Turbulent energy cascades in hydrodynamics and magneto-hydrodynamics

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Piero della Francesca “Playing with Mirror Symmetry” ~ 1450 C.E. Monterchi IT
A TALE ABOUT TRANSFER PROPERTIES OF INVISCID CONSERVED QUANTITIES, KINETIC ENERGY, HELICITY ENSTROPHY, MAGNETIC HELICITY ETC...

Q1: HOW TO PREDICT THE DIRECTION OF THE TRANSFER (FORWARD/BACKWARD) AND ITS ROBUSTNESS UNDER EXTERNAL PERTURBATION (FORCING/BOUNDARY CONDITIONS)?

Q2: HOW MUCH THE FLUCTUATIONS AROUND THE MEAN TRANSFER ARE INTENSE AND SELF-SIMILAR (INTERMITTENCY AND ANOMALOUS SCALING)?

AS A MATTER OF FACT, FOR 3D NAVIER STOKES EQUATIONS, WE DO NOT KNOW HOW TO PREDICT NEITHER THE SIGN OF THE MEAN ENERGY TRANSFER NOR THE INTENSITY OF THE FLUCTUATIONS AROUND IT.
EXPLORING THE ROLE OF MIRROR SIMMETRY

- ROLE OF KINETIC HELICITY IN THE REVERSAL OF THE MEAN ENERGY FLUX IN 3D NAVIER-STOKES (FORWARD/BACKWARD) AND IN THE FORMATION OF REAL-SPACE INTERMITTENCY

- IMPLICATION FOR THE SMALL-SCALES REGULARITY OF THE NAVIER-STOKES SOLUTIONS

- EMPIRICAL OBSERVATION ON ROTATING TURBULENCE

- IMPLICATION FOR THE STATISTICS OF THE REYNOLDS STRESS AND FOR THE SUB-GRID ENERGY TRANSFER IN TURBULENCE MODELING

- ROLE OF MAGNETIC HELICITY IN THE FORMATION OF LARGE AND SMALL SCALES DYNAMO IN MAGNETOHYDRODYNAMICS
Figure 4

Snapshot of the intensity distributions of (a) the energy-dissipation rate $\varepsilon = \varepsilon/(2\nu)$ and (b) the enstrophy $\Omega = \omega^2$ for DNS-ES at $R_\lambda = 675$ in arbitrary units.
PARISI-FRISCH MULTIFRACTAL PREDICTION FOR ACCELERATION

The graph illustrates the multifractal prediction for acceleration. The formula provided is:

\[ P(a) \sim \int_{h \in I} dh a^{(h-3+D(h))/3} \nu^{7-2h-2D(h)/3} L_0 D(h)+h-3 \sigma_v^{-1} \times \exp \left( -\frac{a^{\frac{2(1+h)}{3}}}{2} \frac{\nu^{\frac{2(1-2h)}{3}}}{2\sigma_v^2} \right) \]
The joint cascade of energy and helicity in three-dimensional turbulence

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The role of helicity in three-dimensional turbulence is, in our opinion, still somewhat mysterious. In particular, it is still unclear how energy and helicity dynamics interact in detail. The role of helicity in geophysical flows has been considered—without being fully resolved—while its appearance and influence in engineering applications is still largely unexplored. We hope that this work will be a helpful step in the direction of better understanding the subtle manifestations of helicity in three-dimensional turbulence.
CONFINEMENT 3D → 2D

Upscale energy transfer in thick turbulent fluid layers

H. Xia, D. Byrne, G. Falkovich and M. Shats

Nat Phys. 2011
How to switch from 3D to 2D?

INVERSE ENERGY CASACE UNDER ROTATION
3D -> 2D
HELICITY ENHANCEMENTS

P.D. Mininni and A. Pouquet.
\[ \partial_t \bar{v} + (\bar{v} \cdot \nabla) \bar{v} = -\nabla \bar{p} - \nabla \cdot \tau(v, v) + \nu \Delta \bar{v} \]

SUB GRID /REYNOLDS STRESS:
\[ \tau_{ii}(v, v) = \bar{v}_i \bar{v}_i - \bar{v}_i \bar{v}_i \]
\[ \partial_t \frac{1}{2} \bar{v}_i \bar{v}_i + \partial_j A_j = -\Pi \]

