Stochastic (intermittent) Spikes and Strong Noise Limit of SDEs.

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Strong Noise Limit of (some) SDEs

— Stochastic differential equations (SDE) of the form:

\[
dX_t = \left( a(X_t) - \lambda^2 b(X_t) \right) dt + \lambda c(X_t) dB_t
\]

with \( a(0) > 0 \) and \( b(0) = 0 = c(0) \) with \( b(x) > 0 \) in a neighbourhood of the origin.

— We are interested in the limit of large lambda (strong noise limit)
  with an appropriate rescaling of \( a(x) \).

— Typical behaviour
  (extracted from monitoring phenomena of quantum systems, here monitored Rabi oscillations)

As the « noise » increases:
- Jumps (quantum jumps) emerge.
- A finer structure survives,
  in between the jumps: the spikes.
- Different from metastability in the weak noise limit of SDE. (Kramer’s or Freidlin & Wentzell’s like).
Strong Noise Limit of (some) SDEs (II)

— The examples we shall discuss:

\[ dX_t = \frac{\lambda^2}{2} (\epsilon - bX_t) dt + \lambda X_t dB_t \text{ with } \lambda \to \infty; \ \epsilon \to 0; \ \lambda^2 \epsilon^{b+1} =: J \text{ fixed} \]

— Claim: In the scaling limit, the solution of this SDE is a fractional power of a reflected Brownian motion parametrised by its local time:

\[ X_t = (b + 1)^{\frac{1}{b+1}} |W_{\tau}|^{\frac{1}{b+1}} \text{ with } J t = 2\Gamma(b + 1) L_\tau, \]

with \( L \) the local time at the origin of the Brownian motion \( W \).

— A geometrical construction (for \( b=0 \)):

A sample of a reflected Brownian motion parametrised with its natural time.

The same example but parametrised by its local time spend at 0.
A Motivation: Quantum Trajectories (I).

— Continuous monitoring of a quantum system (say via interaction with series of probes):
  - extract (random) information.
  - modify (back-action) the Q-state.

\[ d\rho_t = -i[H,\rho_t]dt + \lambda^2 L(\rho_t)dt + \lambda D(\rho_t) dB_t \]

Many examples/realisations: QED, circuit QED, Q-dots, etc...

— Quantum system evolution equation is stochastic:

Shroedinger like deterministic evolution

Measurement back-action scaled with the information rate.

\[ \rightarrow B_t \] is a Brownian motion, a «informative» noise linked to the readout signal.
A Motivation: Quantum Trajectories (II)

— An example: Energy monitoring of a Qu-bit in contact with a thermal bath.

\[ dP_t = \eta (p - P_t)dt + \lambda P_t (1 - P_t)dB_t \]

Thermal relaxation

\[ \text{Effect of monitoring (back-action)} \]

\[ \text{\{competing processes\}} \]

smooth thermal relaxation at small information rate.

jumpy behaviour between energy eigenstates (thermally activated quantum jumps) at high information rate.

— Linearisation (near \( P=0 \)):

\[ dP_t = \eta p dt + \lambda P_t dB_t \]

(i.e. the case \( b=0 \)).
**Strong Noise Limit: Strategy of Analysis/Proof**

— Recall the SDEs we are looking at:

\[ dX_t = \frac{\lambda^2}{2} (\epsilon - bX_t) dt + \lambda X_t \, dB_t \]

with \( \lambda \to \infty; \lambda^2 \epsilon^{b+1} =: J \) fixed

This can be solved explicitly, and \( X \to 0 \) almost surely (at fixed time \( T \)) but there are `spiky times'.

— Strategy of analysis (alias proof):

(i) Show that the tips of the spikes form a point Poisson process.

(ii) Redefine times (via quadratic variation) to decipher the spike structure.

— Remark: change variable and set \( Q_t := \frac{X_t^{b+1}}{b+1} \) then

\[ dQ_t = \frac{\lambda^2}{2} \hat{\epsilon} Q_t^{\frac{b+1}{b+1}} dt + \hat{\lambda} Q_t \, dB_t \]

Not smooth, but nevertheless trajectories well defined (because \( X>0 \)).
Passage Times and a Point Poisson Process (I)

— Two competing effects in: \( dX_t = \frac{\lambda^2}{2} (\epsilon - bX_t)dt + \lambda X_t dB_t \)

— Why spikes survives in the limit?

Look at the distribution of the X-maxima in a time interval T:

\[
\mathbb{P}[X - \text{max on } [0, T] < m] \simeq \left[ \int_0^m dx P_{\text{inv}}(x) \right] \lambda^2 T \simeq \left[ 1 - c \frac{\epsilon^{b+1}}{m^{b+1}} \right] \lambda^2 T \simeq e^{-c(\lambda^2 \epsilon^{b+1}) T/m^{b+1}}.
\]

using/assuming independency after time scale of order \( \lambda^{-2} \).

