Variations on a theme by Cauchy

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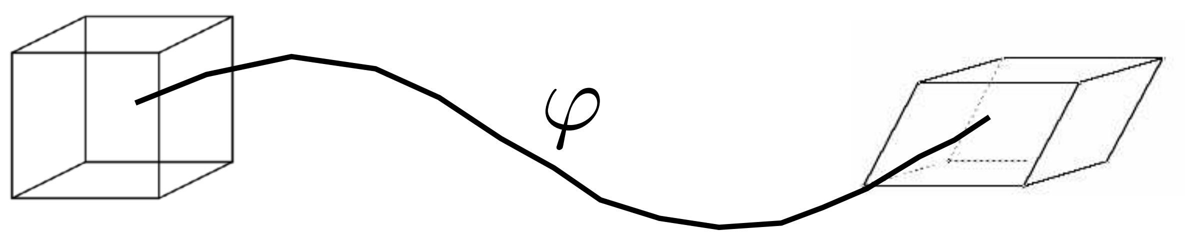
- Euler equations in flat or curved space and differential geometry (DG)
- Cauchy's (1815) Lagrangian invariants equations and Cauchy's vorticity formula from a traditional and DG point of view arxiv.org/abs/1701.01592
- Generalized Cauchy invariants, local helicities galore
- Recursion relations for time-Taylor coefficients and analyticity; the Cauchy-Lagrange numerical method
- A few words about wall-bounded flow and blow-up

Eulerian (spatial) and Lagrangian (material) coordinates

Volume : V Volume : V'



Time $s = 0$
$\boldsymbol{a} = (a, b, c)$
Lagr. coord.

Time $s = t$
$\boldsymbol{x} = (x, y, z)$
Eul. coord.

Lagrangian map :
$\boldsymbol{a} \mapsto \boldsymbol{x} = \varphi(\boldsymbol{a}, t)$.
Jacobian : $J \equiv \det(\partial x_i / \partial a_j) = V'/V$.
Incompressible flow : $J = 1$

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p, \qquad \nabla \cdot \boldsymbol{v} = 0$$

The geometrization of the Lagrangian approach

Vladimir Arnold 1937 - 2010

Arnold (1966) (Ann. Inst. Fourier): The solutions of the incompressible Euler equations extremize the kinetic-energy-based action :

Kolmogorov/Pontryagin

$$A = \frac{1}{2} \int_0^T \int_\Omega \mu(a) g_{ij}(\varphi(a,t)) \dot{\varphi}^i(a,t) \dot{\varphi}^j(a,t) \quad \text{(Euclidean case:} \quad g_{ij} = \delta_{ij},\ \mu(a) = d^3 a),$$

with the constraints $\varphi(a, 0) = a$ and given $\varphi(a, T)$.

From a DG point of view, vorticity is an invariant

When changing from Eulerian to Lagrangian coordinates, a scalar function, say a temperature θ must just be compose with the Lagrangian map φ :

$$\theta^{\text{Lagrangian}} = \theta^{\text{Eulerian}} \circ \varphi$$

When working with differential forms one must take into account the change in elements of length, surface, volume, …, to obtain the *pullback* (Eulerian to Lagrangian) and the *pushforward* (Lagrangian to Eulerian) transformations.

Consider the velocity 1-form $v^\flat := \vec{v} \cdot \overrightarrow{dx}$. It satisfies

$$\partial_t v^\flat + \pounds_v v^\flat + d\left(p - \frac{1}{2}(v, v)_g\right) = 0,$$

which can be written in more traditional (Euclidean) fluid mechanics notation

$$\partial_t \vec{v} + \frac{1}{2}\vec{\nabla}|\vec{v}|^2 + \vec{v} \cdot \vec{\nabla}\vec{v} + \vec{\nabla}\left(p - \frac{|\vec{v}|^2}{2}\right) = 0$$

The vorticity 2-form $\omega := dv^\flat = \sum_{i<j}\left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i}\right) dx^i \wedge dx^j$

thus satisfies $\partial_t \omega + \pounds_v \omega = 0$. The vorticity is Lie-advection invariant!
A first instance is Helmholtz (1858). But actually goes back to Cauchy.

