Anomalous Dissipation, Spontaneous Stochasticity and Onsager's Conjecture

Theodore D. Drivas and Gregory L. Eyink

Department of Applied Mathematics and Statistics
Whiting School of Engineering, The Johns Hopkins University

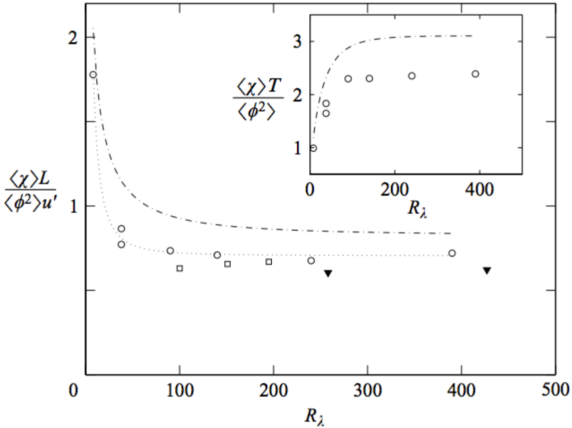
TDM2017: Turbulent Dissipation, Mixing and Predictability

PART I:
Scalar Mixing and Spontaneous Stochasticity

Turbulent flows are extremely effective at both:

a) Dissipating energy (e.g. “Zeroth Law”, Onsager’s Conjecture)

$$\chi = \kappa|\nabla\theta|^2$$



Donzis, D. A., K. R. Sreenivasan, and PK Yeung. Journal of Fluid Mechanics 532 (2005): 199-216.

Turbulent flows are extremely effective at both:

b) Mixing (e.g. Richardson Dispersion, Spontaneous Stochasticity)

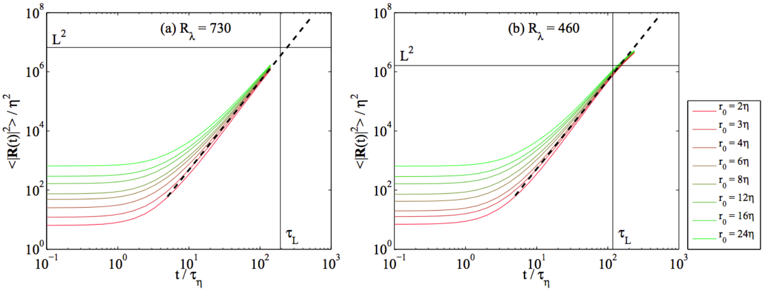


Figure 2. Time-evolution of the mean-squared distance for $R_\lambda = 730$ (a) and $R_\lambda = 460$ (b) for various initial separations r_0 as labeled. The horizontal and vertical solid lines represent the integral scale L and its associated turnover time τ_L, respectively. The dashed line corresponds to the explosive Richardson-Obukhov law (3) with $g = 0.52$.

Bitane R, Holger H, and Bec J, Journal of Turbulence 14.2 (2013): 23-45.

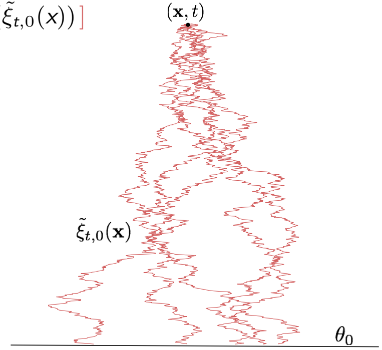
STOCHASTIC LAGRANGIAN REPRESENTATION OF PASSIVE SCALAR

$$\partial_t \theta + \mathbf{u}^\nu \cdot \nabla \theta = \kappa \Delta \theta, \quad \theta|_{t=0} = \theta_0$$

Feynman-Kac formula with backwards stochastic trajectories:

$$\mathrm{d}\tilde{\xi}_{t,s}(x) = \mathbf{u}^\nu(\tilde{\xi}_{t,s}(x), s)\, \mathrm{d}s + \sqrt{2\kappa}\, \hat{\mathrm{d}}\widetilde{\mathbf{W}}_s,$$

$$\theta(x,t) = \mathbb{E}\left[\theta_0(\tilde{\xi}_{t,0}(x))\right]$$



P. Saffman (1960), Sawford et a. (2005)
K. Sreenivasan and J. Schumacher (2010).

KRAICHNAN MODEL (BROWNIAN FLOWS)

Advection of a scalar field θ by a rough Brownian vector field on $\mathbb{T}^n$:

$$\mathrm{d}\theta + (\bar{\mathbf{u}} \cdot \nabla\theta - \kappa\Delta\theta)\,\mathrm{d}t = \mathrm{d}\mathbf{U}^\nu \circ \nabla\theta, \quad \theta(0) = \theta_0 \in C^b(\mathcal{D}).$$

For $0 < \alpha < 1$, the velocities $\mathbf{u}\mathrm{d}t = \bar{\mathbf{u}}\mathrm{d}t + \mathrm{d}\mathbf{U}$ are built to satisfy:

$$\mathbb{E}\,|\mathbf{U}^\nu(\mathbf{x},t) - \mathbf{U}^\nu(\mathbf{x}',t')|^2 = t \wedge t' \times \begin{cases} A|\mathbf{x}-\mathbf{x}'|^{2\alpha} & \ell_\nu \ll |\mathbf{x}-\mathbf{x}'| \ll L \\ B|\mathbf{x}-\mathbf{x}'|^2 & |\mathbf{x}-\mathbf{x}'| \ll \ell_\nu \end{cases}.$$

