# A TALE of SPONTANEOUS STOCHASTICITY

Krzysztof Gawędzki
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”Ask no questions, and you’ll be told no lies”
*Charles Dickens*

- **The Holy Grail of the developed turbulence:**

  The limit $\nu \to 0$ of the $3d$ **Navier-Stokes** ensemble with stochastic homogeneous and isotropic large scale forcing exists and shows on short distances a persistent direct energy cascade and a universal scaling

- We are pretty far from proving such a statement but the studies of the inertial-range properties of large Reynolds number $Re$ flows indicate that it may be true.

- One of the anticipated properties of such a $Re = \infty$ ensemble would be that its typical velocities are rough, with **Hölder** exponents less than $1/3$, at least locally, as otherwise the injected energy could not be dissipated (**Onsager** 1949, **Duchon-Robert** 2000)

- **Spontaneous stochasticity** is another layer on top of the Holy Grail, a conjecture stating that in typical velocities of the $Re = \infty$ ensemble the **Lagrangian flow** is stochastic rather than deterministic

**Question:** Why adding another layer to an already bold conjecture?

**Answer:** Because it may add an essential element for proving the Holy Grail, capturing an important property of high $Re$ flows verifiable today

- The idea of **spontaneous stochasticity** has some similarity but is different from the other attempts to introduce stochastic elements at infinite $Re$**:**

  - measure-valued solutions of the **Euler equation** of **DiPerna-Majda** CMP **108** (1987)

  - generalized flows of **Brenier** J. AMS **2** (1989)

  - multiphase and sticky generalized flows of **Shnirelman** CMP **210** (2000)

  The main difference is that the stochasticity concerns only the Lagrangian flow in deterministic rough velocities, so the ODE rather than the PDE aspect

  On the other hand, the applications of the idea to nonlinear PDE remain on the heuristic or conjectural level

- **Lagrangian flow**

  - For a **smooth** velocity field $\boldsymbol{v}(t,\boldsymbol{r})$ the Lagrangian flow $(s,\boldsymbol{r}) \mapsto \boldsymbol{R}_{t,\boldsymbol{r}}(s)$ is defined by the ODE

$$\frac{d\boldsymbol{R}_{t,\boldsymbol{r}}(s)}{ds} = \boldsymbol{v}(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))\,, \qquad \boldsymbol{R}_{t,\boldsymbol{r}}(t) = \boldsymbol{r}$$

  - For a **rough** (non-**Lipschitz**) in space velocity field $\boldsymbol{v}(t,\boldsymbol{r})$ some limiting procedure is required

    **1.** one can consider a noisy Lagrangian flow solving the SDE

$$d\boldsymbol{R}^{\kappa}_{t,\boldsymbol{r}}(s) = \boldsymbol{v}(s,\boldsymbol{R}^{\kappa}_{t,\boldsymbol{r}}(s))\,ds + \sqrt{2\kappa}\,d\underbrace{\boldsymbol{\beta}}_{\text{Brownian motion}}(s)\,, \qquad \boldsymbol{R}^{\kappa}_{t,\boldsymbol{r}}(t) = \boldsymbol{r}$$

    that exists for $\kappa > 0$ even for rough (e.g. spatially **Hölder**) velocities and has transition probabilities

$$P^{\kappa}_{t,\boldsymbol{r}}(s,\boldsymbol{R}\,|\,\boldsymbol{v}) = \mathbb{E}\;\delta(\boldsymbol{R}-\boldsymbol{R}^{\kappa}_{t,\boldsymbol{r}}(s))$$

    **2.** one can use velocity $\boldsymbol{v}_\eta(t,\boldsymbol{r})$ smoothed on a small scale $\eta$ and consider the usual Lagrangian flow $\boldsymbol{R}^{\eta}_{t,\boldsymbol{r}}(s)$ for $\boldsymbol{v}_\eta$ setting

$$P^{\eta}_{t,\boldsymbol{r}}(s,\boldsymbol{R}\,|\,\boldsymbol{v}) = \delta(\boldsymbol{R}-\boldsymbol{R}^{\eta}_{t,\boldsymbol{r}}(s))$$

We shall say that the Lagrangian flow in velocity field $\boldsymbol{v}(t,\boldsymbol{r})$ is stochastic if one of the limits

$$P_{t,\boldsymbol{r}}(s,\boldsymbol{R}\,|\,\boldsymbol{v}) = \begin{cases} \lim\limits_{\kappa\to 0} P^{\kappa}_{t,\boldsymbol{r}}(s,\boldsymbol{R}\,|\,\boldsymbol{v}) \\ \lim\limits_{\eta\to 0} P^{\eta}_{t,\boldsymbol{r}}(s,\boldsymbol{R}\,|\,\boldsymbol{v}) \end{cases}$$

exists in a sufficiently weak sense but is not concentrated at a single $\boldsymbol{R} = \boldsymbol{R}_{t,\boldsymbol{r}}(s)$ for each $t,\boldsymbol{r},s$

Such a behavior could be only possible in non-**Lipschitz** velocities where there may be many solutions of the Lagrangian ODE with a common initial or final value

**Remarks.** **1.** Physically, $\kappa$ would represent the molecular diffusivity and $\eta$ the viscous Kolmogorov scale. The $\kappa\to 0$ limit corresponds to the vanishing **Prandtl number** $Pr = \frac{\nu}{\kappa}$ and the $\eta\to 0$ one to $Pr=\infty$

**2.** The two limits may be different pointing to the **Prandtl** number dependence of the limiting Lagrangian flow

- **Why should we expect such behavior in $Re=\infty$ velocities?**

  - The first indication comes from the 1926 **Richardson law** for the Lagrangian dispersion $\rho(s)=|\boldsymbol{R}_{0,\boldsymbol{r}}(s)-\boldsymbol{R}_{0,\boldsymbol{r}'}(s)|$