The Cauchy (1815) invariants equation in DG notation

Use a dot for the Lagrangian derivative $\partial_t + v_k\partial_k$, Euler eq. reads $\ddot{x}_k = -\partial_k p$. Multiply by dx^k and sum over k : $\ddot{x}_k dx^k = -dp$.
Denote the Lagrangian time derivative by D_t when needed, rewrite as

$$D_t\left(\dot{x}_k dx^k\right) - d\frac{|\dot{x}|^2}{2} = -dp.$$

Apply the exterior derivative d, use $dd = 0$ and integrate over time from 0 to t:
The Cauchy invariants equation $d\dot{x}_k \wedge dx^k = d(v_{0k}da^k) = \omega_0$.
In traditional fluid mechanics notation it is $\vec{\nabla}^{\mathrm{L}}\dot{x}_k \times \vec{\nabla}^{\mathrm{L}}x_k = \vec{\omega_0}$.

By elementary manipulations of the Jacobian matrix $\vec{\nabla}^{\mathrm{L}}x$ Cauchy obtains his famous vorticity formula $\vec{\omega} = \vec{\omega_0} \cdot \vec{\nabla}^{\mathrm{L}}x$.

Actually, both the Cauchy invariants equation and his vorticity formula are instances, related by Hodge duality, of a general result on Lie-advection invariant exact p-forms.

The generalized Cauchy invariants equation

We now work with a prescribed time-dependent velocity field v_t, along which some p-form γ is Lie-advected. If γ is exact, one has a *Generalized Cauchy invariants equation* and (by Hodge duality) a *Generalized Cauchy formula.* Specifically, let

$$\gamma = d\alpha \quad \text{and} \quad \partial_t \gamma_t + \pounds_{v_t} \gamma_t = 0. \qquad \frac{d}{dt}\varphi_t = v(t, \varphi_t), \quad \varphi_0 = \text{Identity},$$

then $$\frac{1}{(p-1)!}\,\delta^{i_1\ldots i_{p-1}}_{j_1\ldots j_{p-1}} d\alpha_{i_1\ldots i_{p-1}} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{p-1}} = \gamma_0, \quad x = \varphi_t,$$

where $$\delta^{j_1\ldots j_p}_{i_1\ldots i_p} = \begin{vmatrix} \delta^{j_1}_{i_1} & \ldots & \delta^{j_1}_{i_p} \\ \vdots & \ddots & \vdots \\ \delta^{j_p}_{i_1} & \ldots & \delta^{j_p}_{i_p} \end{vmatrix}$$

and $$\frac{1}{(p-1)!}\,\delta^{i_1\ldots i_{p-1}}_{j_1\ldots j_{p-1}} \star \left(d\alpha_{i_1\ldots i_{p-1}} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{p-1}}\right) = \star\gamma_0,$$

where $\star$ denotes the Hodge dual operator.

Local helicities in hydrodynamics and MHD

- In 3D hydrodynamics and MHD there are known integral invariants, such as the kinetic helicity $\int d^3x\,\vec{v}\cdot\vec{\omega}$, the magnetic helicity $\int d^3x\,\vec{A}\cdot\vec{B}$ and the cross-helicity $\int d^3x\,\vec{v}\cdot\vec{B}$.
- All these helicities have *local* counterparts, material invariants, that is Lie-advected (LA) 0-forms, Hodge duals of LA 3-forms, namely the local helicity $\sigma = u \wedge \omega$, the cross helicity $\xi = u \wedge B$ and the Elsasser local magnetic helicity $h = \mathcal{A} \wedge B$. (All forms are LA.)
- Here, ω and B denote the vorticity and magnetic 2-forms, u (resp. $\mathcal{A}$) are the LA Clebsch (resp. magnetic vector potential) 1-forms, taken equal to the velocity $v^\flat$ (resp. the magnetic vector potential A) 1-forms at $t = 0$. Note that $\omega = dv^\flat = du$ and $B = dA = d\mathcal{A}$.
- One also has the following generalized Cauchy invariants equations:

$$dA_k \wedge dx^k = B_0 = dA_0 \quad \text{and} \quad \frac{1}{2}\delta^{kl}_{ij}\, d\pi_{kl} \wedge dx^i \wedge dx^j = \sigma_0,$$

where $\sigma = u \wedge \omega = d\pi$ (if the Poincaré lemma applies).

