Weak vorticity formulation for incompressible 2D Euler in domains with boundary

Helena J. Nussenzveig Lopes

Instituto de Matemática
Univ. Fed. Rio de Janeiro (UFRJ)
BRASIL

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Collaborators:

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Dragoş Iftimie (U. Lyon I)

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Milton C. Lopes Filho (UFRJ)

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Dragoş Iftimie (U. Lyon I)

Milton C. Lopes Filho (UFRJ)

Franck Sueur (U. Bordeaux)

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OBS. Actually, iff.

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Mechanism to generate small scales: ∂ layer + Kelvin-Helmholtz instability

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Mechanism to generate small scales: ∂ layer + Kelvin-Helmholtz instability

Seek **framework** for *weak solutions* of 2D Euler, in domains with (rigid) boundary, vortex sheet regularity, **which allow tracking vorticity dynamics.**

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$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p, & \text{in } \Omega \times (0, \infty) \\ \text{div } u = 0, & \text{in } \Omega \times [0, \infty) \\ u(x,t) \cdot \hat{n}(x) = 0, & \text{on } \partial\Omega \times [0, \infty) \\ u(x,0) = u_0(x), & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{1}$$

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If u is a weak solution then possible to recover pressure $p \in L^\infty_{loc}((0,+\infty); \mathcal{D}'(\Omega))$.

Theorem (J.-M. Delort, JAMS, 1991)

Let $u_0 \in L^2_\sigma(\Omega)$ be such that $\omega_0 =$ curl $u_0 \in \mathcal{BM}_+(\Omega)$. Then there exists (at least one!) weak solution $u \in L^\infty_{loc}((0,\infty); L^2_\sigma(\Omega)) \cap C^0_{loc}([0,\infty); H^{-L}(\Omega))$ of (1) with initial velocity u_0.

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No tracking vortex dynamics or “conserved quantities".

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Question: what survives in (6) for flows with vortex sheet regularity?

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Lemma

Let $u \in L^1_{loc}(\Omega)$ such that $\omega = \text{ curl } u \in \mathcal{BM}(\Omega)$, bounded measure. Then the circulation of u around $\partial\Omega$ is well-defined through the formula:

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The proof of this proposition is not trivial.

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Use Delort's argument with insight from "Schochet-proof". Ingredients:

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- ω is the curl in the sense of *distributions*, hence no mass at boundary.

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Test functions for weak vorticity: φ such that $\nabla\varphi$ is compactly supported in space and time. I.e., φ constant *in neighborhood* of $\partial\Omega$.

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In Lopes-Filho-NL-Xin established existence of boundary coupled weak solution for half-plane. How? No mass going towards boundary (needed new *a priori* estimate).

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Set $m = m(t) = \mu(\partial\Omega)$.

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This *cannot* be controlled/excluded by *a priori* estimates!

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Similarly for torque:

$$\int_{\Gamma_j} p(x - \bar{x}_j)^\perp \cdot \hat{n} \, dS;$$

where $\bar{x}_j$ is the center of mass.

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