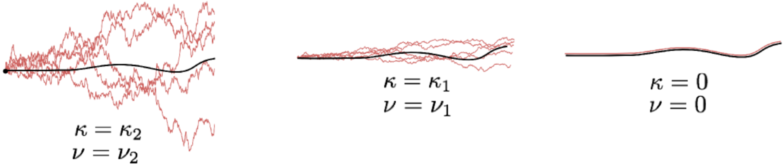
Bernard, Gawędzki and Kupiainen (1998) found that anomalous dissipation is explained by spontaneous stochasticity:

$$p^{\nu,\kappa}(\mathbf{x}',t'|\mathbf{x},t) = \mathbb{E}\left[\delta^d(\mathbf{x}' - \tilde{\boldsymbol{\xi}}^{\nu,\kappa}_{t,t'}(\mathbf{x}))\right] \xrightarrow{\nu,\kappa\to 0} p^*(\mathbf{x}',t'|\mathbf{x},t) \neq \delta^d(\mathbf{x}' - \boldsymbol{\xi}^{0,0}_{t,t'}(\mathbf{x})).$$

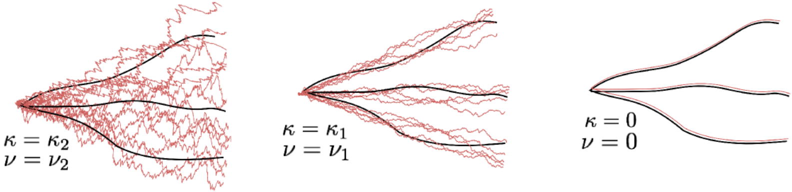
See also [E & Vanden-Eijnden 00, 01]. Later, [Le Jan & Raimond 02, 04] proved the existence of $\mathbb{P}^*_{\mathbf{x},t}[\mathrm{d}\boldsymbol{\xi}]$, family of Markov transition probabilities, conditioned $\mathbf{u}$, related to the flow $\mathrm{d}\boldsymbol{\xi}_t = \mathbf{u}(\boldsymbol{\xi}_t\circ, t)\mathrm{d}t$. The unique dissipative weak solution is:

$$\vartheta(\mathbf{x},t) = \int \mathbb{P}^*_{\mathbf{x},t}[\mathrm{d}\boldsymbol{\xi}]\;\vartheta_0(\boldsymbol{\xi}_{-t}) = \mathbb{E}^*_{\mathbf{x}}\left[\vartheta_0(\tilde{\boldsymbol{\xi}}_{-t})\right].$$

NO SPONTANEOUS STOCHASTICITY



SPONTANEOUS STOCHASTICITY



BEYOND THE KRAICHNAN MODEL

QUESTION 1: Does the same picture hold for Navier-Stokes? Scalar sources?

QUESTION 2: What is the effect of walls? Different boundary conditions?

FLUCTUATION-DISSIPATION RELATION WITH NO WALLS

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,0}(\mathbf{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,s}(\mathbf{x}), s)\, \mathrm{d}s\right]\right\rangle_\Omega = \kappa \int_0^t ds \left\langle |\nabla\theta^\nu(s)|^2\right\rangle_\Omega.$$

Balance between scalar dissipation and the input of scalar fluctuations from the initial scalar field and the scalar sources, as sampled by backwards trajectories.

FLUCTUATION-DISSIPATION RELATION WITH NO WALLS

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,0}(\mathbf{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,s}(\mathbf{x}), s)\, \mathrm{d}s\right]\right\rangle_\Omega = \kappa \int_0^t ds \left\langle |\nabla \theta^\nu(s)|^2 \right\rangle_\Omega .$$

Balance between scalar dissipation and the input of scalar fluctuations from the initial scalar field and the scalar sources, as sampled by backwards trajectories.

THEOREM: Spontaneous stochasticity is necessary and sufficient for anomalous dissipation of passive scalars.

Idea of Proof:

$$\lim_{k\to\infty} \left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\nu_k,\kappa_k}_{t,0}(\mathbf{x}))\right]\right\rangle_\Omega = \int d^dx \int d^dx_0 \int d^dx_0'\; \theta_0(\mathbf{x}_0)\theta_0(\mathbf{x}_0')$$
$$\times \left[p_2^*(\mathbf{x}_0, 0; \mathbf{x}_0', 0|\mathbf{x}, t) - p^*(\mathbf{x}_0, 0|\mathbf{x}, t)p^*(\mathbf{x}_0', 0|\mathbf{x}, t)\right],$$

where $p_2^*(\mathbf{x}_0, 0, \mathbf{x}_0', 0|\mathbf{x}, t) \equiv \delta^d(\mathbf{x}_0 - \mathbf{x}_0')p^*(\mathbf{x}_0, 0|\mathbf{x}, t)$.

QUESTION 1: Same picture for Navier-Stokes? Scalar sources? YES!

FLUCTUATION-DISSIPATION RELATION WITH NO WALLS

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,0}(\mathbf{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,s}(\mathbf{x}), s)\, \mathrm{d}s\right]\right\rangle_\Omega = \kappa \int_0^t ds \left\langle |\nabla\theta^\nu(s)|^2 \right\rangle_\Omega .$$

Balance between scalar dissipation and the input of scalar fluctuations from the initial scalar field and the scalar sources, as sampled by backwards trajectories.

THEOREM: Spontaneous stochasticity is necessary and sufficient for anomalous dissipation of passive scalars. Morally true for active scalars!

Idea of Proof:

$$\lim_{k\to\infty} \left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\nu_k,\kappa_k}_{t,0}(\mathbf{x}))\right] \right\rangle_\Omega = \int d^d x \int d^d x_0 \int d^d x_0'\ \theta_0(\mathbf{x}_0)\theta_0(\mathbf{x}_0')$$
$$\times\left[p_2^*(\mathbf{x}_0, 0; \mathbf{x}_0', 0|\mathbf{x}, t) - p^*(\mathbf{x}_0, 0|\mathbf{x}, t)p^*(\mathbf{x}_0', 0|\mathbf{x}, t)\right],$$

where $p_2^*(\mathbf{x}_0, 0, \mathbf{x}_0', 0|\mathbf{x}, t) \equiv \delta^d(\mathbf{x}_0 - \mathbf{x}_0')p^*(\mathbf{x}_0, 0|\mathbf{x}, t)$.

