Asymptotics for Magnetostrophic turbulence in the Earth's fluid core

Susan Friedlander, USC

Vlad Vicol, Walter Rusin, Anthony Suen
Nathan Glatt-Holtz, Juraj Foldes, Geordie Richards
Layering of the Earth.

- Inner Core
- Outer Core
- Mantle
- Asthenosphere
- Lithosphere
- Crust

Temperature and pressure increase with depth.

Mesosphere: hot but stronger due to high pressure
Asthenosphere: hot, weak, plastic
Lithosphere: cool, rigid, brittle

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MHD system: Coriolis, Lorentz, Gravity

\[ N^2 \left[ R_0 (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \hat{\mathbf{e}}_3 \times \mathbf{u} \right] \]

\[ = -\nabla \mathbf{p} + \hat{\mathbf{e}}_2 \cdot \nabla \mathbf{b} + R_m b \cdot \nabla \mathbf{b} + N^2 \Theta \hat{\mathbf{e}}_3 + \nu \Delta \mathbf{u} \]

\[ R_m \left[ \partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} - b \cdot \nabla \mathbf{u} \right] = \hat{\mathbf{e}}_2 \cdot \nabla \mathbf{u} + \Delta \mathbf{b} \]

\[ \partial_t \Theta + \mathbf{u} \cdot \nabla \Theta = \kappa \Delta \Theta + S \]

\[ \nabla \cdot \mathbf{u} = 0 \quad \nabla \cdot \mathbf{b} = 0 \]

\( \mathbf{u} \) velocity, \( \mathbf{b} \) magnetic field, \( \Theta \) temperature

\( \mathbf{p} \) magnetic + fluid pressure

dimensionless parameters

\( N^2, R_0, R_m, \nu, \kappa \)

\( O(1), O(10^{-3}), O(10^{-2}) \) very small

\( \kappa \) very small
Moffatt - Coper Model: leading order.

\[ N^2 \hat{e}_3 \times u = - \nabla P + \hat{e}_2 \cdot \nabla b + N^2 \hat{e}_3 + \nu \Delta u \]

\[ 0 = \hat{e}_2 \cdot \nabla u + \Delta b \]

\[ \partial_t \Theta + u \cdot \nabla \Theta = \kappa \Delta \Theta + S \]

\[ \nabla \cdot u = 0, \quad \nabla \cdot b = 0 \]

\[ \left\{ \left[ \gamma \Delta^2 - (\hat{e}_2 \cdot V)^2 \right]^2 + N^4 (\hat{e}_3 \cdot V)^2 \right\} u \]

\[ = - N^2 \left[ \gamma \Delta^2 - (\hat{e}_2 \cdot V)^2 \right] \nabla \times (\hat{e}_3 \times \nabla \Theta) \]

\[ + N^4 (\hat{e}_3 \cdot V) \Delta (\hat{e}_3 \times \nabla \Theta) \]

Domain: \( \mathbb{T}^3 \times (0, \infty) \) \( \Theta \) has zero vertical mean.
Magnetogostrophic Equation (MG)

$$\frac{\partial}{\partial t} \theta + (u \cdot \nabla) \theta = K \Delta \theta + S$$

$$u = M[\theta]$$

Components of Fourier Multiplier Symbol $M(k)$

$$\hat{M}_1 = \frac{[N^4 k_2 k_3 |k|^2 - N^2 k_1 k_2 (k_2^2 + \nu |k|^4)]}{D}$$

$$\hat{M}_2 = \frac{[-N^4 k_1 k_3 |k|^2 - N^2 k_2 k_3 (k_2^2 + \nu |k|^4)]}{D}$$

$$\hat{M}_3 = \frac{[N^2 (k_1^2 + k_2^2) (k_2^2 + \nu |k|^4)]}{D}$$

where $D = N^4 |k|^2 k_3^2 + (\nu |k|^4 + k_2^2)^2$

recall $k_3 \neq 0$
Observations about the symbol

1) anisotropic, even in wave number \( k \)
2) \( \nu > 0 \); \( \hat{M}(k) \sim 1/k^2 \) as \( k \to \infty \)
3) \( \nu = 0 \):