SUB GRID ENERGY TRANSFER:
\[ \Pi = -\tau_{ij} \bar{S}_{ij} \quad \bar{S}_{ij} = \frac{1}{2} (\partial_i \bar{v}_j + \partial_j \bar{v}_i) \]
Helicity and singular structures in fluid dynamics

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This contribution is part of the special series of Inaugural Articles by members Contributed by H. Keith Moffatt, January 14, 2014 (sent for review December

Helicity is, like energy, a quadratic invariant of the Euler equations of ideal fluid flow, although, unlike energy, it is not sign definite. In physical terms, it represents the degree of linkage of the vortex lines of a flow, conserved when conditions are such that these vortex lines are frozen in the fluid. Some basic properties of helicity are reviewed, with particular reference to (i) its crucial role in the dynamo excitation of magnetic fields in cosmic systems; (ii) its bearing on the existence of Euler flows of arbitrarily complex streamline topology; (iii) the constraining role of the analogous magnetic helicity in the determination of stable knotted minimum-energy magnetostatic structures; and (iv) its role in depleting nonlinearity in the Navier-Stokes equations, with implications for the coherent structures and energy cascade of turbulence. In a final section, some singular phenomena in low Reynolds number flows are briefly described.

Helicity conservation by flow across scales in reconnecting vortex links and knots

Martin W. Scheeler, Dustin Kleckner, Davide Proment, Gordon L. Kindlmann, and William T. M.

The conjecture that helicity in fluid mechanics, but the nature of this conservation in the presence of dissipation has proven difficult to resolve. Making use of recent advances, we create vortex knots and links in viscous fluids and simulated superfluids and track their geometry through topology-changing reconnections. We find that the reassociation of vortex lines through a reconnection enables the transfer of helicity from links and knots to helical coils. This process is remarkably efficient, owing to the antiparallel orientation spontaneously adopted by the reconnecting vortices. Using a new method for quantifying the spatial helicity spectrum, we find that the reconnection process can be viewed as transferring helicity between scales, rather than dissipating it. We also infer the presence of geometric deformations that convert helical coils into even smaller scale twist, where it may ultimately be dissipated. Our results suggest that helicity conservation plays an important role in fluids and related fields, even in the presence of dissipation.
Q: CAN WE DISSECT 3D NS EQUATIONS TO EXTRACT INTERESTING INFORMATION FROM ITS ELEMENTARY CONSTITUENTS?

\[
\begin{align*}
\partial_t v + (v \cdot \partial) v &= -\partial P + \nu \Delta v + F \\
\partial \cdot v &= 0 \\
+ \text{Boundary Conditions}
\end{align*}
\]
The Beltrami Spectrum for Incompressible Fluid Flows

Peter Constantin\textsuperscript{1,\*} and Andrew Majda\textsuperscript{2,\**}

\[ u(k) = u^+(k)h^+(k) + u^-(k)h^-(k) \]

\[ h^\pm = \hat{\nu} \times \hat{k} \pm i\hat{\nu} \]

\[ \hat{\nu} = z \times k / \|z \times k\| \]

\[ ik \times h^\pm = \pm kh^\pm \]

\[ \begin{cases} E = \sum_k |u^+(k)|^2 + |u^-(k)|^2; \\ H = \sum_k k(|u^+(k)|^2 - |u^-(k)|^2). \end{cases} \]
\[
\frac{d}{dt} u^s_k(k, t) \quad (s_k = \pm 1)
\]

\[
\frac{d}{dt} u^s_k(k) + \nu k^2 u^s_k(k) = \sum_{k+p+q=0} \sum_{s_p,s_q} g_k,p,q (s_p p - s_q q) \times [u^{s_p}(p) u^{s_q}(q)]^*. 
\] (15)