QUESTION 1: Same picture for Navier-Stokes? Scalar sources? YES!

WALL BOUNDED FLOWS WITH IMPOSED SCALAR FLUX

$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = \kappa\triangle\theta + S \qquad \text{for} \quad \mathbf{x}\in\Omega,$$

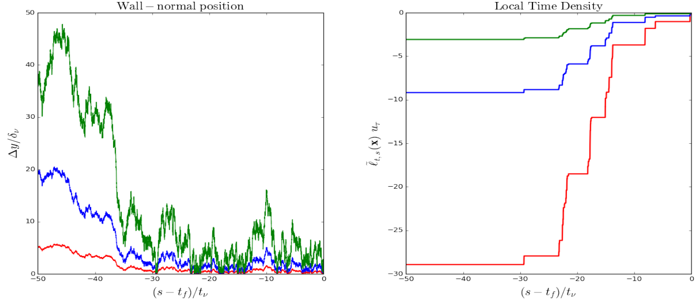
$$-\kappa\frac{\partial\theta}{\partial n} = g \qquad \text{for} \quad \mathbf{x}\in\partial\Omega.$$

Define stochastic trajectories which reflect off the boundary of the domain:

$$\hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}) = \mathbf{u}^{\nu}(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\,\mathrm{d}s + \sqrt{2\kappa}\,\hat{\mathrm{d}}\mathbf{W}_s - \kappa\boldsymbol{n}(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\,\hat{\mathrm{d}}\tilde{\ell}_{t,s}(\mathbf{x})$$

where the *boundary local time* is:

$$\tilde{\ell}_{t,s}(\mathbf{x}) = \int_t^s dr\,\delta(\mathrm{dist}(\tilde{\boldsymbol{\xi}}_{t,r}(\mathbf{x}),\partial\Omega)) \equiv \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_t^s dr\,\chi_{\partial\Omega_\varepsilon}(\tilde{\boldsymbol{\xi}}_{t,r}(\mathbf{x})), \qquad s<t,$$



WALL BOUNDED FLOWS WITH IMPOSED SCALAR FLUX

$$\begin{aligned} \partial_t \theta + \mathbf{u} \cdot \nabla \theta &= \kappa \triangle \theta + S \qquad &&\text{for} \quad \mathbf{x} \in \Omega, \\ -\kappa \frac{\partial \theta}{\partial n} &= g \qquad &&\text{for} \quad \mathbf{x} \in \partial\Omega. \end{aligned}$$

Define stochastic trajectories which reflect off the boundary of the domain:

$$\hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}) = \mathbf{u}^{\nu}(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\, \mathrm{d}s + \sqrt{2\kappa}\, \hat{\mathrm{d}}\mathbf{W}_s - \kappa \boldsymbol{n}(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\, \hat{\mathrm{d}}\tilde{\ell}_{t,s}(\mathbf{x})$$

Feynman-Kac formula, e.g. (Freidlin, 1985):

$$\theta(\mathbf{x}, t) = \mathbb{E}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t \mathrm{d}s\, S(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s) + \int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\, \hat{\mathrm{d}}\tilde{\ell}_{t,s}\right].$$

Fluctuation dissipation relation:

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t \mathrm{d}s\, S(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s) + \int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\hat{\mathrm{d}}\tilde{\ell}_{t,s}\right]\right\rangle_{\Omega} = \kappa \int_0^t \mathrm{d}s \left\langle |\nabla \theta(s)|^2 \right\rangle_{\Omega}$$

WALL BOUNDED FLOWS WITH IMPOSED SCALAR FLUX

$$\partial_t \theta + \mathbf{u}\cdot\nabla\theta = \kappa\triangle\theta + S \qquad \text{for} \quad \mathbf{x}\in\Omega,$$
$$-\kappa\frac{\partial\theta}{\partial n} = g \qquad \text{for} \quad \mathbf{x}\in\partial\Omega.$$

Define stochastic trajectories which reflect off the boundary of the domain:

$$\hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}) = \mathbf{u}^{\nu}(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\,\mathrm{d}s + \sqrt{2\kappa}\,\hat{\mathrm{d}}\mathbf{W}_s - \kappa\boldsymbol{n}(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\,\hat{\mathrm{d}}\tilde{\ell}_{t,s}(\mathbf{x})$$

Feynman-Kac formula, e.g. (Freidlin, 1985):

$$\theta(\mathbf{x},t) = \mathbb{E}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t \mathrm{d}s\, S(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s) + \int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\,\hat{\mathrm{d}}\tilde{\ell}_{t,s}\right].$$

Fluctuation dissipation relation:

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t \mathrm{d}s\, S(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s) + \int_0^t g(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\hat{\mathrm{d}}\tilde{\ell}_{t,s}\right]\right\rangle_{\Omega}$$
$$= \kappa\int_0^t \mathrm{d}s\,\langle|\nabla\theta(s)|^2\rangle_{\Omega}$$

REMARK: FDR also holds for Dirichlet conditions with $g = -\kappa\frac{\partial\theta}{\partial n}$.

WALL BOUNDED FLOWS WITH ZERO FLUX

For zero flux conditions (stirring milk into coffee) our fluctuation-dissipation relation reads

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t \mathrm{d}s\, S(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\right]\right\rangle_\Omega = \kappa \int_0^t \mathrm{d}s \left\langle |\nabla\theta(s)|^2 \right\rangle_\Omega$$

and we have equivalence of anomalous dissipation & spontaneous stochasticity.