Recursion relations derived from Cauchy's invariants

$$\sum_{k=1}^{3} \nabla^{\mathrm{L}} \dot{x}_k \times \nabla^{\mathrm{L}} x_k = \boldsymbol{\omega}_0$$

$$\det(\nabla^{\mathrm{L}} \boldsymbol{x}) = 1$$

- Introduce the displacement: $\boldsymbol{\xi} := \mathbf{x} - \mathbf{a}$

$$\nabla^{\mathrm{L}} \times \dot{\boldsymbol{\xi}} + \nabla^{\mathrm{L}} \dot{\xi}_k \times \nabla^{\mathrm{L}} \xi_k = \boldsymbol{\omega}_0$$

$$\det\left(\mathbf{I} + \nabla^{\mathrm{L}} \boldsymbol{\xi}\right) = 1 \quad \text{or} \quad \nabla^{\mathrm{L}} \cdot \boldsymbol{\xi} + \frac{1}{2}\Big[(\nabla^{\mathrm{L}} \cdot \boldsymbol{\xi})^2 - \mathrm{tr}\big(\nabla^{\mathrm{L}} \boldsymbol{\xi}\big)^2\Big] + \det(\nabla^{\mathrm{L}} \boldsymbol{\xi}) = 0$$

- Expand (formally) in powers of t: $\boldsymbol{\xi} = \sum_{n=1}^{\infty} t^n \boldsymbol{\xi}^{(n)}$, and determine coefficients of various powers:

$$n \nabla^{\mathrm{L}} \times \boldsymbol{\xi}^{(n)} + \sum_{r+s=n} r \nabla^{\mathrm{L}} \xi_k^{(r)} \times \nabla^{\mathrm{L}} \xi_k^{(s)} = \boldsymbol{\omega}_0 \delta_{n1}, \quad n = 1, 2, \ldots$$

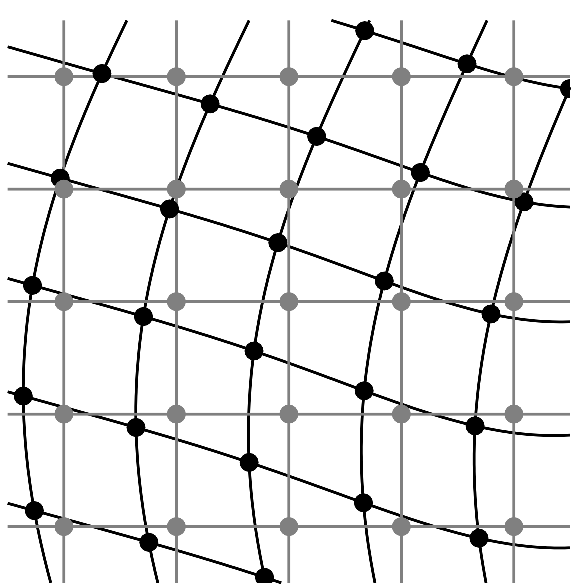
$$\nabla^{\mathrm{L}} \cdot \boldsymbol{\xi}^{(n)} + \frac{1}{2} \sum_{r+s=n} \Big[\nabla^{\mathrm{L}} \cdot \xi^{(r)}\, \nabla^{\mathrm{L}} \cdot \xi^{(s)} - \mathrm{tr}\big(\nabla^{\mathrm{L}} \xi^{(r)} \nabla \xi^{(s)}\big)\Big] + \sum_{r+s+\sigma=n} \nabla^{\mathrm{L}} \xi_1^{(r)} \cdot \left(\nabla^{\mathrm{L}} \xi_2^{(s)} \times \nabla^{\mathrm{L}} \xi_3^{(\sigma)}\right) = 0$$