Eight different types of interaction between three modes \(u^s_k(k), u^s_p(p),\) and \(u^s_q(q)\) with \(|k| < |p| < |q|\) are allowed according to the value of the triplet \((s_k, s_p, s_q)\)

\[
\dot{u}^s_k = r(s_p p - s_q q) \frac{s_k k + s_p p + s_q q}{p} (u^{s_p}(p) u^{s_q}(q))^*,
\]

\[
\dot{u}^s_p = r(s_q q - s_k k) \frac{s_k k + s_p p + s_q q}{p} (u^{s_q}(q) u^{s_k}(k))^*,
\]

\[
\dot{u}^s_q = r(s_k k - s_p p) \frac{s_k k + s_p p + s_q q}{p} (u^{s_k}(k) u^{s_p}(p))^*.
\]
HELICAL TRIADIC INTERACTION IN THE NAVIER-STOKES EQS

\[ u^+(k) \quad u^+(p) \quad u^+(q) \]

\[ u^-(k) \quad u^-(p) \quad u^-(q) \]
\[
\begin{align*}
E &= \sum_k |u^+(k)|^2 + |u^-(k)|^2; \\
H &= \sum_k k(|u^+(k)|^2 - |u^-(k)|^2).
\end{align*}
\]
TRIADIC INTERACTION IN REWEIGHTED NAVIER-STOKES EQUATIONS

\[ 0 \leq \lambda \leq 1 \]
TRIADIC INTERACTION IN STOCHASTICALLY DECIMATED NAVIER_STOKES EQUATIONS

\[ 0 \leq \lambda \leq 1 \]

\[ \text{Prob} \propto \lambda \]

\[ \text{Prob} \propto \lambda^2 \]
\[ P^{\pm} \equiv \frac{h^{\pm} \otimes h^{\pm}}{h^{\pm} \cdot h^{\pm}}. \quad v^{\pm}(x) \equiv \sum_{k} P^{\pm} u(k); \]

\[ u(k) = u^{+}(k) h^{+}(k) + u^{-}(k) h^{-}(k) \]

\[ \partial_t v^{+} + P^{+} B[v^{+}, v^{+}] = \nu \Delta v^{+} + f^{+} \]
\[ E = \sum_k |u^+(k)|^2 + |u^-(k)|^2; \]
\[ H = \sum_k k(|u^+(k)|^2 - |u^-(k)|^2). \]
LARGE SCALES FORCING: DIRECT HELICITY CASCADE

HOMOCHIRAL 3D NAVIER STOKES EQS.

L. B., S. Musacchio and F. Toschi

VANISHING INTERMITTENCY

forcing

FLUX HELICITY

\( U_{\text{rms}} \)
On the Global Regularity of a Helical-Decimated Version of the 3D Navier-Stokes Equations

We study the global regularity, for all time and all initial data in $H^{1/2}$, of a recently introduced decimated version of the incompressible 3D Navier-Stokes (dNS) equations. The model is based on a projection of the dynamical evolution of Navier-Stokes (NS) equations into the subspace where helicity (the $L^2$-scalar product of velocity and vorticity) is sign-definite. The presence of a second (beside energy) sign-definite inviscid conserved quadratic quantity, which is equivalent to the $H^{1/2}$-Sobolev norm, allows us to demonstrate global existence and uniqueness, of space-periodic solutions, together with continuity with respect to the initial conditions, for this decimated 3D model. This is achieved thanks to the establishment of two new estimates, for this 3D model, which show that the $H^{1/2}$ and the time average of the square of the $H^{3/2}$ norms of the velocity field remain finite. Such two additional bounds are known, in the spirit of the work of H. Fujita and T. Kato (Arch. Ration. Mech. Anal. 16:269–315, 1964; Rend. Semin. Mat. Univ. Padova 32:243–260, 1962), to be sufficient for showing well-posedness for the 3D NS equations. Furthermore, they are directly linked to the helicity evolution for the dNS model, and therefore with a clear physical meaning and consequences.
EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS OF THE HELICAL-DECIMATED NSE

\[\begin{aligned}
    \partial_t v^+ &= \mathcal{P}^+(-v^+ \cdot \nabla v^+ - \nabla p^+) + \nu \Delta v^+ + f^+ \\
    \nabla \cdot v^+ &= 0
\end{aligned}\]

HILBERT-NORM COINCIDES WITH THE SIGN-DEFINITE HELICITY

\[||g||_{H^{1/2}} = \sum_k k |g(k)|^2\]

