Finite time blowup constructions for modifications of the Navier-Stokes and Euler equations

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The Navier-Stokes equations for viscous, incompressible three-dimensional fluid flow are given as

$$i\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p$$
$$\nabla \cdot u = 0$$

where $u : [0, T) \times \mathbf{R}^3 \to \mathbf{R}^3$ is the velocity field, $p : [0, T) \times \mathbf{R}^3 \to \mathbf{R}$ is the pressure field, and $\nu > 0$ is the viscosity. The Euler equations are the limiting inviscid case $\nu = 0$ of the Navier-Stokes equations.

- We have the infamous global regularity problem for the Navier-Stokes equations: given smooth, compactly supported, divergence-free initial data $u_0 : \mathbf{R}^3 \to \mathbf{R}^3$, does there exist a global smooth solution $u : [0, +\infty) \times \mathbf{R}^3 \to \mathbf{R}^3$ (and $p : [0, +\infty) \times \mathbf{R}^3 \to \mathbf{R}$) of the Navier-Stokes equations (with finite energy, to avoid pathologies at spatial infinity)?
- One can also ask the same question for the Euler equations, although the answer here is widely believed to be negative; the Navier-Stokes equation should be better behaved due to dissipative effects.

In both the Navier-Stokes and Euler equations, local existence is known, and if global regularity fails, there must be blowup at some finite maximal time of (smooth) existence $0 < T_* < +\infty$. Many blowup criteria are known, we list just two:

- (Escuriaza-Seregin-Sverak 2003) If global regularity fails for a solution to Navier-Stokes, then $\limsup_{t \to T_*^-} \|u(t)\|_{L^3(\mathbf{R}^3)} = +\infty$.
- (Beale-Kato-Majda 1984) If global regularity fails for a solution to Euler, then $\int_0^{T_*} \|u(t)\|_{L^\infty(\mathbf{R}^3)}\, dt = +\infty$.

Solutions to both the Navier-Stokes and Euler equations enjoy the energy identity

$$\|u(T)\|_{L^2(\mathbf{R}^3)}^2 + 2\nu \int_0^T \|\nabla u(t)\|_{L^2(\mathbf{R}^3)}^2 \, dt = \|u(0)\|_{L^2(\mathbf{R}^3)}^2$$

which provides *a priori* control on the $L_t^\infty L_x^2$ norm of the velocity field. However, this is not enough to imply global regularity due to the supercritical nature of this bound.

This can be understood heuristically by scaling or dimensional analysis.

- Consider a solution u to Navier-Stokes with viscosity $\nu = 1$. At a given time t, suppose that the solution has "amplitude" $A = A(t)$ and "frequency" $N = N(t)$. Then the dissipative term $\nu \Delta u$ would be expected to be of size AN^2, while the transport term $(u \cdot \nabla)u$ would be expected to be of size $A^2 N$.
- Thus: if $A \ll N$, we expect dissipation to dominate and the behaviour to be like the linear heat equation; but if $A \gg N$ we expect nonlinear behaviour.

On the other hand, the uncertainty principle tells us that the energy $\frac{1}{2}\|u(t)\|_2^2$ of the solution should be roughly A^2/N^3 or larger (because the solution must be large on at least a ball of radius $\sim 1/N$). So the energy inequality gives the bound

$$A \ll N^{3/2}.$$

This does not prevent the nonlinear regime $A \gg N$ from occurring. If A is as large as $N^{3/2}$, one could conceivably double the amplitude in time as short as $\frac{A}{A^2 N} \sim N^{-5/2}$, which on iteration could lead to finite time blowup.

This suggests a “maximally nonlinear” blowup scenario for Navier-Stokes:

- At a given time t, the solution u has a large frequency $N(t)$, is concentrated in a ball at the dual spatial scale $1/N(t)$ with amplitude $\sim N(t)^{3/2}$.
- In time $\sim N(t)^{-5/2}$, the solution compresses its energy into a ball of half the scale $1/2N(t)$, twice the frequency $2N(t)$, and with amplitude $\sim (2N(t))^{3/2}$.
- This process iterates, and the time steps form a convergent geometric series, to obtain finite time blowup.

- The precise scenario described on the previous slide cannot *quite* happen, due to the non-zero energy dissipation at each stage of the process. However, approximations to this scenario in which the amplitude is $\sim N(t)^{3/2-\varepsilon}$ rather than $\sim N(t)^{3/2}$ (and the time between stages comparable to $N(t)^{-5/2+\varepsilon}$ rather than $N(t)^{-5/2}$ are not yet ruled out.
- This would be a near-maximally efficient cascade of energy from low frequency modes to high frequency modes.

Of course, it is still open whether the Navier-Stokes equations admit this sort of finite time blowup scenario. However, we have

Theorem (T., 2015), informal version

There exists an “averaged” Navier-Stokes equation that obeys the energy identity and admits smooth solutions that blow up in finite time, basically according to the scenario just described.

Previous work of Montgomery-Smith, Chemin-Gallagher-Paicu, Li-Sinai obtained similar results but without energy conservation.