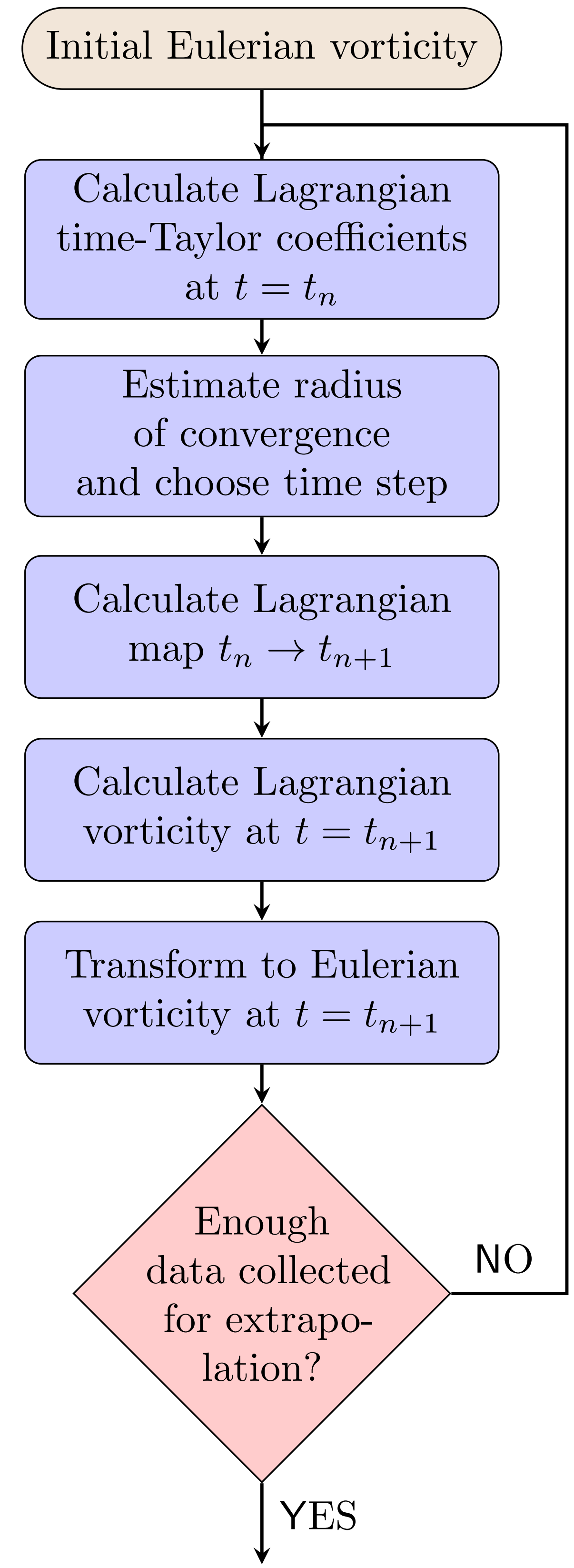
- In the presence of a solid impermeable boundary, these recursion relations for the time-Taylor coefficients $\xi^{(n)}$ must be supplemented by conditions expressing the invariance of the boundary under the Lagrangian flow.

- The recursion relations can be used to obtain bounds on the time-Taylor coefficients and prove analyticity in time of the Lagrangian particle trajectories, but also to derive novel semi-Lagrangian numerical integration schemes.

The Lagrangian algorithm built on the Cauchy invariants

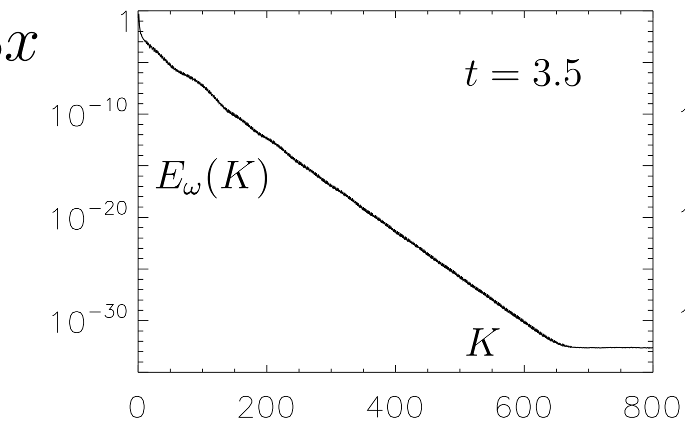
- In 1928 Courant, Friedrichs and Lewy showed that numerical solutions of hyperbolic PDE's by simple finite difference methods are subject to the constraint $\Delta t < \Delta x / V_{\max}$.
- In hydrodynamics, this affects Eulerian but not Lagrangian algorithms. The use of high-order Lagrangian time-Taylor expansions allows us to study, e.g. blowup, by semi-Lagrangian high-order numerical schemes.