On the curved regions in Fourier space
\( k_3 = O(1) \), \( k_1 \sim k_2 \) and \( |k_1| \to \infty \)
\( \hat{M}(k) \sim C k_1 \)

i.e. when \( \nu > 0 \) \( M \) is smoothing of degree 2
when \( \nu = 0 \) \( M \) is singular of degree -1.
Hierarchy of active scalar equations

\[ \partial_t \Theta + u \cdot \nabla \Theta = (K \Delta^2 \Theta) \]

1. Inviscid (\( \nu = 0 \)) MG: Singular order 1
   \( K = 0 \), Hadamard ill-posed
   \( K > 0 \), critical case, globally well posed

2. SGG: Singular order zero
   \( K = 0 \), open
   \( K > 0 \), critical case, globally well posed

3. 2D Euler in vorticity form: smoothing degree 1
   well posed.

4. Viscous MG: smoothing degree 2
   "better" than 2D Euler
Inviscid, nondiffusive \( \text{MG}_6 \) (\( \nu = 0, \kappa = 0 \))

Friedlander-Vicol (2011)

1) Instability of \( \text{MG}_6 \) linearised about a particular steady state
   Use continued fractions to construct an eigenvalue with arbitrarily large real part

2) Ill-posedness in Sobolev spaces for the full nonlinear problem follows by showing the solution map is not Lipschitz with respect to initial data.
   Result requires the derivative loss and the fact that \( \mathcal{M} \) is even.
* Inviscid critical MG. \((v=0, K>0, \alpha=1)\)

**Friedlander & Vicol (2011)**

1) **Linear parabolic PDE**

\[
\partial_t \theta + \nabla \cdot \nabla \theta = K \Delta \theta, \quad \nabla \cdot v = 0
\]

\(v \in L^2_t L^2_x \cap L^\infty_t BMO^{-1}_x\)

Then weak solutions are Hölder continuous.

Proof uses De Giorgi iteration.

2) **Use this result to prove that Leray-Hopf weak solutions to an active scalar nonlinear PDE of the type \(\ast\) are classical solutions.**
Viscous, nondiffusive M6r (r>0, κ=0)

Friedlander & Suen (2015)

Th 1. Let $\Theta_0 \in L^3$. There exists a unique —
    global weak solution to
$$\Theta_t + u \cdot \nabla \Theta = 0, \quad u = M_r[\Theta]$$
    Such that $\Theta \in BC((0,\infty); L^3)$, $u \in C((0,\infty); W^{3,3})$.
    In particular, $\Theta(\cdot, t) \rightarrow \Theta_0$ weakly as $t \rightarrow 0^+$.

Th 2. Let $\Theta_0 \in W^{5,3}$, $s \geq 0$. There exists a —
    unique solution $\Theta(\cdot, t) \in W^{3,3}$ for all $t \geq 0$.
    In particular, for $s=1$
$$\|\nabla \Theta\|_{L^3} \leq C_1 \|\nabla \Theta_0\|_{L^3} \exp(tC_2 \|\Theta_0\|_{W^{3,3}})$$
Note: $L^3$ is the critical Lebesgue space with respect to the natural scaling of the MG system in the sense that if $\Theta(x, t)$ is a solution, then $\Theta_{\lambda}(x, t) = \lambda^3 \Theta(\lambda x, \lambda^2 t)$ is also a solution with corresponding drift velocity given by $V_{\lambda}(x, t) = \lambda V(\lambda x, \lambda^2 t) = M[\Theta_{\lambda}]$ for $\lambda > 0$.

The theorems require no smallness condition.
Viscous, diffusive MG ($r > 0$, $k > 0$)

**Th 3.** There exists a unique global in time mild solution to

$$\theta_t + u \cdot \nabla \theta = k \Delta \theta, \quad u = M_r[\theta], \quad \theta_0 \in L^3$$

such that $\theta \in BC((0, \infty); L^3)$

$$t^{\frac{5}{2} + \frac{1}{2} - \frac{3}{2p}} \theta \in C((0, \infty); W^{3,p}), \quad s \in [0, 1], \quad p \in (3, \infty).$$

In particular, $\theta(\cdot, t) \to \theta_0$ in $L^3$ as $t \to 0^+$ and $\|\theta(\cdot, t)\|_{W^{3,p}} \to 0$ as $t \to \infty$.