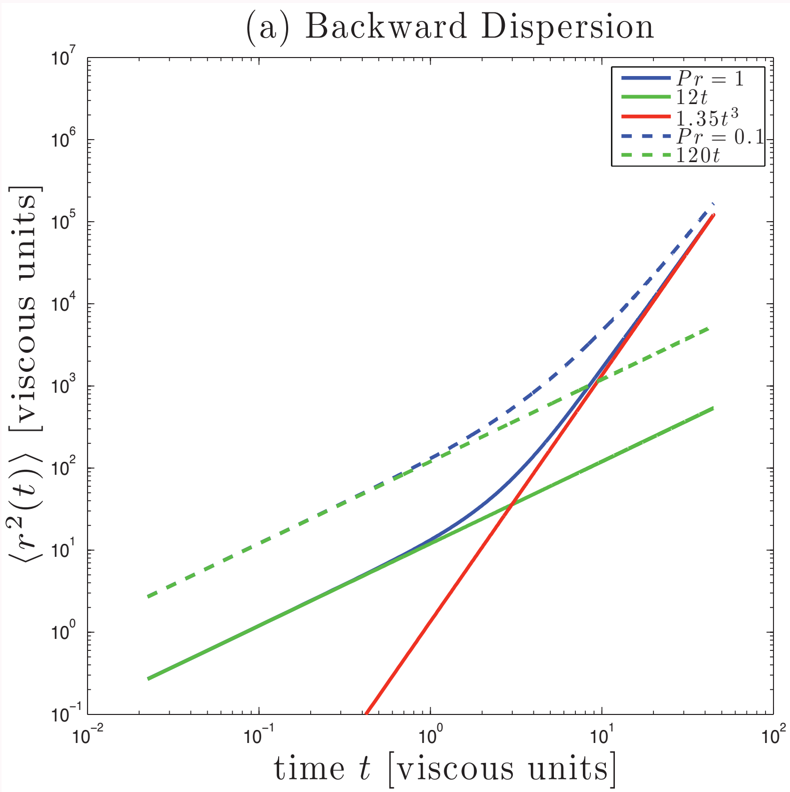
    $$\left\langle\rho(s)^2\right\rangle\,\propto\,\epsilon\,s^3$$

    for large $s$ with the coefficient independent of $\rho(0)$

    Extended to the smallest scales, this implies that Lagrangian trajectories starting arbitrarily close separate to $O(1)$ distance in finite time, a behavior impossible for trajectories determined by the initial point

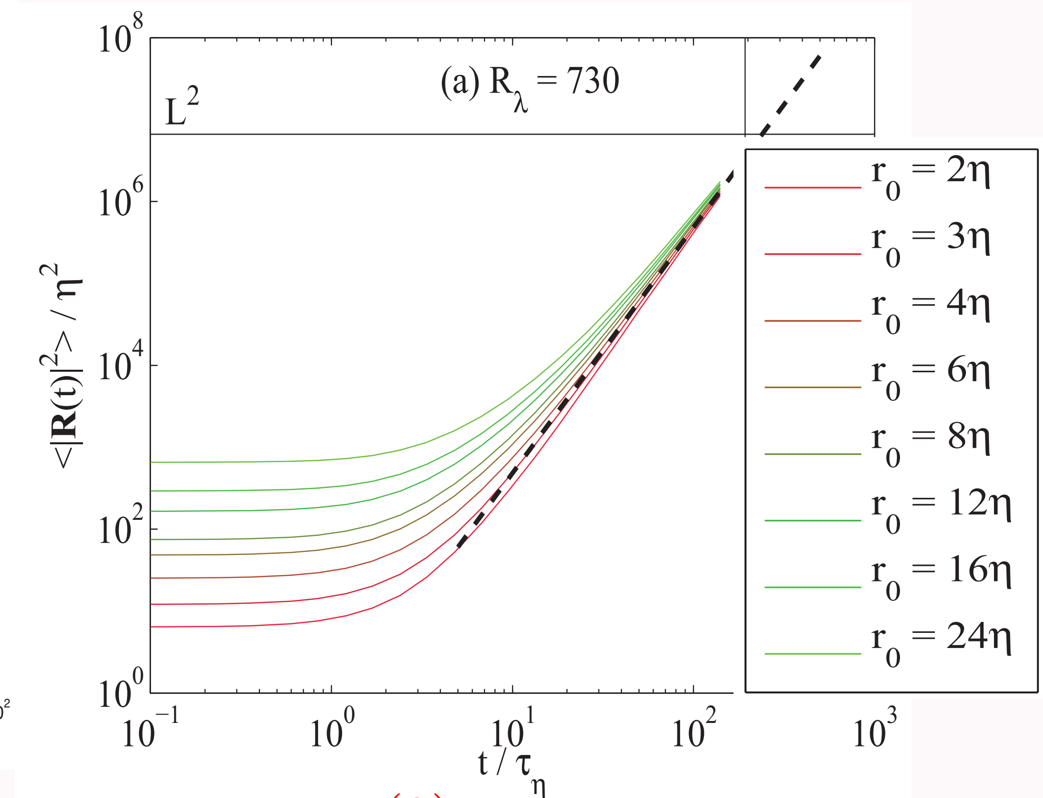
    $\big($In contrast, in chaotic dynamical systems $\rho(s)\propto\mathrm{e}^{\lambda s}\rho(0)$ for small $\rho$ taking longer and longer to separate the smaller $\rho(0)\big)$

