathbb{C}^m = \prod_{x \in \pi} \frac{m + c(x)}{h(x)}$$

In particular

$$\dim S_\pi \mathbb{C}^m \leq (d + 1)^{\binom{m}{2}}.$$

$$\dim[\pi] = \frac{d!}{\prod_{x \in \pi} h(x)}$$

$\dim(\mathbb{C}^m)^{\otimes d} = m^d$ is exponential in d ,

Cauchy and Pieri

Cauchy: As $GL(A) \times GL(B)$ -module,

$$S^d(A \otimes B) = \bigoplus_{|\pi|=d, \ell(\pi) \leq \min\{\mathbf{a}, \mathbf{b}\}} S_{\pi} A \otimes S_{\pi} B$$

$$\Lambda^d(A \otimes B) = \bigoplus_{|\pi|=d, \ell(\pi) \leq \min\{\mathbf{a}, \mathbf{b}\}} S_{\pi} A \otimes S_{\pi'} B$$

$$\text{Pieri: } S_{(5,2,1)} V \otimes V = S_{(6,2,1)} V \oplus S_{(5,3,1)} V \oplus S_{(5,2,2)} V \oplus S_{(5,2,1,1)} V$$

Picture:

More generally $S_{\pi} V \otimes S^k V$, $S_{\pi} V \otimes \Lambda^k V$

Equations for secant varieties of Segre varieties

$T \in A \otimes B \otimes C$, $\mathbf{R}(T) \geq \text{rank}(T_A : A^* \rightarrow B \otimes C)$ and permuted statements. $\leadsto$

$$\Lambda^{r+1} A^* \otimes \Lambda^{r+1} (B \otimes C)^* \subset I_{r+1}(\sigma_r(\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))).$$

General idea: $X \subset \mathbb{P}M$, $f : M \hookrightarrow U \otimes W$ linear. If $\forall x \in \hat{X}$, $\text{rank}(f(x)) \leq t$, then for all $[m] \in \sigma_r(X)$, $\text{rank}(f(m)) \leq rt$ i.e., size $rt + 1$ minors of image spans a subspace of $I_{rt+1}(\sigma_r(X))$

How to find good ones? If M : irred. G -module and X : G -variety, look for G -module inclusions: systematic.

$$\text{Ex. } A \subset \text{Hom}(\Lambda^p A, \Lambda^{p+1} A) = \Lambda^p A^* \otimes \Lambda^{p+1} A \leadsto$$

$$A \otimes B \otimes C \subset \text{Hom}(\Lambda^p A \otimes B^*, \Lambda^{p+1} A \otimes C)$$

Advantageous to restrict to $2p + 1$ dimensional subspaces.

Matrix multiplication and Koszul flattening

$$M_{\langle n \rangle} \in A \otimes B \otimes C = (U^* \otimes V) \otimes (V^* \otimes W) \otimes (W^* \otimes U)$$

As a tensor $M_{\langle n \rangle}$ is $\text{Id}_U \otimes \text{Id}_V \otimes \text{Id}_W$ re-ordered.

Koszul flattening map is

$$(M_{\langle U, V, W \rangle})_A^{\wedge p} : \Lambda^p(U^* \otimes V) \otimes (V \otimes W^*) \rightarrow \Lambda^{p+1}(U^* \otimes V) \otimes (W^* \otimes U).$$

Presence of $\text{Id}_W = \text{Id}_{W^*} \Rightarrow$ map factors as $(M_{\langle U, V, \mathbb{C}^1 \rangle})_A^{\wedge p} \otimes \text{Id}_{W^*}$, where

$$(M_{\langle u, v, 1 \rangle})_A^{\wedge p} : V \otimes \Lambda^p(U^* \otimes V) \rightarrow \Lambda^{p+1}(U^* \otimes V) \otimes U.$$

$$v \otimes (\xi^1 \otimes e_1) \wedge \cdots \wedge (\xi^p \otimes e_p) \mapsto$$

$$\sum_{s=1}^u u_s \otimes (u^s \otimes v) \wedge (\xi^1 \otimes e_1) \wedge \cdots \wedge (\xi^p \otimes e_p).$$

Here $u_1, \dots, u_u$: basis of U with dual basis $u^1, \dots, u^u$ of U^* , so $\text{Id}_U = \sum_{s=1}^u u^s \otimes u_s$.

Matrix multiplication and Koszul flattening

Strategic restriction $\Rightarrow \underline{\mathbf{R}}(M_{\langle n \rangle}) \geq 2\mathbf{n}^2 - \mathbf{n}$ (L-Ottaviani 2015)

(previous $\frac{3}{2}\mathbf{n}^2 + \frac{1}{2}\mathbf{n} - 1$ (Lickteig 1985, after Strassen 1983))

Next time: Further uses of representation theory

Thank you for your attention

For more on **tensors**, their geometry and applications, resp. **geometry and complexity**, resp. **recent developments**:

