A TALE of
SPONTANEOUS STOCHASTICITY

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”Ask no questions, and you’ll be told no lies”
Charles Dickens

The Holy Grail of the developed turbulence:

The limit $\nu \to 0$ of the 3d Navier-Stokes ensemble with stochastic homogeneous and isotropic large scale forcing exists and shows on short distances a persistent direct energy cascade and a universal scaling
We are pretty far from proving such a statement but the studies of the inertial-range properties of large Reynolds number $Re$ flows indicate that it may be true.

One of the anticipated properties of such a $Re = \infty$ ensemble would be that its typical velocities are rough, with Hölder exponents less than $1/3$, at least locally, as otherwise the injected energy could not be dissipated (Onsager 1949, Duchon-Robert 2000).

**Spontaneous stochasticity** is another layer on top of the Holy Grail, a conjecture stating that in typical velocities of the $Re = \infty$ ensemble the Lagrangian flow is stochastic rather than deterministic.

**Question:** Why adding another layer to an already bold conjecture?

**Answer:** Because it may add an essential element for proving the Holy Grail, capturing an important property of high $Re$ flows verifiable today.
The idea of *spontaneous stochasticity* has some similarity but is different from the other attempts to introduce stochastic elements at infinite $Re$:

- generalized flows of Brenier J. AMS 2 (1989)
- multiphase and sticky generalized flows of Shnirelman CMP 210 (2000)

The main difference is that the stochasticity concerns only the Lagrangian flow in deterministic rough velocities, so the ODE rather than the PDE aspect

On the other hand, the applications of the idea to nonlinear PDE remain on the heuristic or conjectural level
• **Lagrangian flow**
  
  • For a **smooth** velocity field \( \mathbf{v}(t, \mathbf{r}) \) the Lagrangian flow \((s, \mathbf{r}) \mapsto \mathbf{R}_t, \mathbf{r}(s)\) is defined by the ODE
  
  \[
  \frac{d\mathbf{R}_t, \mathbf{r}(s)}{ds} = \mathbf{v}(s, \mathbf{R}_t, \mathbf{r}(s)), \quad \mathbf{R}_t, \mathbf{r}(t) = \mathbf{r}
  \]

  • For a **rough** (non-Lipschitz) in space velocity field \( \mathbf{v}(t, \mathbf{r}) \) some limiting procedure is required

  1. one can consider a noisy Lagrangian flow solving the SDE

  \[
  d\mathbf{R}_{t, \mathbf{r}}^\kappa(s) = \mathbf{v}(s, \mathbf{R}_{t, \mathbf{r}}^\kappa(s)) \, ds + \sqrt{2\kappa} \, d\mathbf{\beta}(s), \quad \mathbf{R}_{t, \mathbf{r}}^\kappa(t) = \mathbf{r}
  \]

  that exists for \( \kappa > 0 \) even for rough (e.g. spatially Hölder) velocities and has transition probabilities

  \[
  P_{t, \mathbf{r}}^\kappa(s, \mathbf{R} | \mathbf{v}) = \mathbb{E} \, \delta(\mathbf{R} - \mathbf{R}_{t, \mathbf{r}}^\kappa(s))
  \]

  2. one can use velocity \( \mathbf{v}_\eta(t, \mathbf{r}) \) smoothed on a small scale \( \eta \)

  and consider the usual Lagrangian flow \( \mathbf{R}_{t, \mathbf{r}}^\eta(s) \) for \( \mathbf{v}_\eta \) setting

  \[
  P_{t, \mathbf{r}}^\eta(s, \mathbf{R} | \mathbf{v}) = \delta(\mathbf{R} - \mathbf{R}_{t, \mathbf{r}}^\eta(s))
  \]
We shall say that the Lagrangian flow in velocity field \( \mathbf{v}(t, r) \) is stochastic if one of the limits

\[
P_{t, r}(s, \mathbf{R} \mid \mathbf{v}) = \begin{cases} \lim_{\kappa \to 0} P_{t, r}^\kappa(s, \mathbf{R} \mid \mathbf{v}) \\ \lim_{\eta \to 0} P_{t, r}^\eta(s, \mathbf{R} \mid \mathbf{v}) \end{cases}
\]

exists in a sufficiently weak sense but is not concentrated at a single \( \mathbf{R} = R_{t, r}(s) \) for each \( t, r, s \).

Such a behavior could be only possible in non-Lipschitz velocities where there may be many solutions of the Lagrangian ODE with a common initial or final value.

**Remarks.** 1. Physically, \( \kappa \) would represent the molecular diffusivity and \( \eta \) the viscous Kolmogorov scale. The \( \kappa \to 0 \) limit corresponds to the vanishing Prandtl number \( Pr = \frac{\nu}{\kappa} \) and the \( \eta \to 0 \) one to \( Pr = \infty \).

2. The two limits may be different pointing to the Prandtl number dependence of the limiting Lagrangian flow.
• Why should we expect such behavior in \( Re = \infty \) velocities?

• The first indication comes from the 1926 Richardson law for the Lagrangian dispersion \( \rho(s) = |R_{0,r}(s) - R_{0,r'}(s)| \)

\[
\langle \rho(s)^2 \rangle \propto \epsilon s^3
\]

for large \( s \) with the coefficient independent of \( \rho(0) \)

Extended to the smallest scales, this implies that Lagrangian trajectories starting arbitrarily close separate to \( O(1) \) distance in finite time, a behavior impossible for trajectories determined by the initial point

(In contrast, in chaotic dynamical systems \( \rho(s) \propto e^{\lambda s} \rho(0) \) for small \( \rho \) taking longer and longer to separate the smaller \( \rho(0) \))

• For a recent discussions of the pair dispersion statistics in developed turbulence see Thalabard et al. JFM 755 R4 (2014) that questions the Richardson diffusion but not the \( s^3 \) finite-time separation
\[ \langle \rho(s)^2 \rangle \text{ at } Re_\lambda = 433 \]

from [Eyink PR E 83 (2011)]

\[ \rho(0) \text{-dependence} \]

from [Bitane et al. J. Turb. 14 (2013)]
Further indication came from the study of the Lagrangian flow in the Kraichnan homogeneous and isotropic ensemble of velocities that are Hölder in space but decorrelated in time. This is a Gaussian ensemble with mean zero and 2-point function

\[ \langle v^i(t, r) v^j(t', r') \rangle = (D_0 \delta^{ij} - D^{ij}(r - r')) \delta(t - t') \]

with the isotropic tensor \( D^{ij}(r) \) characterized by the Hölder exponent \( \alpha \) and the compressibility \( \varphi \) with \( 0 < \alpha, \varphi < 1 \)

\[ D^{ij}(r) \propto r^{2\alpha} \quad \text{for small } r, \quad \varphi = \frac{\partial_i \partial_j D^{ij}(r)}{\partial_i \partial_i D^{jj}(r)} \]

It was observed in Bernard-G.-Kupiainen JSP 90 (1998) in the \( \varphi = 0 \) version of this model that the probability of the pair dispersion \( P_0,\rho_0(s, \rho) \) has a non-zero \( \rho_0 \to 0 \) limit leading to the Richardson type law

\[ \langle \rho(s)^2 \rangle \propto s^{\frac{1}{1-\alpha}} \]

that holds for all \( s > 0 \) for \( \rho_0 = 0 \)
As was stressed, that leads to the breakdown of deterministic Lagrangian flow in typical velocities and its replacement by a stochastic one, leading to a direct cascade of advected tracer field with a dissipative anomaly at $\kappa = 0$.

It was then shown in G.-Vergassola Physica D 138 (2000) that such a behavior persists in Kraichnan velocities with $\varphi < \frac{d}{4\alpha^2}$ whereas for $\varphi > \frac{d}{4\alpha^2}$ one recovers a deterministic Lagrangian flow at $\kappa = 0$ with trajectories collapsing together in finite time, leading to an inverse cascade of advected tracer field without dissipative anomaly.

Explosive separation of trajectories versus implosive collapse (from Falkovich-G.-Vergassola Rev. Mod. Phys. 73 (2001))
In E-Vanden Eijnden PNAS 97 (2000) and Physica D 152153 (2001) it was shown that for \( \frac{d-2+2\alpha}{4\alpha} < \varphi < \frac{d}{4\alpha^2} \) the \( \kappa \to 0 \) and \( \eta \to 0 \) Lagrangian flows differ, the first being stochastic whereas the second deterministic with collapsing trajectories.

In G.-Horvai JSP 116 (2004) the \( Pr \)-dependence of the limiting flow was revisited and a “sticky” limiting Lagrangian flow was constructed for intermediate \( \varphi \) by fine-tuning \( \kappa \) and \( \eta \).

In Le Jan - Raimond C. R. Acad. Sci. 327 (1998) the transition probabilities \( P_{t,r}(s, R \mid v) \) were rigorously constructed and in Ann. Probab. 30 (2002) and 32 (2004) the Lagrangian flows in Kraichnan velocities were connected to “nonclassical noises” and the relation between different limiting flows for intermediate \( \varphi \) was clarified.

The Kraichnan model is time-reversal invariant unlike the real turbulence where the backward-in-time Richardson separation is considerably faster (Sawford et al. Phys. Fluids 17 (2005), Eyink PR E 83 (2011), Buaria et al. J. Fluid Mech. 799 (2016)).
Lagrangian flow in rough velocities beyond Kraichnan model

In Chaves et al. JSP 113 (2003) we analyzed the Lagrangian flow in Gaussian self-similar ensemble of Hölder velocities correlated in time arguing for scaling laws but without definitive results.

In Eyink-Drivas JSP 158 (2014) the backward-in-time Lagrangian flow for the Burgers velocities was shown to become stochastic when $\nu \to 0$ and $Pr$-dependent unlike the forward-in-time deterministic flow with trajectories coalescing onto the shocks.

The same authors showed in arXiv:1509.04941 the appearance of stochastic Hamiltonian flows in the semiclassical limit of the 1d Schrödinger equation with a rough potential.

The works of Dubédat (2009) and of Miller-Sheffield (2012-2013) studied the Lagrangian flow in time-frozen 2d compressible velocities

$$\nu(r) = (\cos\left(\frac{1}{\chi}\phi(r) + \theta\right), \sin\left(\frac{1}{\chi}\phi(r) + \theta\right))$$

for $\chi > 0$, $0 < \theta \leq 2\pi$ and the random massless free field $\phi(r)$. Their flow (probably the $\eta \to 0$ one) is deterministic with trajectories looking locally like $SLE_\kappa$-curves for $0 < \kappa \leq 4$ and $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ that collapse together when meeting.
From Miller-Sheffield arXiv:1201.1496v2 [math.PR]

\[ \kappa = \frac{1}{2}, \] blue lines: \[ \theta = \frac{\pi}{4}, \] magenta lines: \[ \theta = -\frac{\pi}{4} \]
Terminology

In Bernard et al. (1998) we talked about “intrinsically probabilistic character of the Lagrangian flow”

I used the terms “breakdown of deterministic Lagrangian flow”, “stochastic flow” and “fuzzy trajectories” in 1998 reviews.

E-Vanden-Eijnden (2000) coined the name “intrinsic stochasticity”.


We employed the term “spontaneous stochasticity” in the review Falkovich et al. RMP 73 (2001). The name was then repeatedly used in Eyink’s papers.

The deterministic flows with trajectories collapsing together are generally called “coalescent flows”
Lagrangian formulation of hydrodynamic equations

Consider for a smooth velocity field $\mathbf{v}(t, r)$ with the Lagrangian flow $(s, r) \mapsto \mathbf{R}_{t, r}(s)$ the matrix

$$(W^i_j)_{t, r}(s) = \frac{\partial R^i_t, r(s)}{\partial r^j}$$

that propagates the infinitesimal dispersion of Lagrangian trajectories $\delta R_{t, r}(s) = W_{t, r}(s) \delta r$. It satisfies the ODE

$$\frac{dW_{t, r}(s)}{ds} = (\nabla \mathbf{v})^T(s, \mathbf{R}_{t, r}(s)))W_{t, r}(s), \quad W_{t, r}(t) = I$$