  - For a recent discussions of the pair dispersion statistics in developed turbulence see **Thalabard** *et al.* JFM 755 R4 (2014) that questions the **Richardson** diffusion but not the $s^3$ finite-time separation




$\langle \rho(s)^2 \rangle$ at $Re_\lambda = 433$

from **Eyink** PR E 83 (2011)




$\rho(0)$-dependence

from **Bitane** *et al.* J. Turb. 14 (2013)

- Further indication came from the study of the Lagrangian flow in the **Kraichnan** homogeneous and isotropic **ensemble** of velocities that are **Hölder** in space but decorrelated in time

  This is a Gaussian ensemble with mean zero and $2$-point function

  $$\big\langle v^i(t,\boldsymbol{r})\, v^j(t',\boldsymbol{r}')\big\rangle \,=\, \big(D_0\delta^{ij} - D^{ij}(\boldsymbol{r}-\boldsymbol{r}')\big)\,\delta(t-t')$$

  with the isotropic tensor $D^{ij}(\boldsymbol{r})$ characterized by the **Hölder exponent** $\alpha$ and the **compressibility** $\wp$ with $0<\alpha,\wp<1$

  $$D^{ij}(\boldsymbol{r}) \,\propto\, r^{2\alpha} \quad \text{for small} \quad r\,, \qquad \wp = \frac{\partial_i\partial_j D^{ij}(\boldsymbol{r})}{\partial_i\partial_i D^{jj}(\boldsymbol{r})}$$

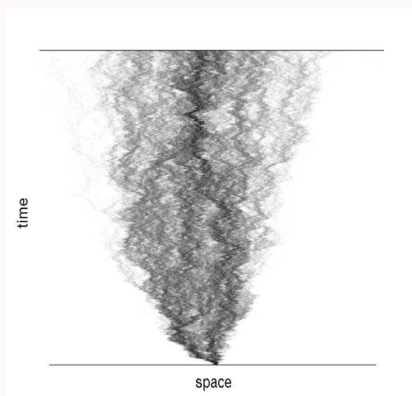
  It was observed in **Bernard**-**G.**-**Kupiainen** JSP 90 (1998) in the $\wp=0$ version of this model that the probability of the pair dispersion $P_{0,\rho_0}(s,\rho)$ has a non-zero $\rho_0\to 0$ limit leading to the **Richardson** type law

  $$\big\langle \rho(s)^2\big\rangle \,\propto\, s^{\frac{1}{1-\alpha}}$$

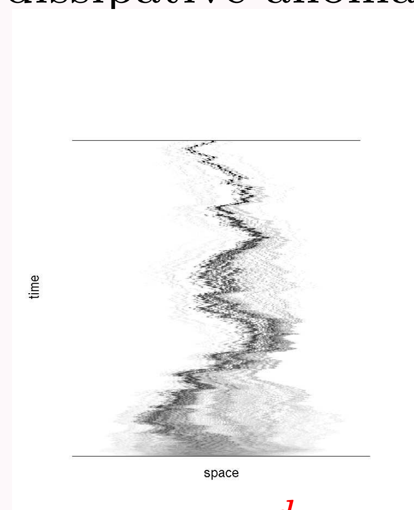
  that holds for all $s>0$ for $\rho_0=0$

As was stressed, that leads to the breakdown of deterministic Lagrangian flow in typical velocities and its replacement by a stochastic one, leading to a direct cascade of advected tracer field with a dissipative anomaly at $\kappa = 0$

It was then shown in **G.-Vergassola** Physica D 138 (2000) that such a behavior persists in **Kraichnan** velocities with $\wp < \frac{d}{4\alpha^2}$ whereas for $\wp > \frac{d}{4\alpha^2}$ one recovers a deterministic Lagrangian flow at $\kappa = 0$ with trajectories collapsing together in finite time, leading to an inverse cascade of advected tracer field without dissipative anomaly




$\wp < \frac{d}{4\alpha^2}$




$\wp > \frac{d}{4\alpha^2}$

Explosive separation of trajectories versus implosive collapse (from **Falkovich-G.-Vergassola** Rev. Mod. Phys. 73 (2001))

In **E - Vanden Eijnden** PNAS 97 (2000) and Physica D 152153 (2001) it was shown that for $\frac{d-2+2\alpha}{4\alpha} < \wp < \frac{d}{4\alpha^2}$ the $\kappa \to 0$ and $\eta \to 0$ Lagrangian flows differ, the first being stochastic whereas the second deterministic with collapsing trajectories

In **G. - Horvai** JSP 116 (2004) the $Pr$-dependence of the limiting flow was revisited and a "sticky" limiting Lagrangian flow was constructed for intermediate $\wp$ by fine-tuning $\kappa$ and $\eta$

In **Le Jan - Raimond** C. R. Acad. Sci. 327 (1998) the transition probabilities $P_{t,\boldsymbol{r}}(s, \boldsymbol{R} \,|\, \boldsymbol{v})$ were rigorously constructed and in Ann. Probab. 30 (2002) and 32 (2004) the Lagrangian flows in **Krachnan** velocities were connected to "nonclassical noises" and the relation between different limiting flows for intermediate $\wp$ was clarified

The **Kraichnan** model is time-reversal invariant unlike the real turbulence where the backward-in-time **Richardson** separation is considerably faster (**Sawford** *et al.* Phys. Fluids 17 (2005), **Eyink** PR E 83 (2011), **Buaria** *et al.* J. Fluid Mech. 799 (2016))

- **Lagrangian flow in rough velocities beyond Kraichnan model**

In **Chaves** *et al.* JSP 113 (2003) we analyzed the Lagrangian flow in Gaussian self-similar ensemble of **Hölder** velocities correlated in time arguing for scaling laws but without definitive results

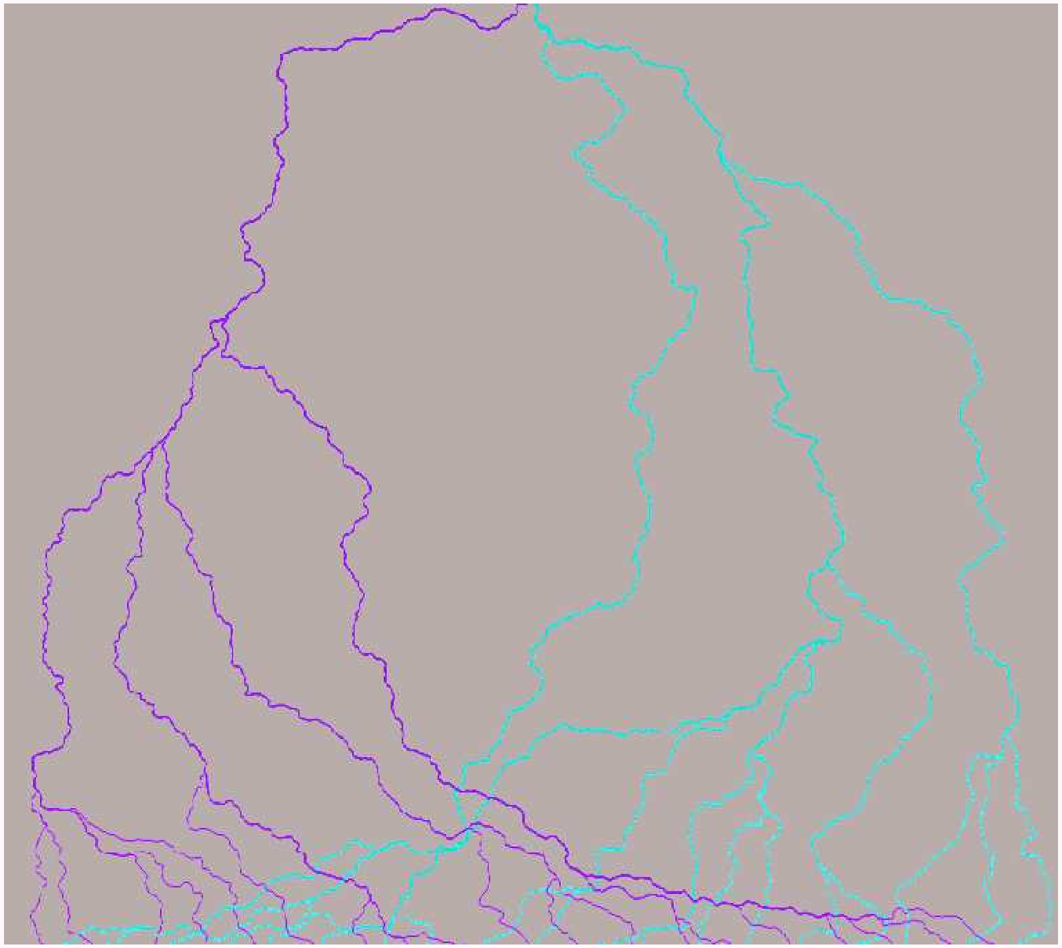
In **Eyink-Drivas** JSP 158 (2014) the backward-in-time Lagrangian flow for the **Burgers** velocities was shown to become stochastic when $\nu \to 0$ and $Pr$-dependent unlike the forward-in-time deterministic flow with trajectories coalescing onto the shocks

The same authors showed in arXiv:1509.04941 the appearance of stochastic Hamiltonian flows in the semiclassical limit of the $1d$ **Schrödinger** equation with a rough potential

The works of **Dubédat** (2009) and of **Miller-Sheffield** (2012-2013) studied the Lagrangian flow in **time-frozen** $2d$ compressible velocities

$$\boldsymbol{v}(\boldsymbol{r}) = \big(\cos(\tfrac{1}{\chi}\phi(\boldsymbol{r})+\theta), \sin(\tfrac{1}{\chi}\phi(\boldsymbol{r})+\theta)\big)$$

for $\chi > 0$, $0 < \theta \leq 2\pi$ and the random massless free field $\phi(\boldsymbol{r})$. Their flow (probably the $\eta \to 0$ one) is deterministic with trajectories looking locally like $\mathbf{SLE}_\kappa$-curves for $0 < \kappa \leq 4$ and $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ that collapse together when meeting



From **Miller-Sheffield** arXiv:1201.1496v2 [math.PR]

$\kappa = \frac{1}{2}$, blue lines: $\theta = \frac{\pi}{4}$, magenta lines: $\theta = -\frac{\pi}{4}$

- **Terminology**

  In **Bernard** *et al.* (1998) we talked about "intrinsically probabilistic character of the Lagrangian flow"

  I used the terms "breakdown of deterministic Lagrangian flow", "stochastic flow" and "fuzzy trajectories" in 1998 reviews

  **E-Vanden-Eijnden** (2000) coined the name "intrinsic stochasticity"

  **Le Jan-Raimond** (2002, 2004) talk about "statistical solutions" and "flow of kernels"

  We employed the term "spontaneous stochasticity" in the review **Falkovich** *et al.* RMP **73** (2001). The name was then repeatedly used in **Eyink**'s papers

  The deterministic flows with trajectories collapsing together are generally called "coalescent flows"

- **Lagrangian formulation of hydrodynamic equations**

  - Consider for a smooth velocity field $\boldsymbol{v}(t,\boldsymbol{r})$ with the Lagrangian flow $(s,\boldsymbol{r}) \mapsto \boldsymbol{R}_{t,\boldsymbol{r}}(s)$ the matrix

$$(W^i{}_j)_{t,\boldsymbol{r}}(s) \,=\, \frac{\partial R^i_{t,\boldsymbol{r}}(s)}{\partial r^j}$$

    that propagates the infinitesimal dispersion of Lagrangian trajectories $\delta R_{t,\boldsymbol{r}}(s) = W_{t,\boldsymbol{r}}(s)\,\delta\boldsymbol{r}$. It satisfies the ODE

$$\frac{dW_{t,\boldsymbol{r}}(s)}{ds} \,=\, (\boldsymbol{\nabla v})^T(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))\,W_{t,\boldsymbol{r}}(s)\,, \qquad W_{t,\boldsymbol{r}}(t) = I$$

    In smooth flows the eigenvalues of $\frac{1}{s}\ln W^T_{t,\boldsymbol{r}}W_{t,\boldsymbol{r}}(s)$ give when $s\to\infty$ the **Lyapunov** exponents of the flow and their large deviations encode the multifractal structure of advection, see **Grassberger** *et al.* JSP 51 (1988), **Bec** *et al.* PRL 92 (2004)

- The $3d$ hydrodynamic equations

$$\partial_t \theta + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta = 0$$ advection of scalar

$$\partial_t \boldsymbol{B} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) = 0$$ advection of magnetic field

$$\partial_t \boldsymbol{\omega} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = 0$$ Euler equation for vorticity

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} p$$ Euler equation for velocity

with $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ and with $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$ in the last two equations have equivalent Lagrangian reformulation: for any times $s,t$

$$\begin{aligned}
&\theta(s,R_{t,\boldsymbol{r}}(s)) = \theta(t,\boldsymbol{r})\\
&\det(W_{t.\boldsymbol{r}}(s))\,W_{t,\boldsymbol{r}}(s)^{-1}\boldsymbol{B}(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s)) = \boldsymbol{B}(t,\boldsymbol{r})\\
&W_{t,\boldsymbol{r}}(s)^{-1}\boldsymbol{\omega}(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s)) = \boldsymbol{\omega}(t,\boldsymbol{r})\\
&P(\boldsymbol{r})\Big(\boldsymbol{v}(s;\boldsymbol{R}_{t,\boldsymbol{r}}(s))\,W_{t,\boldsymbol{r}}(s)\Big) = \boldsymbol{v}(t,\boldsymbol{r})
\end{aligned}$$

where $P^i{}_j(\boldsymbol{r}) = \boldsymbol{\delta}^i_j - \frac{\nabla_{r^i}\nabla_{r^j}}{\boldsymbol{\nabla}^2_{\boldsymbol{r}}}$ is the transverse projector

**Proof** for the **Euler** equation for velocity**:**

$$\frac{d}{ds}\Big(\boldsymbol{v}(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))\,W_{t,\boldsymbol{r}}(s)\Big) \;=\; \Big((\partial_s\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\Big)(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))\,W_{t,\boldsymbol{r}}(s)$$
$$+\,\boldsymbol{v}(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))(\boldsymbol{\nabla}\boldsymbol{v})^T(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))\,W_{t,\boldsymbol{r}}(s)$$
$$=\;\Big(-(\boldsymbol{\nabla}p)(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s)) + \tfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{v}^2)(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))\Big)\,W_{t,\boldsymbol{r}}(s)$$
$$=\;\boldsymbol{\nabla}_{\boldsymbol{r}}\Big(-p(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s)) + \tfrac{1}{2}\boldsymbol{v}^2(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))\Big)$$

Upon applying $P(\boldsymbol{r})$ this gives

$$P(\boldsymbol{r})\Big(\boldsymbol{v}(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s))\,W_{t,\boldsymbol{r}}(s)\Big) \;=\; \text{const.}$$
$$=\;P(\boldsymbol{r})\Big(\boldsymbol{v}(t,\boldsymbol{R}_{t,\boldsymbol{r}}(t))\,W_{t,\boldsymbol{r}}(t)\Big) \;=\; \boldsymbol{v}(t,\boldsymbol{r})$$

$\blacksquare$

(The application of $\boldsymbol{\nabla}_{\boldsymbol{r}}\times$ instead of $P(\boldsymbol{r})$ would give **Cauchy**'s 1815/1827 formulation of the **Euler** equation, see **Frisch**-**Villone** Eur. Phys. J. 39 (2014))

- As a consequence of the Lagrangian formulation of the **Euler** equation

$$\int \boldsymbol{u}(\boldsymbol{r}) \cdot \Big(\boldsymbol{v}(s; \boldsymbol{R}_{t,\boldsymbol{r}}(s))\, W_{t,\boldsymbol{r}}(s)\Big)\, d\boldsymbol{r} \;=\; \int \boldsymbol{u}(r) \cdot \boldsymbol{v}(t, \boldsymbol{r})\, d\boldsymbol{r}$$