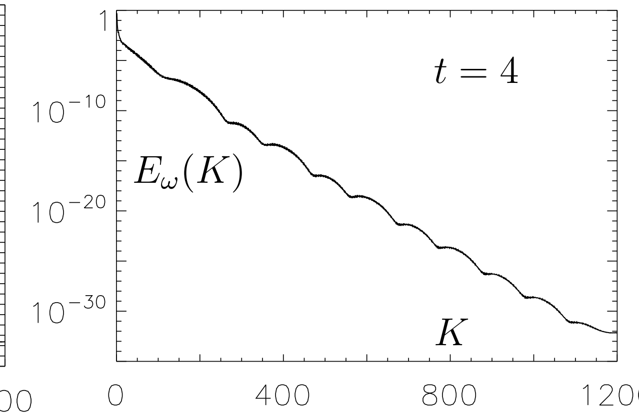


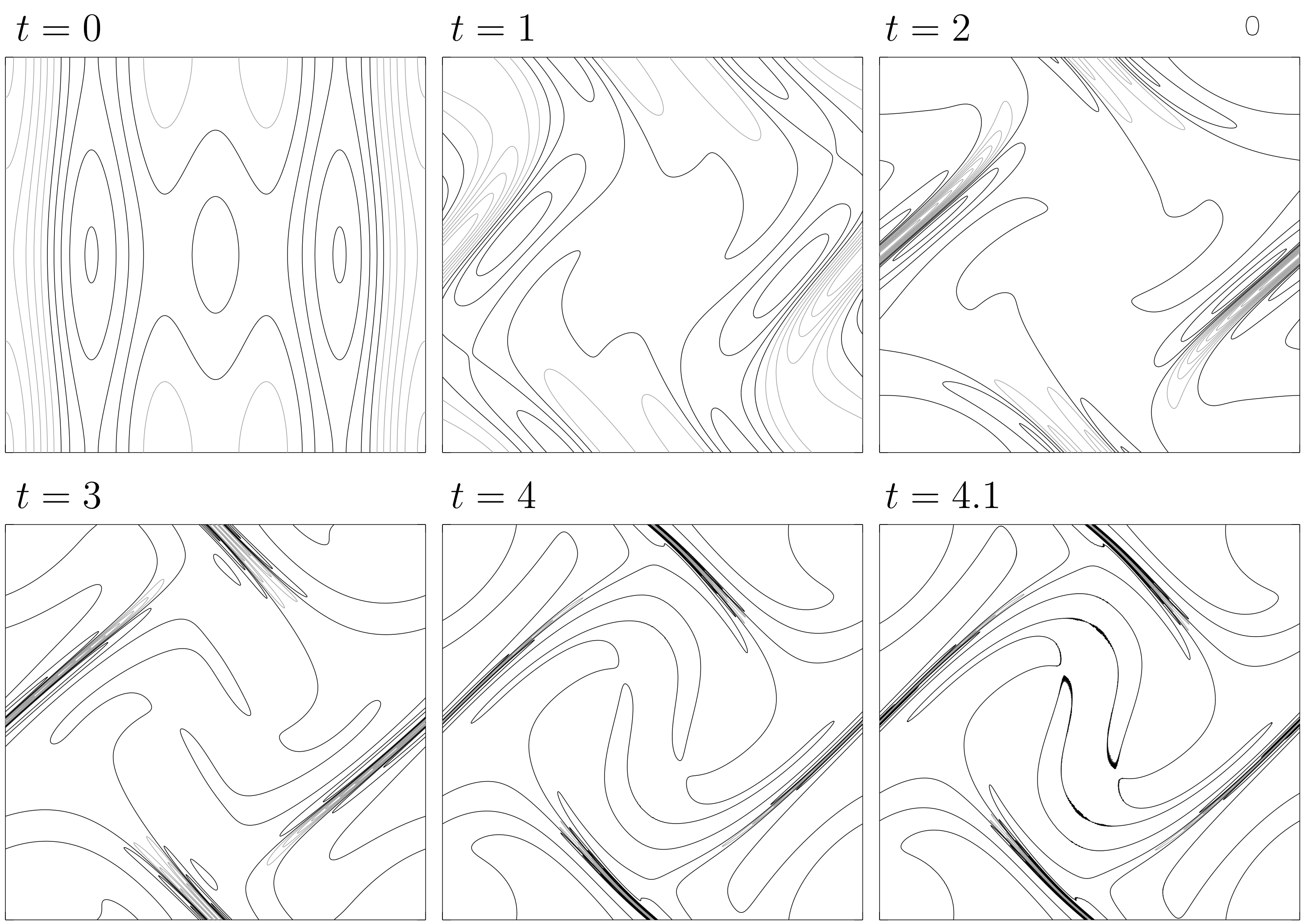


Switching from Eulerian to Lagrangian computations can result in speed up of several orders of magnitude

$$\boldsymbol{\omega}^{(\mathrm{init})} = \cos x + \cos y + 0.6\cos 2x + 0.2\cos 3x$$