In smooth flows the eigenvalues of $\frac{1}{s} \ln W_{t, r}^T W_{t, r}(s)$ give when $s \to \infty$ the Lyapunov exponents of the flow and their large deviations encode the multifractal structure of advection, see Grassberger et al. JSP 51 (1988), Bec et al. PRL 92 (2004)
• The 3d hydrodynamic equations

\[ \partial_t \theta + (v \cdot \nabla)\theta = 0 \]  
\[ \partial_t B - \nabla \times (v \times B) = 0 \]  
\[ \partial_t \omega - \nabla \times (v \times \omega) = 0 \]  
\[ \partial_t v + (v \cdot \nabla)v = -\nabla p \]

advection of scalar  
advection of magnetic field  
Euler equation for vorticity  
Euler equation for velocity

with \( \nabla \cdot B = 0 \) and with \( \nabla \cdot v = 0 \) in the last two equations have equivalent Lagrangian reformulation: for any times \( s, t \)

\[ \theta(s, R_t, r(s)) = \theta(t, r) \]  
\[ \det(W_{t,r}(s)) W_{t,r}(s)^{-1} B(s, R_{t,r}(s)) = B(t, r) \]  
\[ W_{t,r}(s)^{-1} \omega(s, R_{t,r}(s)) = \omega(t, r) \]  
\[ P(r) \left( v(s; R_{t,r}(s)) W_{t,r}(s) \right) = v(t, r) \]

where \( P^i_j(r) = \delta^i_j - \frac{\nabla^i r \nabla^j r}{\nabla^2 r} \) is the transverse projector
Proof for the **Euler** equation for velocity:

\[
\frac{d}{ds} \left( v(s, R_t, r(s)) W_{t, r}(s) \right) = \left( (\partial_s v) + (v \cdot \nabla)v \right)(s, R_t, r(s)) W_{t, r}(s) \\
+ v(s, R_t, r(s))(\nabla v)^T(s, R_t, r(s)) W_{t, r}(s)
\]

\[
= \left( - (\nabla p)(s, R_t, r(s)) + \frac{1}{2} \nabla (v^2)(s, R_t, r(s)) \right) W_{t, r}(s)
\]

\[
= \nabla_r \left( - p(s, R_t, r(s)) + \frac{1}{2} v^2(s, R_t, r(s)) \right)
\]

Upon applying \( P(r) \) this gives

\[
P(r) \left( v(s, R_t, r(s)) W_{t, r}(s) \right) = \text{const.}
\]

\[
= P(r) \left( v(t, R_t, r(t)) W_{t, r}(t) \right) = v(t, r)
\]

(The application of \( \nabla_r \times \) instead of \( P(r) \) would give Cauchy's 1815/1827 formulation of the **Euler** equation, see Frisch-Villone Eur. Phys. J. 39 (2014))
As a consequence of the Lagrangian formulation of the Euler equation

\[
\int u(r) \cdot \left( v(s; R_t, r(s)) W_{t, r}(s) \right) dr = \int u(r) \cdot v(t, r) dr
\]

whenever \( \nabla \cdot u = 0 \)

Taking \( u(r) = \int \delta(r - \ell(\sigma)) \frac{d\ell(\sigma)}{d\sigma} d\sigma \) for a closed loop \( \sigma \mapsto \ell(\sigma) \) one gets the Kelvin Theorem on conservation of velocity circulation

\[
\int v(s, R_t, \ell(\sigma)(s)) \cdot \frac{dR_s, \ell(\sigma)(s)}{d\sigma} d\sigma = \int v(t, \ell(\sigma)) \cdot \frac{d\ell(\sigma)}{d\sigma} d\sigma
\]

that may be also rewritten in Helmholtz’s form as a conservation of the flux of vorticity across a surface \( (\sigma^1, \sigma^2) \mapsto S(\sigma^1, \sigma^2) \) with \( \partial S = \ell \)

\[
\int \omega(s, R_t, S(\bar{\sigma})) \cdot \left( \frac{\partial R_t, S(\bar{\sigma})(s)}{\partial \sigma^1} \times \frac{\partial R_t, S(\bar{\sigma})(s)}{\partial \sigma^2} \right) d\bar{\sigma} = \int \omega(t, S(\bar{\sigma})) \cdot \left( \frac{\partial S(\bar{\sigma})}{\partial \sigma^1} \times \frac{\partial S(\bar{\sigma})}{\partial \sigma^2} \right) d\bar{\sigma}
\]

with its magnetic field counterpart where \( \omega \) is replaced by \( B \) called the Alfvén Theorem
Backward in time Lagrangian evolution of a contour