WALL BOUNDED FLOWS WITH IMPOSED SCALAR FLUX

For general flux conditions the situation is more complicated. For example, consider the heat equation on $\mathbb{R}^+$ with constant flux J at $x = 0$ and $\theta_0, S = 0$. Local time densities may be explicitly calculated and:

$$\theta(x, t) = -J\,\mathbb{E}[\tilde{\ell}_{t,0}^{x=0}(x)] \sim J\sqrt{\frac{t}{\kappa}}\varphi\left(\frac{x}{\sqrt{\kappa t}}\right)$$

for a suitable scaling function φ. Scalar boundary layer of thickness $\sim \sqrt{\kappa t}$ near $x = 0$ where the field diverges as $\sim J\sqrt{t/\kappa}$. Dissipation is non-vanishing (and divergent!) though there is clearly no spontaneous stochasticity:

$$\left\langle \kappa|\nabla\theta(x, t)|^2 \right\rangle_\Omega \sim J^2\sqrt{\frac{t}{\kappa}} \xrightarrow{\kappa\to 0} \infty!$$

Thin scalar boundary layers near walls provide another mechanism for non-vanishing dissipation!

There is no longer an equivalence between SS and AD, nevertheless our FDR is still valid and can give important information.

APPLICATION: RAYLEIGH BÉNARD CONVECTION:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \beta g \, T \hat{\mathbf{z}}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \kappa \Delta T, \qquad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}\big|_{z=\pm H/2} = 0$$

with fixed flux boundary conditions

$$-\kappa \frac{\partial T}{\partial z}\bigg|_{z=\pm H/2} = J$$ imposed flux models poorly conducting plates.

Enhancement of vertical heat transport measured by the Nusselt number:

$$Nu = \frac{\text{total heat flux}}{\text{flux due to thermal conduction}}. \qquad Nu = Nu(Ra, Pr)?$$

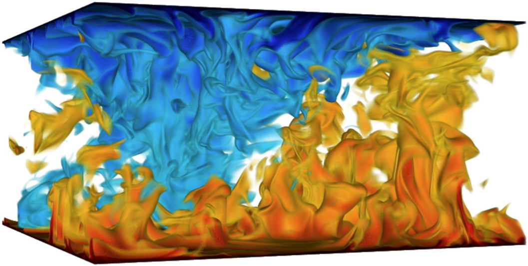


Figure depicts *fixed temperature RB convection*. Erwin P. van der Poel & Rodolfo Ostilla Mónico, Livermore National

FLUCTUATION-DISSIPATION IN RAYLEIGH BÉNARD CONVECTION

$$\frac{Nu}{\sqrt{Ra\,Pr}} = \frac{\langle \kappa |\nabla T|^2 \rangle_{V,\infty}}{(\Delta T)^2 U/H} \tag{2}$$

Classical theory of Kraichnan-Spiegel predicts $Nu_{KS} \sim C \cdot Ra^{1/2} Pr^{1/2}$.
We derived the following steady-state fluctuation-dissipation relation:

$$\langle \kappa |\nabla T|^2 \rangle_{V,\infty} \quad = \quad \lim_{t\to\infty} \frac{1}{2t} \left\langle \mathrm{Var}\left[J\left(\tilde{\ell}^{top}_{t,0} - \tilde{\ell}^{bot}_{t,0} \right) \right] \right\rangle_V$$

$$\lim_{t\to\infty} \frac{J^2}{2t} \mathrm{Cov}\left[\tilde{\ell}^{\lambda}_{t,0}(\mathbf{x}), \tilde{\ell}^{\lambda'}_{t,0}(\mathbf{x}) \right] = \frac{J^2}{H^2} \int_{-\infty}^{0} ds \, \left(Hp_z(\lambda H/2, s|\lambda' H/2, 0) - 1 \right)$$
$$:= \frac{J^2}{H^2} \tau^{\lambda\lambda'}_{mix} .$$

Long-time average of the thermal dissipation is entirely due to statistical correlations of the incidences of single fluid particles on the top and bottom walls at distinct times. Thus there is a relation with to mixing time

$$\langle \kappa |\nabla T|^2 \rangle_{V,\infty} = \frac{J^2}{H^2} \tau_{mix}, \qquad \Longrightarrow \qquad \frac{\tau_{mix}}{\tau_{ff}} = \frac{\sqrt{RaPr}}{Nu}$$

What are the consequences of our exact relation?

$$\frac{\tau_{mix}}{\tau_{ff}} \sim \frac{Nu_{KS}}{Nu}, \qquad \tau_{mix} = \int_{-\infty}^{0} dt\,(H\langle c(t)\rangle_{bot} - 1) + \int_{-\infty}^{0} dt\,(H\langle c(t)\rangle_{top} - 1)$$
$$- \int_{-\infty}^{0} dt\,(H\langle c(t)\rangle_{bot} - 1) - \int_{-\infty}^{0} dt\,(H\langle c(t)\rangle_{top} - 1)$$

With τ_{ff} = free-fall time, $\partial_t c + \mathbf{u} \cdot \nabla c = -\kappa \Delta c$ and $\lim_{t \to -\infty} \langle c(t) \rangle_{top/bot} = 1/H$.

MODIFIED FROM:
Two-dimensional convection simulation with $\texttt{res} = 7680 \times 4320$, $Ra = 10^{13}$, $Pr = 1, \Gamma = 16:9$.
J. Lülff, M. Wilczek, A. Daitche, 'Turbulence Team Münster' YouTube channel,
`http://www.youtube.com/user/turbulenceteamms`, 2012.