whenever $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$

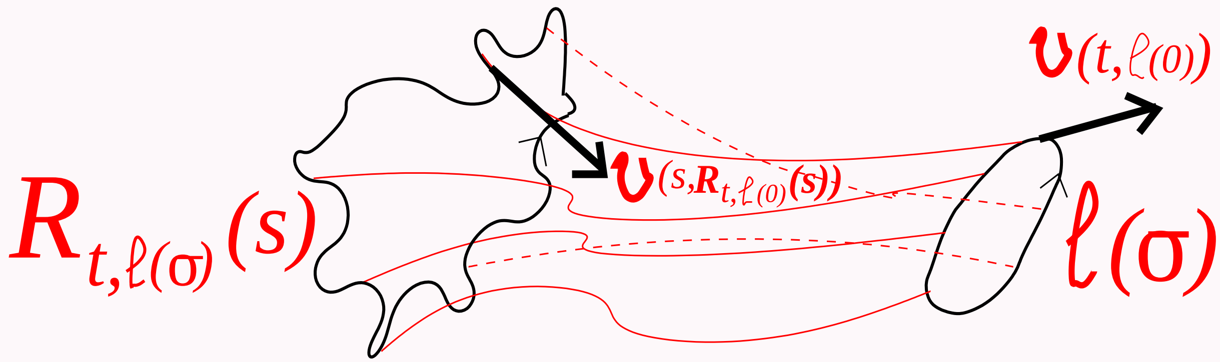
- Taking $\boldsymbol{u}(\boldsymbol{r}) = \int \delta(\boldsymbol{r} - \boldsymbol{\ell}(\sigma))\, \frac{d\boldsymbol{\ell}(\sigma)}{d\sigma}\, d\sigma$ for a closed loop $\sigma \longmapsto \boldsymbol{\ell}(\sigma)$ one gets the **Kelvin Theorem** on conservation of velocity circulation

$$\int \boldsymbol{v}(s, R_{t,\boldsymbol{\ell}(\sigma)}(s)) \cdot \frac{d\boldsymbol{R}_{s,\boldsymbol{\ell}(\sigma)}(s)}{d\sigma}\, d\sigma \;=\; \int \boldsymbol{v}(t, \boldsymbol{\ell}(\sigma)) \cdot \frac{d\boldsymbol{\ell}(\sigma)}{d\sigma}\, d\sigma$$

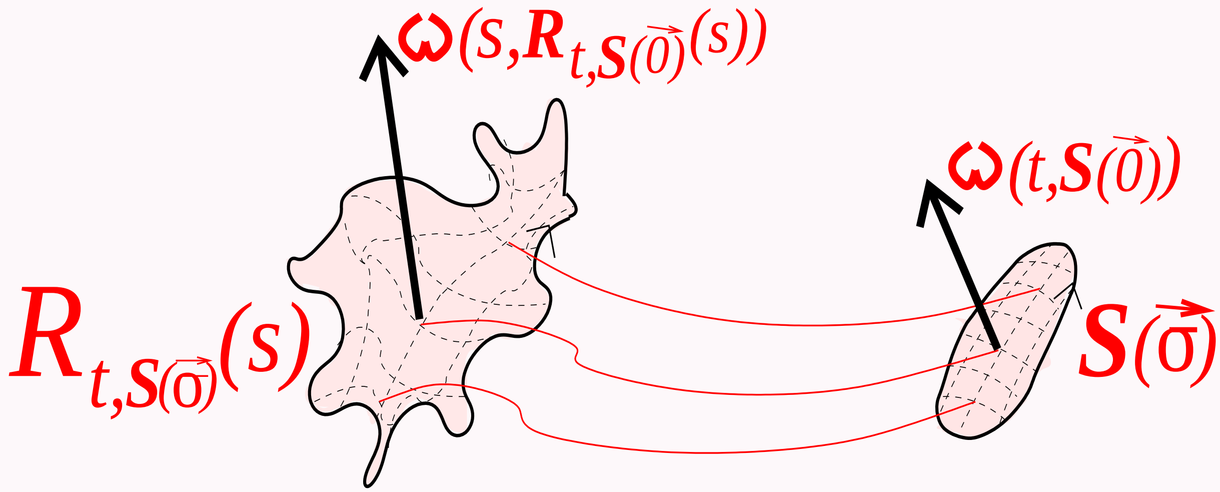
that may be also rewritten in **Helmholtz**'s form as a conservation of the flux of vorticity across a surface $(\sigma^1, \sigma^2) \longmapsto \boldsymbol{S}(\sigma^1, \sigma^2)$ with $\partial \boldsymbol{S} = \boldsymbol{\ell}$

$$\begin{aligned} &\int \boldsymbol{\omega}(s, \boldsymbol{R}_{t,\boldsymbol{S}(\vec{\sigma})}) \cdot \Big(\frac{\partial \boldsymbol{R}_{t,\boldsymbol{S}(\vec{\sigma})}(s)}{\partial \sigma^1} \times \frac{\partial \boldsymbol{R}_{t,\boldsymbol{S}(\vec{\sigma})}(s)}{\partial \sigma^2}\Big)\, d\vec{\sigma} \\ &= \int \boldsymbol{\omega}(t, \boldsymbol{S}(\vec{\sigma})) \cdot \Big(\frac{\partial \boldsymbol{S}(\vec{\sigma})}{\partial \sigma^1} \times \frac{\partial \boldsymbol{S}(\vec{\sigma})}{\partial \sigma^2}\Big)\, d\vec{\sigma} \end{aligned}$$

with its magnetic field counterpart where $\boldsymbol{\omega}$ is replaced by $\boldsymbol{B}$ called the **Alfvén Theorem**




Backward in time Lagrangian evolution of a contour




Backward in time Lagrangian evolution of a surface

- **Probabilistic Lagrangian interpretation of dissipative hydrodynamic equations**

$$\partial_t \theta + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta - \kappa\boldsymbol{\nabla}^2\theta = 0$$ advection−diffusion of scalar

$$\partial_t \boldsymbol{B} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) - \kappa\boldsymbol{\nabla}^2\boldsymbol{B} = 0$$ advection−diffusion of magnetic field

$$\partial_t \boldsymbol{\omega} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) - \nu\boldsymbol{\nabla}^2\boldsymbol{\omega} = 0$$ Navier−Stokes equation for vorticity

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \nu\boldsymbol{\nabla}^2\boldsymbol{v} = -\boldsymbol{\nabla}p$$ Navier−Stokes equation for velocity

also have equivalent Lagrangian reformulation but with the stochastic Lagrangian flows

$$d\boldsymbol{R}_{t,\boldsymbol{r}}(s) = \boldsymbol{v}(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s)\,ds + \left\{\begin{matrix}\sqrt{2\kappa}\\ \sqrt{2\nu}\end{matrix}\right\} d\underset{\text{Brownian motion}}{\boldsymbol{\beta}}(s)\,, \qquad \boldsymbol{R}_{t,\boldsymbol{r}}(t) = \boldsymbol{r}$$

with the same equation as before for $W_{t,\boldsymbol{r}}(s)$ but $\big(\boldsymbol{R}_{t,\boldsymbol{r}}(s), W_{t,\boldsymbol{r}}(s)\big)$ becoming a random process

- The Lagrangian equivalent of the hydrodynamic equations with dissipation takes now the form: for any $s \leq t$

$$
\begin{aligned}
&\mathbb{E}\ \theta(s, \boldsymbol{R}_{t,\boldsymbol{r}}(s)) \ = \ \theta(t, \boldsymbol{r}) \\
&\mathbb{E}\ \det(W_{t.\boldsymbol{r}}(s))\, W_{t,\boldsymbol{r}}(s)^{-1} \boldsymbol{B}(s, \boldsymbol{R}_{t,\boldsymbol{r}}(s)) \ = \ \boldsymbol{B}(t, \boldsymbol{r}) \\
&\mathbb{E}\ W_{t,\boldsymbol{r}}(s)^{-1} \boldsymbol{\omega}(s, \boldsymbol{R}_{t,\boldsymbol{r}}(s)) \ = \ \boldsymbol{\omega}(t, \boldsymbol{r}) \\
&\mathbb{E}\ P(\boldsymbol{r})\Big(\boldsymbol{v}(s; \boldsymbol{R}_{t,\boldsymbol{r}}(s))\, W_{t,\boldsymbol{r}}(s)\Big) \ = \ \boldsymbol{v}(t, \boldsymbol{r})
\end{aligned}
$$

  where $\mathbb{E}$ denotes the expectation with respect to the Brownian noise

- Note that the dissipation fixes the time direction

- The last reformulation of the Navier-Stokes equations is due to **Constantin-Iyer** (2008) and it is equivalent to the **Stochastic Kelvin Thm** (**Eyink** 2008)

$$
\mathbb{E} \int \boldsymbol{v}(s, R_{t,\boldsymbol{\ell}(\sigma)}(s)) \cdot \frac{d\boldsymbol{R}_{s,\ell(\sigma)}(s)}{d\sigma}\, d\sigma \ = \ \int \boldsymbol{v}(t, \boldsymbol{\ell}(\sigma)) \cdot \frac{d\boldsymbol{\ell}(\sigma)}{d\sigma}\, d\sigma
$$

or its **Helmholtz**'s form

$$\mathbb{E} \int \boldsymbol{\omega}(s, \boldsymbol{R}_{t,\boldsymbol{S}(\vec{\sigma})}) \cdot \Big( \frac{\partial \boldsymbol{R}_{t,\boldsymbol{S}(\vec{\sigma})}(s)}{\partial \sigma^1} \times \frac{\partial \boldsymbol{R}_{t,\boldsymbol{S}(\vec{\sigma})}(s)}{\partial \sigma^2} \Big) \, d\vec{\sigma}$$
$$= \int \boldsymbol{\omega}(t, \boldsymbol{S}(\vec{\sigma})) \cdot \Big( \frac{\partial \boldsymbol{S}(\vec{\sigma})}{\partial \sigma^1} \times \frac{\partial \boldsymbol{S}(\vec{\sigma})}{\partial \sigma^2} \Big) \, d\vec{\sigma}$$

(and similarly for the magnetic field for which one obtains the **Stochastic Alfvén Thm**)

- One may also rephrase the Lagrangian formulation of the dissipative hydrodynamic equations as the condition that the random processes parameterized by time $s \leq t$

$$\theta(s, \boldsymbol{R}_{t,\boldsymbol{r}}(s))$$
$$\det(W_{t.\boldsymbol{r}}(s)) \, W_{t,\boldsymbol{r}}(s)^{-1} \boldsymbol{B}(s, \boldsymbol{R}_{t,\boldsymbol{r}}(s))$$
$$W_{t,\boldsymbol{r}}(s)^{-1} \boldsymbol{\omega}(s, \boldsymbol{R}_{t,\boldsymbol{r}}(s))$$
$$\int \boldsymbol{v}(s, \boldsymbol{R}_{t,\boldsymbol{\ell}(\sigma)}(s)) \cdot \frac{d\boldsymbol{R}_{s,\boldsymbol{\ell}(\sigma)}(s)}{d\sigma} \, d\sigma$$

are backward-in-time **martingales** for all $t$, $\boldsymbol{r}$ and all loops $\boldsymbol{\ell}(\sigma)$