— Claim: The tips of the spikes form a point Poisson process with intensity

\[
d\nu = \hat{J} dt \frac{dq(x)}{q(x)^2} = \frac{(b + 1)^2}{2\Gamma(b + 1)} J dt \frac{dx}{x^{b+2}}
\]

with \( q(x) = x^{b+1}/(b + 1) \)
Passage Times and a Point Poisson Process (II)

The argument consisting in reconstructing the transition/passage time distribution from the point Poisson process.

Passage times $T_{y \rightarrow z}$ from $y$ to $z>y$, and their generating function:

$$\mathbb{E}[e^{-\sigma T_{y \rightarrow z}}] = \frac{\psi(y, \sigma)}{\psi(z, \sigma)}$$

(by Markov property)

$\psi(x, \sigma)$ satisfies a 2nd order Schroedinger like equation (by a martingale prop.)

In the scaling limit $\lambda \rightarrow \infty$; $\lambda^2 e^{b+1} =: J$ fixed, $\psi(x, \sigma)$ is linear in sigma:

$$\psi(x, \sigma) = \hat{J} + \sigma q(x)$$

with $q(x) = x^{b+1}/(b + 1)$

Comparison with spike tips as a point Poisson process of intensity $d\nu = dt \, d\hat{\nu}(x)$

$$\mathbb{E}[e^{-\sigma T_{y \rightarrow z}}] = \frac{1 + \sigma \hat{\nu}([y, \infty])^{-1}}{1 + \sigma \hat{\nu}([z, \infty])^{-1}}$$

(two contributions)

Hence:

$$d\nu = \hat{J} dt \, \frac{dq(x)}{q^2(x)}$$
Effective Time and Local Time (I)

— To decipher the structure of the spikes we may use a time variable which is sensitive to the variation of the process:
  Say, a new clock which clicks only if the process varies in a sensible way.

— Recall $dQ_t = \frac{\dot{\lambda}^2}{2} \dot{Q}_t^{b+1} dt + \hat{\lambda} Q_t dB_t$ for $Q_t := \frac{X_t^{b+1}}{b+1}$

We define the new time as the quadratic variation of $Q$:

Claim: Let $d\tau = \hat{\lambda}^2 Q_t^2 dt$

In the scaling limit $\lambda \to \infty; \lambda^2 \epsilon^{b+1} =: J$ fixed

Because $Q$ is a martingale at $\epsilon=0$ and the drift is active only at $Q=0$.

(to ensure positivity).

— Let $dW_\tau = \hat{\lambda} Q_t dB_t = \lambda X_t^{b+1} dB_t$, then $W$ is a new Brownian motion, and $dQ_\tau = dL_\tau + dW_\tau$ with $dL_\tau = \frac{\dot{\epsilon}}{2} Q_\tau^{-\frac{b+3}{b+1}} d\tau$

By the Skorokhod's theorem [and the uniqueness of the decomposition $f(x) = g(x)-l(x)$] applied to the Brownian motion $W$.

The equation $dQ_\tau = dL_\tau + dW_\tau$ then reduces to Tanaka's equation.
Effective Time and Local Time: Reconstruction.

— To reconstruct we have to determine the relation between the new time \( \tau \) and the original physical time \( t \) (using \( dL_\tau = \frac{\lambda^2}{2} \epsilon X_t^b dt \) plus ergodicity).

**Claim:** In the scaling limit \( \lambda \to \infty; \lambda^2 \epsilon^{b+1} =: J \) fixed

\( L_\tau \) is the \( \tilde{W}_\tau \) local time at the origin and \( J dt = 2\Gamma(b + 1) dL_\tau \),

This yields: \( X_t = (b + 1) \frac{1}{b+1} |\tilde{W}_\tau|^{\frac{1}{b+1}} \) parametrised by the local time.

— Since \( Q \) is a reflected Brownian motion parametrised by its local time at the origin, its maxima form a point Poisson process with intensity \( \tilde{J} dt dq/q^2 \).

— But more information on the internal structure of the jumps (anomalous operators).
**Conclusion / Remarks:**

- **Strong noise limit of SDES jump processes**
  (between unstable states) and spiky processes (as aborted jumps).

- **Generic (enough...) phenomena** but different from
  weak noise (Kramer's like) behaviour.

- Deciphering the fine/internal structures of stochastic jumps by **parametrising**
  the process with its **quadratic variation** instead of its linear natural « clock » time.

- .... Any application in turbulence ??....
... Dissipation in turbulence:

Questions: i) Is there any (relevant) scaling law in the height distribution?
ii) What is the statistics of the dissipation profile when parametrised by the quadratic variation?