Backward in time Lagrangian evolution of a surface
• Probabilistic Lagrangian interpretation of dissipative hydrodynamic equations

\[
\begin{align*}
\partial_t \theta + (\mathbf{v} \cdot \nabla) \theta - \kappa \nabla^2 \theta &= 0 & \text{advection–diffusion of scalar} \\
\partial_t B - \nabla \times (\mathbf{v} \times B) - \kappa \nabla^2 B &= 0 & \text{advection–diffusion of magnetic field} \\
\partial_t \omega - \nabla \times (\mathbf{v} \times \omega) - \nu \nabla^2 \omega &= 0 & \text{Navier–Stokes equation for vorticity} \\
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \nabla^2 \mathbf{v} &= -\nabla p & \text{Navier–Stokes equation for velocity}
\end{align*}
\]

also have equivalent Lagrangian reformulation but with the stochastic Lagrangian flows

\[
dR_{t,r}(s) = \mathbf{v}(s, R_{t,r}(s)) \, ds + \left\{ \frac{\sqrt{2\kappa}}{\sqrt{2\nu}} \right\} d\beta(s), \quad R_{t,r}(t) = r
\]

Brownian motion

with the same equation as before for \( W_{t,r}(s) \) but \((R_{t,r}(s), W_{t,r}(s))\) becoming a random process
The Lagrangian equivalent of the hydrodynamic equations with dissipation takes now the form: for any $s \leq t$

\[
\mathbb{E} \, \theta(s, R_t, r(s)) = \theta(t, r)
\]

\[
\mathbb{E} \, \det(W_{t,r}(s)) W_{t,r}(s)^{-1} B(s, R_t, r(s)) = B(t, r)
\]

\[
\mathbb{E} \, W_{t,r}(s)^{-1} \omega(s, R_t, r(s)) = \omega(t, r)
\]

\[
\mathbb{E} \, P(r) \left( \nu(s; R_t, r(s)) W_{t,r}(s) \right) = \nu(t, r)
\]

where $\mathbb{E}$ denotes the expectation with respect to the Brownian noise.

Note that the dissipation fixes the time direction.

The last reformulation of the Navier-Stokes equations is due to Constantin-Iyer (2008) and it is equivalent to the Stochastic Kelvin Thm (Eyink 2008)

\[
\mathbb{E} \, \int \nu(s, R_t, \ell(\sigma)(s)) \cdot \frac{dR_s, \ell(\sigma)(s)}{d\sigma} \, d\sigma = \int \nu(t, \ell(\sigma)) \cdot \frac{d\ell(\sigma)}{d\sigma} \, d\sigma
\]
or its Helmholtz’s form

\[
\mathbb{E} \int \omega(s, R_t, S(\vec{\sigma})) \cdot \left( \frac{\partial R_t, S(\vec{\sigma})(s)}{\partial \sigma^1} \times \frac{\partial R_t, S(\vec{\sigma})(s)}{\partial \sigma^2} \right) \, d\vec{\sigma}
\]

\[
= \int \omega(t, S(\vec{\sigma})) \cdot \left( \frac{\partial S(\vec{\sigma})}{\partial \sigma^1} \times \frac{\partial S(\vec{\sigma})}{\partial \sigma^2} \right) \, d\vec{\sigma}
\]

(and similarly for the magnetic field for which one obtains the Stochastic Alfvén Thm)

- One may also rephrase the Lagrangian formulation of the dissipative hydrodynamic equations as the condition that the random processes parameterized by time \( s \leq t \)

\[
\theta(s, R_t, r(s))
\]
\[
det(W_{t, r}(s)) \, W_{t, r}(s)^{-1} \, B(s, R_t, r(s))
\]
\[
W_{t, r}(s)^{-1} \omega(s, R_t, r(s))
\]
\[
\int \nu(s, R_t, \ell(\sigma)(s)) \cdot \frac{dR_s, \ell(\sigma)(s)}{d\sigma} \, d\sigma
\]

are backward-in-time martingales for all \( t, r \) and all loops \( \ell(\sigma) \)
• **Sources** are straightforward to include. E.g. for the scalar advection-diffusion

\[
\partial_t \theta + (\mathbf{v} \cdot \nabla) \theta - \kappa \nabla^2 \theta = g
\]

one has

\[
\mathbb{E} \left[ \theta(s, R_t, r(s)) + \int_s^t g(\tau, R_t, r(\tau)) \, d\tau \right] = \theta(t, r)
\]