What “averaged” means is a bit technical. By applying the Leray projection P to divergence-free vector fields, one can rewrite the Navier-Stokes equation as

$$\partial_t u = B(u,u) + \nu \Delta u$$

where $B(u,u) := -P((u \cdot \nabla)u)$ is a certain bilinear operator.

The averaged equation takes the form

$$\partial_t u = \tilde{B}(u, u) + \nu \Delta u$$

where $\tilde{B}(u, u)$ is an average of operators of the form $T_1 B(T_2 u, T_3 u)$, where T_1, T_2, T_3 are "mild" operators such as rotations, dilations by a factor close to 1, or Fourier multipliers of order 0. The operator $\tilde{B}$ obeys almost all the function space estimates that B does, and as such inherits almost all of the known local existence theory.

- The specific averaged operator $\tilde{B}$ used to prove the above theorem is "programmed" to resemble a shell model for Navier-Stokes introduced by Katz and Pavlovic in 2005.
- At a given time t, the finite time blowup solution is mostly concentrated in a ball of radius $1/N(t)$ with amplitude $N(t)^{3/2-\varepsilon}$, is largely stable for time $\sim N(t)^{-5/2+\varepsilon}$, and then suddenly transfers almost all of its energy to the next frequency scale $2N(t)$, concentrating to a smaller ball of radius $1/2N(t)$.
- This process iterates to obtain finite time blowup. The dissipative effects of viscosity are negligible in this scenario.

The averaged Navier-Stokes equation is highly non-physical; while it obeys the energy identity and the function space estimates of the true Navier-Stokes equation, it does not have any analogue of the vorticity equation

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega$$

that the true Navier-Stokes equation satisfies.

- In the case of the Euler equation, the vorticity equation

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u$$

implies many further properties of the evolution: Kelvin's circulation theorem, conservation of helicity, transport of vortex lines, conservation of impulse, etc.

- Are all these properties sufficient to prevent blowup of the Euler equations?

It appears not:

Theorem (T., 2016), informal version

There exists a “modified” Euler equation obeying the energy identity, vorticity equation, circulation theorem, conservation of helicity, incompressibility, and transport of vortex lines, and has the same “scaling strength” as Euler, but which admits smooth solutions that blow up in finite time.

Blowup for other models of Euler equations have been obtained by Constantin-Lax-Majda, Katz-Pavlovic, Hou-Lei, Hou-Shi-Wang, and others, but this appears to be the first such model that is incompressible and obeys a vorticity equation.

What modifications are made to the Euler equation?
In vorticity-stream form, the Euler equations form a system

$$\partial_t\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u$$
$$u = B\omega$$

where $B := -\Delta^{-1}(\nabla\times)$ is the Biot-Savart operator (a Fourier multiplier of order -1).

The modification takes the form

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u$$
$$u = \tilde{B}\omega$$

where $\tilde{B}$ is an (exotic) self-adjoint pseudodifferential operator of order -1 taking values in divergence-free vector fields.

- The requirement that the modified Biot-Savart operator $\tilde{B}$ be self-adjoint is needed in order to have an energy conservation law. Without it, blowup is significantly easier to establish.
- A toy non-self-adjoint model is the one-dimensional active scalar system

$$\partial_t \theta + u \partial_x \theta = 0$$
$$u(t,x) = -\theta(t,2x)$$

which resembles Burgers' equation, but with a non-local (and non-self-adjoint) dependence of the velocity u on the active scalar θ.

$$\partial_t\theta + u\partial_x\theta = 0$$
$$u(t,x) = -\theta(t,2x)$$

- If one has $\theta(0,x) = 1$ for $x \geq 1$ and $\theta(0,0) = 0$, then this equation must blow up by time 1, basically because the level set $\{x : \theta(t,x) = 1\}$ moves leftward at speed at least one, while $\theta(t,0)$ must vanish for all time, causing a collision of characteristics.
- This can be made rigorous by a barrier argument.

- It turns out that a similar barrier argument gives finite time blowup for modification

$$\partial_t\theta + (u \cdot \nabla)\theta = 0$$
$$u = A\theta.$$

of the two-dimensional surface quasi-geostrophic (SQG) equation. Here A is an (exotic) non-self-adjoint pseudo-differential operator of order -1.

- The blowup solution resembles a "patch" solution of SQG where the level set $\{\theta = 1\}$ develops a "corner" singularity in finite time.

- There is an embedding trick using a “two-and-a-half dimensional ansatz” that allows one to encode two-dimensional solutions of non-self-adjoint SQG type equations into three-dimensional solutions of self-adjoint Euler type equations, which will prove the theorem.
- The solutions constructed are expected to blow up on either a line or a circle (depending on whether one uses an embedding coming from Cartesian coordinates or cylindrical coordinates).
- Unfortunately, such blowups cannot occur for Navier-Stokes equations (due to the famous partial regularity results of Caffarelli, Kohn, and Nirenberg).

Thanks for listening!