(Naive) Guess?:
- Let $P(e)$ be the integrated P.D.F. of the dissipation spikes;
- Let $Q(e) = \text{const.}/P(e)$
- Then $Q_v = Q(e_x)$ parametrized by the dissipation quadratic variation $v$
  is (« closed to ») a reflected Brownian motion...
Skorokhod’s decomposition:

\[ F(x) = A(x) - L(x) \]

- \( A(x) \) positive
- \( L(x) > 0 \) & increases only if \( A=0 \).
Spikes and a Poisson Process.

Consider a point Poisson process in $\mathbb{R} \times \mathbb{R}^+$ with intensity $d\nu = dt \, d\hat{\nu}(x)$. The points of this process are associated to the tips of the spikes as usual. Let $\mathcal{N}_{[t_0, t] \times I}$, with $I$ a Borel set in $\mathbb{R}^+$, be the number of point of the process in $[t_0, t] \times I$. There are two contributions to the distribution of the transition times from $y \to z$:
(i) if the spike going above $y$ at initial time $t = 0$ is also going above $z$;
(ii) if the spike going above $y$ at initial time $t = 0$ stops before reaching $z$ but, the first at later spike going above $z$ is at time $t$, up to $dt$.

The probability of the first event is

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{N}_{[0, \delta] \times [z, \infty]} = 1 | \mathcal{N}_{[0, \delta] \times [y, \infty]} = 1] = \frac{\hat{\nu}([z, \infty])}{\hat{\nu}([y, \infty])}$$

The probability of the second event is

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{N}_{[0, \delta] \times [y, z]} = 1, \mathcal{N}_{[0, \delta] \times [z, \infty]} = 0, \mathcal{N}_{(t, t+dt] \times [z, \infty]} = 1 | \mathcal{N}_{[0, \delta] \times [y, \infty]} = 1] = \frac{\hat{\nu}([y, z]) \hat{\nu}([z, \infty]) e^{-\hat{\nu}([z, \infty]) t}}{\hat{\nu}([y, \infty])} dt$$

As a consequence, the probability that the transition time $T_{y \to z}$ be $t$ up to $dt$ is

$$\mathbb{P}[T_{y \to z} \in [t, t + dt]] = \frac{\hat{\nu}([z, \infty])}{\hat{\nu}([y, \infty])} (\delta(t) dt - \hat{\nu}([y, z]) e^{-\hat{\nu}([z, \infty]) t} dt)$$

Remark that this distribution is correctly normalized, thanks to $\hat{\nu}([y, \infty]) = \hat{\nu}([y, z]) + \hat{\nu}([z, \infty])$.

The corresponding generating function $\mathbb{E}[e^{-\sigma T_{y \to z}}]$ is then computable by integration. We get:

$$\mathbb{E}[e^{-\sigma T_{y \to z}}] = \frac{\hat{\nu}([z, \infty]) \hat{\nu}([y, z]) + \sigma}{\hat{\nu}([y, z]) \hat{\nu}([z, \infty]) + \sigma} = \frac{1 + \sigma \hat{\nu}([y, z])^{-1}}{1 + \sigma \hat{\nu}([z, \infty])^{-1}}$$
What is the strong limit of weak measurement?

\[ d\rho = (d\rho)_\text{sys} + (d\rho)_\text{meas}, \quad \text{with} \quad (d\rho)_\text{sys} = (-i[H, \rho] + L_{\text{dissip}}(\rho))dt \]
\[ (d\rho)_\text{meas} = \sigma^2 L_{\text{meas}}(\rho)dt + \sigma D_{\text{meas}}(\rho)dW_t \]

These are stochastic differential equations on the space of density matrix, i.e. for the coordinates 'X' of the quantum states. Trajectories develop jumps at strong coupling.

— Claim:
- « At strong coupling these processes converge to (finite state) Markov chains. »

Hint for a proof (strong noise limit):
— Convergence of the transition kernels:
\[ P_t(X_0, dX) \approx_{\sigma \to \infty} \sum_{\alpha} f_\alpha(X_0) [e^{-tM}]_{\alpha\beta} d\mu_\beta(X) \]

with functions 'f' and measures 'mu' determined by the measurement process and the Markov matrix 'M' by the system evolution.
Comparison with the weak noise limit.

— The **weak noise** limit of SDEs a la Kramer or a la Freidlin & Wentzell:

\[ dX_t = -U'(X_t) \, dt + \epsilon \, dB_t \]

with, say, \( U(x) \) a double well potential.

— The typical passage time to go up-hill (i.e. \( U(z) > U(y) \)):

\[ T_{y \to z} \approx e^{-2(U(z)-U(y))/\epsilon^2} \]

— Scaling limit:

Choose \( x \) and fixed \( T_{x_* \to x} \) finite.

Then, all other passage times are either 0 or infinite.