PART II:
Onsager singularity theorem for compressible turbulence

ONSAGER'S CONJECTURE FOR INCOMPRESSIBLE FLUIDS

Lars Onsager in 1949 proved (essentially, but unpublished):

1) A weak Euler solution satisfying $\mathbf{u} \in C^h$ for $h > 1/3$ must conserve energy.

He also hypothesized:

2) There exists a weak solution $\mathbf{u} \in C^h$ with $h \leq 1/3$ which dissipates energy.

3) Euler solutions of (2) should appear in the limit $\nu \to 0$ of NS solutions.

1) Energy conservation for smooth enough weak Euler solutions:
Eyink (1994); Constantin, E, Titi (1994); Duchon-Robert (2000); Cheskidov-Constantin-Friedlander-Shvydkoy (2008).
We call these results *Onsager Singularity Theorems*.

2) Existence and Non-uniqueness of dissipative Euler Solutions:
Scheffer (1993), Shnirelman (2000); De Lellis & Szekelyhidi (2012): $C_{t,x}^{1/10-\varepsilon}$;
Isett (2012): $C_{t,x}^{1/5-\varepsilon}$; Buckmaster (2013), $C_{t,x}^{1/5-\varepsilon}$ with $C_{t,x}^{1/3-\varepsilon}$ a.e. time;
Buckmaster, De Lellis & Szekelyhidi (2014): $L_t^1 C_x^{1/3-\varepsilon}$;
Buckmaster-Masmoudi-Vicol (2016): $C_t^1 H_x^{1/3-\varepsilon}$; Isett (2016): $C_{t,x}^{1/3-\varepsilon}$

3) Zero-Viscosity Limit:
DiPerna & Majda (1987); P. L. Lions (1996);
Brenier-De Lellis-Szekelyhidi Jr. (2011);
Solutions (2) are not yet known to appear in the inviscid limit.

COMPRESSIBLE EULER SYSTEM

$$\partial_t \varrho + \nabla_x \cdot (\varrho \mathbf{v}) = 0 \tag{4}$$

$$\partial_t (\varrho \mathbf{v}) + \nabla_x \cdot (\varrho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I}) = 0 \tag{5}$$

$$\partial_t E + \nabla_x \cdot ((p + E)\mathbf{v}) = 0 \tag{6}$$

with *equation of state* $p := p(u, \varrho)$.

COMPRESSIBLE NAVIER-STOKES SYSTEM

$$\partial_t \varrho + \nabla_x \cdot (\varrho \mathbf{v}) = 0 \tag{4}$$

$$\partial_t (\varrho \mathbf{v}) + \nabla_x \cdot \left(\varrho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I} + \mathbf{T} \right) = 0 \tag{5}$$

$$\partial_t E + \nabla_x \cdot \left((p + E + \mathbf{T} \cdot \mathbf{u} + \mathbf{q}) \mathbf{v} \right) = 0 \tag{6}$$

with *equation of state* $p := p(u, \varrho)$. The *viscous stress tensor* $\mathbf{T}$ is given by:

$$\mathbf{T} := -2\eta \mathbf{S} - \zeta \mathrm{div}_x \mathbf{v} \mathbf{I} \quad \text{with} \quad \mathbf{S} := \frac{1}{2} \left(\nabla_x \mathbf{v} + (\nabla_x \mathbf{v})^\top - \frac{2}{d} \mathrm{div}_x \mathbf{v} \mathbf{I} \right)$$

$$\mathbf{q} := -\kappa \nabla_x T.$$

with $\eta := \eta(u, \varrho) > 0$, $\zeta := \zeta(u, \varrho) > 0$ and $\kappa := \kappa(u, \varrho) > 0$.

ENERGY BALANCE EQUATIONS:

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{v}|^2 \right) + \nabla_x \cdot \left(\left(p + \frac{1}{2} \varrho |\mathbf{v}|^2 \right) \mathbf{v} + \mathbf{T} \cdot \mathbf{v} \right) = p \, \mathrm{div}_x \mathbf{v} - Q,$$

$$\partial_t u + \nabla_x \cdot (u\mathbf{v} + \mathbf{q}) = Q - p \, \mathrm{div}_x \mathbf{v},$$

where the rate of viscous heating of the fluid is:

$$Q := -\mathbf{T} : \nabla_x \mathbf{v} = 2\eta |\mathbf{S}|^2 + \zeta |\mathrm{div}_x \mathbf{v}|^2.$$

ENERGY BALANCE EQUATIONS:

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{v}|^2 \right) + \nabla_x \cdot \left(\left(p + \frac{1}{2} \varrho |\mathbf{v}|^2 \right) \mathbf{v} + \mathbf{T} \cdot \mathbf{v} \right) = p \, \mathrm{div}_x \mathbf{v} - Q,$$

$$\partial_t u + \nabla_x \cdot (u\mathbf{v} + \mathbf{q}) = Q - p \, \mathrm{div}_x \mathbf{v},$$

where the rate of viscous heating of the fluid is:

$$Q := -\mathbf{T} : \nabla_x \mathbf{v} = 2\eta |\mathbf{S}|^2 + \zeta |\mathrm{div}_x \mathbf{v}|^2.$$

The entropy density $s := s(u, \varrho)$ is thermodynamically related to u and ϱ through the first law of thermodynamics in the form:

$$T \mathrm{d}s = \mathrm{d}u - \mu \mathrm{d}\varrho.$$

with the *chemical potential* $\mu := \mu(u, \varrho)$. It follows that:

$$\partial_t s + \nabla_x \cdot \left(s\mathbf{v} + \frac{\mathbf{q}}{T} \right) = \frac{Q}{T} + \Sigma_\kappa =: \Sigma \geq 0,$$

where

$$\Sigma_\kappa := -\frac{\mathbf{q} \cdot \nabla_x T}{T^2} = \kappa \frac{|\nabla_x T|^2}{T^2}.$$

We consider limits of solutions $u^\varepsilon, \varrho^\varepsilon, \mathbf{v}^\varepsilon$ with $\eta^\varepsilon = \varepsilon\hat{\eta}$, $\zeta^\varepsilon = \varepsilon\hat{\zeta}$, $\kappa^\varepsilon = \varepsilon\hat{\kappa}$.

ASSUMP. 1: Given $\varepsilon > 0$, we assume there exists a unique smooth solution $u^\varepsilon, \varrho^\varepsilon, \mathbf{v}^\varepsilon$ of the NS equations on $\mathbb{T}^d \times [0, T]$ for a given equation of state.

We assume that $u^\varepsilon, \varrho^\varepsilon, \mathbf{v}^\varepsilon \in L^\infty(\mathbb{T}^d \times [0, T])$ uniformly in $\varepsilon > 0$ and that for some $1 \le p < \infty$ strong limits exist

$$u^\varepsilon \to u, \;\; \varrho^\varepsilon \to \varrho, \;\; \mathbf{v}^\varepsilon \to \mathbf{v} \;\text{ in }\; L^p(\mathbb{T}^d \times [0, T]).$$

We also assume no vacuum states, i.e. ϱ^ε, $\varrho \ge \varrho_0$ for some $\varrho_0 > 0$.

NOTE: Assumption 1 permits limiting fields with jump discontinuities.

ASSUMP. 2: Thermodynamic functions $h = p,\ T,\ s,\ \hat{\eta}, \ldots$ are C^2 in u, ρ.

ASSUMP. 3: The dissipation terms have distributional limits:

$$Q^\varepsilon_\eta := 2\eta^\varepsilon |\mathbf{S}^\varepsilon|^2, \quad Q^\varepsilon_\zeta := \zeta^\varepsilon (\mathrm{div}_x \mathbf{v}^\varepsilon)^2, \qquad Q^\varepsilon := Q^\varepsilon_\eta + Q^\varepsilon_\zeta \xrightarrow{\mathcal{D}} Q$$

$$\Sigma^\varepsilon_\eta := \frac{Q^\varepsilon_\eta}{T^\varepsilon}, \quad \Sigma^\varepsilon_\zeta := \frac{Q^\varepsilon_\eta}{T^\varepsilon}, \quad \Sigma^\varepsilon_\kappa := \kappa^\varepsilon \left| \frac{\nabla_x T^\varepsilon}{T^\varepsilon} \right|^2, \qquad \Sigma^\varepsilon := \Sigma^\varepsilon_\eta + \Sigma^\varepsilon_\zeta + \Sigma^\varepsilon_\kappa \xrightarrow{\mathcal{D}} \Sigma.$$

THEOREM 1: Any strong limits $u, \varrho, \mathbf{v}$ of NS solutions under Assumption 1–3 are weak solutions of the compressible Euler system on $\mathbb{T}^d \times [0, T]$. Furthermore, the following balances hold distributionally on $\mathbb{T}^d \times (0, T)$:

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{v}|^2\right) + \nabla_x \cdot \left(\left(p + \frac{1}{2}\varrho|\mathbf{v}|^2\right)\mathbf{v}\right) = p * \mathrm{div}_x\mathbf{v} - Q$$

$$\partial_t u + \nabla_x \cdot (u\mathbf{v}) = Q - p * \mathrm{div}_x\mathbf{v}$$

$$\partial_t s + \nabla_x \cdot (s\mathbf{v}) = \Sigma$$

with $Q \geq 0$ and $\Sigma \geq 0$ given by Assumption 3 and with

$$p * \mathrm{div}_x\mathbf{v} := \mathcal{D}\text{-}\lim_{\varepsilon\to 0} p^\varepsilon \, \mathrm{div}_x\mathbf{v}^\varepsilon$$

where this distributional limit necessarily exists.

REMARK: Analogous to Theorem of [Duchon & Robert, 00] in incompressible setting. Unlike them, we assume Q, Σ have distributional limits.

REMARK: Shock solutions [Johnson, 14] provide examples for which $Q, \Sigma > 0$. Presumably this can occur even with continuous solutions...

REMARK: Assuming Sutherland's law, i.e. $\kappa(u, \varrho) := \kappa(T(u, \varrho))$, then $\Sigma^\varepsilon \xrightarrow{\mathcal{D}} \Sigma$ is a consequence of the assumed strong convergence.

GENERAL WEAK SOLUTIONS:

weak solution $\Longleftrightarrow$ coarse-grained solution

$$\begin{aligned}
&\partial_t \bar{\varrho}_\ell + \nabla_x \cdot \bar{\mathbf{j}}_\ell = 0,\\
&\partial_t \bar{\mathbf{j}}_\ell + \nabla_x \cdot \left(\overline{(\mathbf{j} \otimes \mathbf{v})}_\ell + \bar{p}_\ell \mathbf{I} \right) = 0,\\
&\partial_t \bar{E}_\ell + \nabla_x \cdot \left(\overline{((E+p)\mathbf{v})}_\ell \right) = 0.
\end{aligned}$$

where $\overline{f_\ell} = G_\ell * f$. Following [H. Aluie (2013)], we deduce :

$$\partial_t \left(\frac{1}{2} \bar{\varrho}_\ell |\tilde{\mathbf{v}}_\ell|^2 \right) + \nabla_x \cdot \mathbf{J}^v_\ell = \bar{p}_\ell \, \overline{\mathrm{div}_x \mathbf{v}}_\ell - Q^{\mathrm{flux}}_\ell$$