- **Sources** are straightforward to include. E.g. for the scalar advection-diffusion

$$\partial_t\theta + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta - \kappa\boldsymbol{\nabla}^2\theta \;=\; g$$

one has

$$\mathbb{E}\;\; \theta(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s)) + \int_s^t g(\tau,\boldsymbol{R}_{t,\boldsymbol{r}}(\tau))\,d\tau \;=\; \theta(t,\boldsymbol{r})$$

and it is

$$\theta(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s)) + \int_s^t g(\tau,\boldsymbol{R}_{t,\boldsymbol{r}}(\tau))\,d\tau$$

which is a backward-in-time **martingale**, i.e. for $s < s' < t$

$$\begin{aligned}
&\mathbb{E}_{[s,s']}\;\; \theta(s,\boldsymbol{R}_{t,\boldsymbol{r}}(s)) \;+\; \int_s^t g(\tau,\boldsymbol{R}_{t,\boldsymbol{r}}(\tau)) \\
&=\; \mathbb{E}_{[s,s']}\;\; \theta(s,\boldsymbol{R}_{s',\boldsymbol{R}_{t,\boldsymbol{r}}(s')}(s)) + \int_s^{s'} g(\tau,\boldsymbol{R}_{s',\boldsymbol{R}_{t,\boldsymbol{r}}(s')}(\tau))\,d\tau \\
&\qquad\qquad\qquad\qquad\qquad + \int_{s'}^t g(\tau,\boldsymbol{R}_{t,\boldsymbol{r}}(\tau))\,d\tau \\
&=\; \theta(s',\boldsymbol{R}_{t,\boldsymbol{r}}(s')) + \int_{s'}^t g(\tau,\boldsymbol{R}_{t,\boldsymbol{r}}(\tau))\,d\tau
\end{aligned}$$

## Spontaneous stochasticity and hydrodynamic equations

- It was stressed in **Eyink** CRAS 7 (2006) (with some numerical evidence in **Chen** *et al.* PRL 97 (2006)) that the spontaneous stochasticity of the Lagrangian flow in the limit $\nu \to 0$ of the **Navier-Stokes** turbulence is compatible with the stochastic **Kelvin-Helmholtz** Thm

  In **Eyink** PR E 83 (2011) a similar picture for the MHD in the limit $\kappa, \nu \to 0$ with persistent stochastic **Alfvén** Thm was developed and the spontaneous stochasticity of the Lagrangian flow was subsequently employed by **Eyink** *et al.* (see e.g. Nature 497 (2013)) as the corner-stone of the magnetic reconnection mechanism in astrophysical plasma

- This requires, however, an extension of the spontaneous stochasticity scenario where not only $R_{t,\boldsymbol{r}}(s)$ but also $W_{s,\boldsymbol{r}}(s)$ would make some stochastic sense in rough turbulent velocities

  We would need that some limiting procedure may be applied to the matrix-valued kernels that enter the probabilistic Lagrangian formulation of the hydrodynamic equations allowing to define

$$\lim_{\kappa,\eta\to 0} \mathbb{E}\; W^{\kappa,\eta}_{t,\boldsymbol{r}}(s)\,\delta(\boldsymbol{R}-R^{\kappa,\eta}_{t,\boldsymbol{r}}(s)) \;\equiv\; F_{t,\boldsymbol{r}}(s,\boldsymbol{R}\,|\,\boldsymbol{v})$$

$$\lim_{\kappa,\eta\to 0} \mathbb{E}\;\det(W^{\kappa,\eta}_{t,\boldsymbol{r}}(s))\,W^{\kappa,\eta}_{t,\boldsymbol{r}}(s)^{-1})\,\delta(\boldsymbol{R}-R^{\kappa,\eta}_{t,\boldsymbol{r}}(s)) \;\equiv\; G_{t,\boldsymbol{r}}(s,\boldsymbol{R})$$

in the velocity-ensemble correlations

The limiting kernels should satisfy the composition laws, the time reversal $G_{t,\boldsymbol{r}}(s,\boldsymbol{R}) = F_{s,\boldsymbol{R}}(t,\boldsymbol{r})$ and satisfy, at least weakly, the relation

$$\partial_{R^i} F_{t,\boldsymbol{r}}(s,\boldsymbol{R}\,|\,\boldsymbol{v})^i{}_j \;=\; -\,\partial_{r^j} P_{s,\boldsymbol{r}}(t,\boldsymbol{R}\,|\,\boldsymbol{v})$$

- In the incompressible **Kraichnan** model the 2-point expectations of such limiting kernels exist and were analyzed in **Celani** *et al.* Proc. R. Soc. A 462 (2006), **Arponen-Horvai** JSP 129 (2007) and in **Eyink-Neto** New J. Phys. 12 (2009) in the context of kinematic dynamo studies initiated by **Kazantsev** JETP 26 (1968) followed by **Vergassola** PR E 53 (1996)

- A more complete theory of such random kernels remains to be constructed even in the **Kraichnan** model

- Some numerical data on the 2-point function of the kernels $G$ in high $Re$ flows may be found in **Eyink** PR E 83 (2011)

- **Conlusions**

- If the stochastic Lagrangian formulation of dissipative hydrodynamic equations is better suited to control their $Re \to \infty$ or $Re, Re_m \to \infty$ limits then the **spontaneous stochasticity** of the Lagrangian flows in such limits may be an essential element of such a control

- The signs of **spontaneous stochasticity** seem visible in numerical simulations of high $Re$ turbulence showing the **Richardson**-type separation of Lagrangian trajectories or the persistence of the stochastic versions of the **Kelvin-Helmholtz** or **Alfvén** laws

- Admittingly, the evidence for the **spontaneous stochasticity** from the high $Re$ flows is not yet overwhelming and calls for further numerical tests

- The **spontaneous stochasticity** requiring the roughness of typical velocities in the $Re \to \infty$ limit should be responsible for the dissipative anomaly assuring the energy dissipation in the inviscid limit

- Even in the **Kraichnan** model, where the **spontaneous stochasticity** has been rigorously established some work remains to be done that could throw more light on this phenomenon

- Some more details on the Lagrangian flows in the **Kraichnan** model are discussed in a reader-friendly fashion in:

  **G.** "Turbulent advection and breakdown of the Lagrangian flow", in *Turbulent Flows*, ed. J. C. Vassilicos, Cambridge University Press 2001, pp. 86-104

"Spontaneity is a meticulously prepared art"

*Oscar Wilde*