CONSERVATION HELICITY: NEW APRIORI BOUND ON THE VELOCITY

\[\frac{1}{2} \partial_t \sum_k k|u^+(k, t)|^2 + \frac{\nu}{2} \sum_k k^3|u^+(k, t)|^2 \leq \frac{1}{2\nu} \sum_k |f^+(k)|^2 k^{-1}.\]

\[\frac{1}{2} \partial_t ||v^+||_{H^{1/2}} + \frac{\nu}{2} ||v^+||_{H^{3/2}} \leq \frac{1}{2\nu} \sum_k |f^+(k)|^2 k^{-1}.\]

\[v^+ \in L_t^{\infty} H^{1/2}_x; \quad \sqrt{\nu} v^+ \in L_t^2 H^{3/2}_x\]

In this section, we are going to use the fact that for the inviscid and unforced dNS system the helicity is formally conserved, and that it is positive-definite quadratic quantity, which is equivalent to the square of the $\dot{H}^{1/2}$–Sobolev norm. Therefore, obtaining uniform (in time) bounds on the helicity enables us to prove the existence of solutions with a higher degree of regularity, provided the initial data is in $\dot{H}^{1/2}$. Furthermore, this additional regularity will allow us to prove, in the next section, the uniqueness of these regular solutions within the class of weak solutions. Let us observe that all the estimates that follow are formal, but can be rigorously justified by obtaining them first for the corresponding solutions of the Galerkin approximating system, and then passing to the limit, modulo subsequences, with $N \to \infty$. Furthermore, it is worth mentioning that similar ideas and estimates can be found in [19, 22] in the study of short time existence and uniqueness of the three-dimensional NS equations with initial data in $H^{1/2}$. The advantage of system over the NS equations is that the $H^{1/2}$ remains finite, which allows to extend the short time existence argument to prove global regularity for all time and all initial data in $H^{1/2}$. Indeed, the fact that helicity is a
Rossby = 2
Rossby = 0.8
Rossby = 0.2
Rossby = 0.1

Ω < Ω_c

Ro < Ro_c

Ω > Ω_c
Inverse cascade at $\Omega = 50$

Inverse flux is brought mainly by +++ and --- triads.

\[
\Pi^{(+ + +)}(k) = \langle u^+_{<k} N(u^+, u^+) \rangle
\]

\[ Ro \sim 0.15 \]
Inverse cascade at $\Omega = 50$

Inverse flux is brought mainly by +++ and --- triads.

$\Pi^{+++}(k) = \langle u^+_{<k} N(u^+, u^+) \rangle$

$\Pi^{tot}(k) = \langle u_{<k} N(u, u) \rangle$

$Ro \sim 0.15$

WITH G. SAHOO AND P. PERLEKAR (unpublished)
\[ u^\alpha(x) \equiv D^\alpha u(x) \equiv \sum_k e^{ikx} D_k^\alpha u_k, \] (4)

where \( D_k^\alpha \equiv (1 - \gamma_k^\alpha) + \gamma_k^\alpha P_k^+ \) and \( \gamma_k^\alpha = 1 \) with probability \( \alpha \) or \( \gamma_k^\alpha = 0 \) with probability \( 1 - \alpha \). The \( \alpha \)-decimated Navier-Stokes equations (\( \alpha \)-NSE) are

\[ \partial_t u^\alpha = D^\alpha [-u^\alpha \cdot \nabla u^\alpha - \nabla p^\alpha] + \nu \Delta u^\alpha, \] (5)

\[ \alpha = 1 - \lambda \]

TRIADIC INTERACTION IN STOCHASTICALLY DECIMATED NAVIER_STOKES EQS

\[ E(k) = E^+(k) + E^-(k) \]
\[ H(k) = k(E^+(k) - E^-(k)) \]

\[ E^+(k) \]
\[ E^-(k) \]

RECOVERY OF MIRROR SYMMETRY

\[ K(r) = \frac{\langle (\delta_r v)^4 \rangle}{\langle (\delta_r v)^2 \rangle^2} - 3 \]
TRIADIC INTERACTION IN REWEIGHTED NAVIER-STOKES Eqs

\[ \mathcal{N} = \lambda (u \times w) + (1 - \lambda) [\mathbb{P}^+(u^+ \times w^+) + \mathbb{P}^-(u^- \times w^-)] \]