with the Favre average $\tilde{\mathbf{v}}_\ell := \overline{(\varrho \mathbf{v})}_\ell / \bar{\varrho}_\ell$ and $\bar{\tau}_\ell(f,g) = \overline{(fg)}_\ell - \bar{f}_\ell \bar{g}_\ell$. Also:

$$\begin{aligned}
\mathbf{J}^v_\ell &:= \left(\frac{1}{2} \bar{\varrho}_\ell |\tilde{\mathbf{u}}_\ell|^2 + \bar{p}_\ell \right) \tilde{\mathbf{u}}_\ell + \bar{\varrho} \tilde{\mathbf{u}}_\ell \cdot \tilde{\tau}_\ell(\mathbf{u},\mathbf{u}) - \frac{\bar{p}_\ell}{\bar{\varrho}_\ell} \bar{\tau}_\ell(\varrho,\mathbf{u}),\\
Q^{\mathrm{flux}}_\ell &:= \frac{\nabla_x \bar{p}_\ell}{\bar{\varrho}_\ell} \cdot \bar{\tau}_\ell(\varrho,\mathbf{u}) - \bar{\varrho}_\ell \nabla_x \tilde{\mathbf{u}}_\ell : \tilde{\tau}_\ell(\mathbf{u},\mathbf{u}).
\end{aligned}$$

The term Q^{flux}_ℓ represents a turbulent cascade of kinetic energy.

KEY IDENTITY: Modified entropy density:

$$\underline{s}_\ell^* := s(\bar{u}_\ell, \underline{\varrho}_\ell) + \underline{\beta}_\ell k_\ell,$$

whose balance equation is derived to be

$$\partial_t \underline{s}_\ell^* + \nabla_x \cdot \mathbf{J}_\ell^{s*} = -I_\ell^{\mathrm{flux}} + \Sigma_\ell^{\mathrm{flux}*}$$

with

$$\begin{aligned}
\mathbf{J}_\ell^{s*} \quad &:= \quad \underline{s}_\ell \bar{\mathbf{u}}_\ell + \underline{\beta}_\ell \bar{\tau}_\ell(u, \mathbf{u}) - \underline{\lambda}_\ell \bar{\tau}_\ell(\varrho, \mathbf{u}), \\
&\qquad\qquad + \underline{\beta}_\ell \left(\frac{1}{2} \bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i) \tilde{\mathbf{u}}_\ell + \bar{\tau}_\ell(p, \mathbf{u}) + \frac{1}{2} \bar{\varrho}_\ell \tilde{\tau}_\ell(v_i, v_i, \mathbf{u}) \right) \\
I_\ell^{\mathrm{flux}} \quad &:= \quad \underline{\beta}_\ell (\bar{p}_\ell - \underline{p}_\ell) \overline{\Theta}_\ell, \\
\Sigma_\ell^{\mathrm{flux}*} \quad &:= \quad \nabla_x \underline{\beta}_\ell \cdot \bar{\tau}_\ell(u, \mathbf{u}) - \nabla_x \underline{\lambda}_\ell \cdot \bar{\tau}_\ell(\varrho, \mathbf{u}) \\
&\qquad\qquad + \underline{\beta}_\ell Q_\ell^{\mathrm{flux}} + \partial_t \underline{\beta}_\ell \, k_\ell + \nabla_x \underline{\beta}_\ell \cdot \mathbf{J}_\ell^k.
\end{aligned}$$

Inverse turbulent cascade of entropy [forward cascade of (neg)entropy].

THEOREM 2: Let $u, \varrho, \mathbf{v}$ be any $L^\infty(\mathbb{T}^d \times [0,T])$ weak solution of the compressible Euler system. Assuming the distributional limit $Q_{\rm flux} := \mathcal{D}\text{-}\lim_{\ell\to 0} Q_\ell^{flux}$ exists, the following balances hold weakly on $\mathbb{T}^d \times (0,T)$:

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{v}|^2\right) + \nabla_x\cdot\left(\left(p+\frac{1}{2}\varrho|\mathbf{v}|^2\right)\mathbf{v}\right) = p\circ {\rm div}_x\mathbf{v} - Q_{\rm flux}$$

$$\partial_t u + \nabla_x\cdot(u\mathbf{v}) = Q_{\rm flux} - p\circ{\rm div}_x\mathbf{v}$$

$$\partial_t s + \nabla_x\cdot(s\mathbf{v}) = \Sigma_{\rm flux}$$

with the definitions

$$\Sigma_{\rm flux} = \mathcal{D}\text{-}\lim_{\ell\to 0}\Sigma_\ell^{\rm flux}, \qquad p\circ{\rm div}_x\mathbf{v} := \mathcal{D}\text{-}\lim_{\ell\to 0}\bar{p}_\ell\overline{({\rm div}_x\mathbf{v})}_\ell$$

where these distributional limits necessarily exist.

REMARK: In general, $\Sigma_{\rm diss} = \Sigma_{\rm flux}$ but $p * {\rm div}_x\mathbf{v} \neq p\circ{\rm div}_x\mathbf{v}$. Thus

$$Q_{\rm diss} = Q_{\rm flux} + \tau(p, {\rm div}_x\mathbf{v}),$$

where $\tau(p,{\rm div}_x\mathbf{v}) = p * {\rm div}_x\mathbf{v} - p\circ{\rm div}_x\mathbf{v}$.