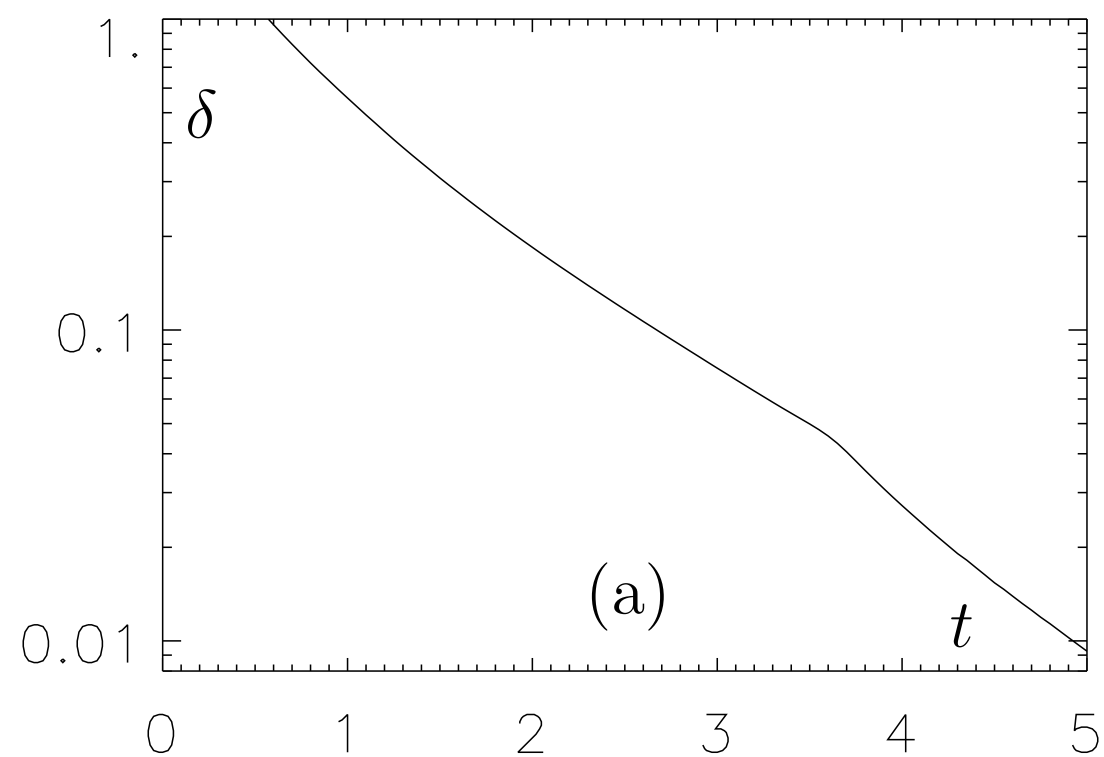






Isolines of the Laplacian of vorticity

Distance $\delta(t)$ to the real domain of the nearest complex-space singularity



Resolution: 8192^2 harmonics

Speed up CL20/RK4 : 200

Some useful references

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