The solution is instantaneously $C^\infty$ smoothed out and in $W^{3,p}$ for all $t > 0$. 


Limit as $k \to 0$

**Th 4.** Let $\Theta_k$ be the solution with $\Theta_0 \in L^3$. Then there exists a sequence $\epsilon \in K_n \subset L^3$ with $(\lim_{n \to \infty} K_n = 0)$ such that

$\Theta_k (\cdot, t) \to \Theta (\cdot, t)$ weakly in $L^3$ as $n \to \infty$

for all $t > 0$ where $\Theta$ is the solution to the non-diffusive MG$_r$ equation.

If also $D\Theta_0 \in L^2$, then for any $T > 0$

$\lim_{k \to 0} k \int_0^T \int_0^T |D\Theta_k|^2 \, dx \, ds = 0$
Stochastically Forced MHD System

$\nu > 0$, $\kappa > 0$, $\lim R_0 \to 0$, $\lim R_m \to 0$

\[ R_0 \left( \partial_t U + U \cdot \nabla U \right) + \hat{e}_3 \times U \]

\[ = -\nabla P + \hat{e}_2 \cdot \nabla B + R_m B \cdot \nabla B + \Theta \hat{e}_3 + \nu \Delta U \]

\[ R_m \left( \partial_t B + U \cdot \nabla B - B \cdot \nabla U \right) = \hat{e}_2 \cdot \nabla U + \Delta B \]

\[ d\Theta + U \cdot \nabla \Theta = \kappa \Delta \Theta + \sigma \, dW \]

White in time, spatially correlated, Gaussian noise $\sigma \, dW$:

\[ \sigma \, dW = \sum_{k \in \mathbb{Z}^3} \alpha_k \sigma_k \, dW^k \]

$\sigma_k$ are $\sin k \cdot x$ and $\cos k \cdot x$.
$W^k$ independent 1-D Brownian motions.
\( x_k \in \mathbb{R} \) are the amplitudes.

Fundamental postulates of turbulence consider energy cascading from large to small spatial scales.

Our system is driven by "spectrally degenerate" stochastic forcing, i.e., noise acts only through a narrow range of low frequencies.

Hypo-elliptic situation substantially more difficult than forcing on all spatial scales.
Martingale Solutions for the Full System

(i) Given any initial probability distribution $\mu$ there exists a stochastic process $(U_j, B_j, \Theta)$ which is weakly continuous and solves the full evolution system.

(ii) For every $R_0$, $R_m > 0$ there exists a stationary Martingale solution that satisfies the uniform moment bound

$$\sup_{R_0, R_m \in (0, N]} \mathbb{E} \exp \left( \eta \left( R_0 \| U \|^2 + R_m \| B \|^2 + \| \Theta \|^2 \right) \right) \leq C_N < \infty$$

Proved using a Galerkin regularization similar to that used for 3D Navier-Stokes.
The $M_{Gr}$ equation is the limit system obtained with $v \gg 1$, $K \gg 1$, $R_o = 0$, $R_m = 0$

Very singular limit

Full system supports initial conditions on all variables $U, B, T$

The limit system ($M_{Gr}$ active scalar) allows initial conditions only on $T$

Multi-time scale analysis with three time scales:

$O(1), O(R_o^{-1}), O(R_m^{-1})$
Limit System:

$M_{G_r}$ active scalar PDE, $r > 0$, $k > 0$.

\[ \partial_t \Theta + (u \cdot \nabla) \Theta = k \Delta \Theta + \sigma dW, \quad \Theta|_{t=0} = \Theta_0 \]

\[ u = M_u[\Theta], \quad b = M_b[\Theta] \]

Components of Fourier multiplier symbol $\hat{M}_u$:

\[ \hat{M}_{u_1} = \left[ k_2 k_3 |k|^2 - k_1 k_3 (k_2^2 + r|k|^4) \right] / D \]

\[ \hat{M}_{u_2} = \left[ -k_1 k_3 |k|^2 - k_2 k_3 (k_2^2 + r|k|^4) \right] / D \]

\[ \hat{M}_{u_3} = \left[ (k_1^2 + k_2^2)(k_2^2 + r|k|^4) \right] / D \]

where

\[ D = |k|^2 k_3^2 + (k_2^2 + r|k|^4)^2 \]

\[ \hat{M}_b = i k_2 \hat{M}_u / |k|^2 \]

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Results for the limit system.

\[ K > 0 \]

Stochastically forced M\(_6\) equation, \( v > 0 \).