REMARK: Shock solutions [Johnson, 14] provide examples with

$$Q_{\rm flux} = 0 \quad \text{and} \quad Q_{\rm diss} = \tau(p,{\rm div}_x\mathbf{v}) > 0\,!$$

An *Onsager singularity theorem* in Besov spaces:

THEOREM 3: If the weak Euler solutions in Theorem 2 satisfy

$$u \in B_p^{\sigma_p^u,\infty}(\mathbf{T}^d \times [0,T]), \ \ \varrho \in B_p^{\sigma_p^\varrho,\infty}(\mathbf{T}^d \times [0,T]), \ \ \mathbf{v} \in B_p^{\sigma_p^v,\infty}(\mathbf{T}^d \times [0,T]),$$

with all three of the following conditions satisfied

$$\begin{aligned} 2\min\{\sigma_p^\varrho, \sigma_p^u\} + \sigma_p^v &> 1 \\ \min\{\sigma_p^\varrho, \sigma_p^u\} + 2\sigma_p^v &> 1 \\ 3\sigma_p^v &> 1 \end{aligned}$$

for any $p \geq 3$, then $Q_{\rm flux} = \Sigma_{\rm flux}$ exist and equal 0. Solutions of Theorem 1 satisfy the exponent conditions have $Q_{\rm diss} = \Sigma_{\rm diss} = 0$ and

$$p * {\rm div}_x \mathbf{v} = p \circ {\rm div}_x \mathbf{v}.$$

REMARK: This result generalizes those of [Constantin-E-Titi, 94] for incompressible NS and [Feireisl et al., 16] for the barotropic setting.

REMARK: Our conditions for $p = 3$ are sharp. Shock solutions with $u, \varrho, \mathbf{v} \in BV \cap L^\infty \subset B_3^{1/3,\infty}(\mathbb{T}^d)$ provide a simple example of dissipative Euler solutions saturating our bounds. For $p > 3$ the question remains open.

Our Theorem 3 is formulated in terms of space-time regularity, whereas the original statement of Onsager and most following works have given necessary conditions for anomalous dissipation in terms of space-regularity only.

THEOREM 4: Let $(u, \varrho, \mathbf{v})$ be any weak Euler solution satisfying $\varrho \geq \varrho_0 > 0$ and $u, \varrho, \mathbf{v} \in L^\infty([0, T] \times \mathbf{T}^d)$ together with:

$$u \in L^\infty([0, T]; B_p^{\sigma_p^u, \infty}(\mathbf{T}^d)),$$
$$\varrho \in L^\infty([0, T]; B_p^{\sigma_p^\varrho, \infty}(\mathbf{T}^d)),$$
$$\mathbf{v} \in L^\infty([0, T]; B_p^{\sigma_p^v, \infty}(\mathbf{T}^d)),$$

for Besov exponents $0 \leq \sigma_p^u, \sigma_q^\varrho, \sigma_q^v \leq 1$. Then the solutions are Besov regular in space-time:

$$u \in B_p^{\min\{\sigma_p^\varrho, \sigma_p^v, \sigma_p^u\}, \infty}(\mathbf{T}^d \times [0, T]),$$
$$\varrho \in B_p^{\min\{\sigma_p^\varrho, \sigma_p^u\}, \infty}(\mathbf{T}^d \times [0, T]),$$
$$\mathbf{v} \in B_p^{\min\{\sigma_p^\varrho, \sigma_p^v, \sigma_p^u\}, \infty}(\mathbf{T}^d \times [0, T]).$$

REMARK: This result is very similar to that obtained in [Isett, 2015] for Hölder-continuous weak solutions of incompressible Euler.

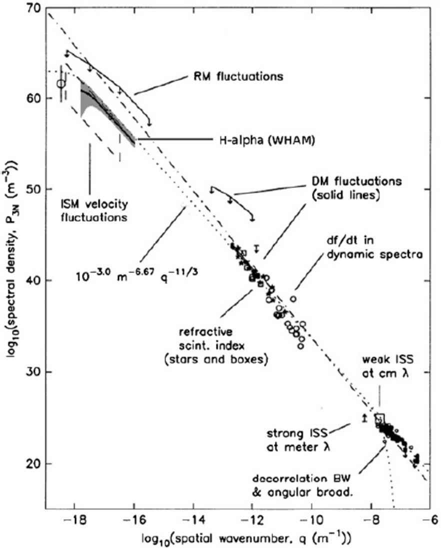


Figure 5. WHAM estimation for electron density overplotted on the figure of the Big Power Law in the sky figure from Armstrong et al. (1995). The range of statistical errors is marked with gray color.

[Chepurnov & Lazarian, 2010]

FUTURE WORK & OPEN QUESTIONS

FIRST HALF: SPONTANEOUS STOCHASTICITY

1) Further explore Lagrangian mechanisms of Nusselt-Rayleigh scaling by measuring mixing time τ_{mix} in Rayleigh-Bénard simulations/experiments.

2) How does spontaneous stochasticity relate to the dissipative anomalous for incompressible Navier-Stokes? Constantin-Iyer representation, Eyink Martingale Hypothesis for circulations.

SECOND HALF: COMPRESSIBLE ONSAGER

1) Weaken Assumption 1. There is evidence that at sufficiently high Mach numbers the limiting mass density ρ as $\varepsilon \to 0$ may exist only as a measure and not as a bounded function [Kim et. al, 2005].

2) Result readily generalize to relativistic Euler equations in Minkowski spacetime. General relativity? Gas and plasma dynamics such as the Vlasov-Maxwell or Boltzmann?

ACKNOWLEDGEMENTS

We thank Charlie Doering, Dan Ginsberg, David Goluskin, Cristian Lalescu, Sam Punshon-Smith and Katepalli Sreenivasan for useful discussions. We also thank Hussein Aluie for sharing unpublished work.

We also thank IPAM, UCLA, where the work in Part 1 of this talk was initiated during the program “Mathematics of Turbulence”.