and it is

\[
\theta(s, R_t, r(s)) + \int_s^t g(\tau, R_t, r(\tau)) \, d\tau
\]

which is a backward-in-time *martingale*, i.e. for \( s < s' < t \)

\[
\mathbb{E}_{[s, s']} \left[ \theta(s, R_t, r(s)) + \int_s^t g(\tau, R_t, r(\tau)) \right]
\]

\[
= \mathbb{E}_{[s, s']} \left[ \theta(s, R_{s'}, R_t, r(s')(s)) + \int_s^{s'} g(\tau, R_{s'}, R_t, r(s')(s)) \, d\tau \right]
\]

\[
+ \int_{s'}^t g(\tau, R_t, r(\tau)) \, d\tau
\]

\[
= \theta(s', R_t, r(s')) + \int_{s'}^t g(\tau, R_t, r(\tau)) \, d\tau
\]
• Spontaneous stochasticity and hydrodynamic equations

• It was stressed in Eyink CRAS 7 (2006) (with some numerical evidence in Chen et al. PRL 97 (2006)) that the spontaneous stochasticity of the Lagrangian flow in the limit $\nu \to 0$ of the Navier-Stokes turbulence is compatible with the stochastic Kelvin-Helmholtz Thm.

In Eyink PR E 83 (2011) a similar picture for the MHD in the limit $\kappa, \nu \to 0$ with persistent stochastic Alfvén Thm was developed and the spontaneous stochasticity of the Lagrangian flow was subsequently employed by Eyink et al. (see e.g. Nature 497 (2013)) as the cornerstone of the magnetic reconnection mechanism in astrophysical plasma.

• This requires, however, an extension of the spontaneous stochasticity scenario where not only $R_{t,r}(s)$ but also $W_{s,r}(s)$ would make some stochastic sense in rough turbulent velocities.

We would need that some limiting procedure may be applied to the matrix-valued kernels that enter the probabilistic Lagrangian formulation of the hydrodynamic equations allowing to define...
\[
\lim_{\kappa, \eta \to 0} \mathbb{E} W_{t, r}^{\kappa, \eta}(s) \delta(R - R_{t, r}^{\kappa, \eta}(s)) \equiv F_{t, r}(s, R \mid \nu)
\]

\[
\lim_{\kappa, \eta \to 0} \mathbb{E} \det(W_{t, r}^{\kappa, \eta}(s)) W_{t, r}^{\kappa, \eta}(s)^{-1} \delta(R - R_{t, r}^{\kappa, \eta}(s)) \equiv G_{t, r}(s, R)
\]

in the velocity-ensemble correlations

The limiting kernels should satisfy the composition laws, the time reversal \( G_{t, r}(s, R) = F_{s, R}(t, r) \) and satisfy, at least weakly, the relation

\[
\partial_{R_i} F_{t, r}(s, R \mid \nu)^i_j = - \partial_{r_j} P_{s, r}(t, R \mid \nu)
\]


- A more complete theory of such random kernels remains to be constructed even in the \textit{Kraichnan} model

- Some numerical data on the 2-point function of the kernels \( G \) in high \( Re \) flows may be found in \textit{Eyink} PR E 83 (2011)
Conclusions

If the stochastic Lagrangian formulation of dissipative hydrodynamic equations is better suited to control their $Re \to \infty$ or $Re, Re_m \to \infty$ limits then the spontaneous stochasticity of the Lagrangian flows in such limits may be an essential element of such a control.

The signs of spontaneous stochasticity seem visible in numerical simulations of high $Re$ turbulence showing the Richardson-type separation of Lagrangian trajectories or the persistence of the stochastic versions of the Kelvin-Helmholtz or Alfvén laws.

Admittingly, the evidence for the spontaneous stochasticity from the high $Re$ flows is not yet overwhelming and calls for further numerical tests.

The spontaneous stochasticity requiring the roughness of typical velocities in the $Re \to \infty$ limit should be responsible for the dissipative anomaly assuring the energy dissipation in the inviscid limit.

Even in the Kraichnan model, where the spontaneous stochasticity has been rigorously established some work remains to be done that could throw more light on this phenomenon.
Some more details on the Lagrangian flows in the Kraichnan model are discussed in a reader-friendly fashion in:


“Spontaneity is a meticulously prepared art”

Oscar Wilde