Well-posedness:

The PDE possesses unique, pathwise solutions which satisfy exponential moment bounds.

Extension of known results in the deterministic case.
Markovian Dynamics of the limit system.

Use the framework of Hairer-Mattingly

1. Show that a form of the Hormander bracket condition is satisfied
2. Verify a form of asymptotic Strong Feller
3. An irreducibility condition
4. Certain exponential moment bounds

We infer that the contractivity property of H-M is satisfied in a suitably chosen Wasserstein metric W.
Th: Let $\{P_t\}_{t\geq 0}$ be the Markov semigroup associated to the MG equation. Then $\{P_t\}_{t\geq 0}$ is contractive in $W$:

$$W(M_1 P_t, M_2 P_t) \leq C e^{-\tau t} W(M_1, M_2), \quad t \geq 0$$

It then follows that $\{P_t\}_{t\geq 0}$ possesses a unique ergodic (invariant) measure $\mu$. Furthermore $\mu$ satisfies attraction properties:

- it is exponentially mixing
- obeys a strong law of large numbers
- obeys a central limit system
Finite time convergence as $R_0, R_m \to 0$.

We use the powerful observation of Hairer-Mattingly [08] that if one can establish a contraction property for the limit system the convergence of statistically steady states can be reduced to convergence of solutions of the full system on finite time scales.

Key observation for our problem is to show that a difference in initial conditions on $U$ and $B$ has negligible effect on $\Theta$, namely algebraic order in $R_0 + R_m$.

Note: the convergence for the SSS do not require uniform convergence in $U$ and $B$ up to the initial time $t = 0$. 18a
Asymptotics for the full evolution system

Denote variables \( \Theta, U, B \) for the full system and \( \tilde{\Theta}, \tilde{U}, \tilde{B} \) for the limit system.

Results as \( R_0 \to 0, \ R_m \to 0 \)

(i) Assume \( \| \Theta(0) - \tilde{\Theta}(0) \| \to 0 \)

Then for any \( t > 0 \) \( \exists \, \delta > 0 \) and \( p > \delta \) so that

\[
\mathbb{E} \sup_{s \in [0,t]} \| \Theta(s) - \tilde{\Theta}(s) \|^p \leq C \left( \| \Theta(0) - \tilde{\Theta}(0) \| + R_0 + R_m \right)^\delta \to 0
\]

and furthermore

\[
\mathbb{E} \int_0^t \| U(s, B(s) - M_{\tilde{U}, \tilde{b}}) \|^2 \, ds \leq C \left( \| \Theta(0) - \tilde{\Theta}(0) \| + R_0 + R_m \right)^\delta \to 0
\]
(ii) Convergence of statistically steady states (invariant measures)

- For every \( R_0, R_m > 0 \) the full system possesses at least one SSS \( M_{R_0,R_m} \) which satisfies certain exponential moment bounds independent of \( R_0, R_m \).

- Any collection \( M_{R_0,R_m} \) converges to \( \mu \) at an algebraic rate in a suitable metric \( W \).

- In particular, for any sufficiently regular observable \( \psi \)

\[
\left| \int \psi(U,B,\theta) \, dM_{R_0,R_m} - \int \psi(M_{u,b}(\theta), \theta) \, d\mu \right| \to 0
\]

as \( R_0, R_m \to 0 \)
Conclusion

Our analysis demonstrates that the unique invariant statistics for the limit equation (i.e. stochastically forced MHD) approximate any reasonable invariant statistics of the full MHD equations describing dynamo action in the Earth's fluid core. Thus we give a rigorous foundation for the Moffatt and Loper model of Magnetostrophic Turbulence.
Thank You!

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