FORCING

BACKWARD ENERGY FLUX

FORWARD ENERGY FLUX

HOMO CHIRAL

FULL NS

WITH G. SAHOO AND A. ALEXAKIS (SUBMITTED TO PRL 2016)
Large-scale magnetic fields in MHD

\[
\frac{\partial}{\partial t} u = - \frac{1}{\rho} \nabla P - (u \cdot \nabla) u + \frac{1}{\rho} (\nabla \times b) \times b + \nu \Delta u
\]

\[
\frac{\partial}{\partial t} b = (b \cdot \nabla) u - (u \cdot \nabla) b + \eta \Delta b
\]

\[\nabla \cdot u = 0 \text{ and } \nabla \cdot b = 0\]

\[
H_m(t) = \int_V d\mathbf{x} \ a(\mathbf{x}, t) \cdot b(\mathbf{x}, t) \rightarrow \text{inverse cascade}
\]

\[
H_k(t) = \int_V d\mathbf{x} \ u(\mathbf{x}, t) \cdot \omega(\mathbf{x}, t) \rightarrow \text{dynamo action (e.g. } \alpha\text{-effect)}
\]
Helical Fourier decomposition

\[
(\partial_t + \nu k^2)u_{k}^{s^*_k} = \frac{1}{2} \sum_{k+p+q=0} \left( \sum_{s_p, s_q} g_{sp, s_q}^{s_k} (s_p p - s_q q) u_{p}^{s_p} u_{q}^{s_q} \right. \\
- \left. \sum_{\sigma_p, \sigma_q} g_{\sigma_p \sigma_q}^{s_k} (\sigma_p p - \sigma_q q) b_{p}^{\sigma_p} b_{q}^{\sigma_q} \right)
\]

\[
(\partial_t + \eta k^2)b_{k}^{s^*_k} = \frac{\sigma_k k}{2} \sum_{k+p+q=0} \left( \sum_{\sigma_p, s_q} g_{\sigma_p s_q}^{s_k} b_{p}^{\sigma_p} u_{q}^{s_q} - \sum_{s_p, \sigma_q} g_{s_p \sigma_q}^{s_k} u_{p}^{s_p} b_{q}^{\sigma_q} \right)
\]

\[
E_{kin} = \sum_{k} |u_{k}^{+}|^2 + |u_{k}^{-}|^2 \quad H_{kin} = \sum_{k} k(|u_{k}^{+}|^2 - |u_{k}^{-}|^2)
\]

\[
E_{mag} = \sum_{k} (|b_{k}^{+}|^2 + |b_{k}^{-}|^2) \quad H_{mag} = \sum_{k} k^{-1}(|b_{k}^{+}|^2 - |b_{k}^{-}|^2)
\]
Generic two-triads system

\[ \partial_t u_{k}^{s_k^*} = \frac{1}{2} \left( g_{sp}^{sk} (s_p p - s_q q) u_{p}^{s_p} u_{q}^{s_q} - g_{\sigma_p \sigma_q}^{sk} (\sigma_p p - \sigma_q q) b_{p}^{\sigma_p} b_{q}^{\sigma_q} \right) \]

\[ \partial_t b_{k}^{\sigma_k^*} = \frac{\sigma_k k}{2} \left( g_{\sigma_p \sigma_q}^{sk} b_{p}^{\sigma_p} u_{q}^{s_q} - g_{sp}^{sk} u_{p}^{s_p} b_{q}^{\sigma_q} \right) \]
Stability analysis

\[ \partial_t u_k^{s_k} = g_k^{s_k} s_p - s_q \ u_p^{s_p} u_q^{s_q} - g_k^{s_k} (\sigma_p \sigma_p - \sigma_q \sigma_q) \ b_p^{\sigma_p} b_q^{\sigma_q} \]

\[ \partial_t u^{s_p} = g_s^{s_p} s_q - s_k \ k \ u_q^{s_q} u_k^{s_k} - g_s^{s_p} (\sigma_q \sigma_k - \sigma_k \sigma_q) \ b_q^{\sigma_q} b_k^{\sigma_k} \]

\[ \partial_t u^{s_q} = g_{s_k}^{s_q} s_k k - s_p \ p \ u_k^{s_k} u_q^{s_q} - g_{s_k}^{s_q} (\sigma_k \sigma_p - \sigma_p \sigma_k) \ b_k^{\sigma_k} b_p^{\sigma_p} \]

\[ \partial_t b_k^{\sigma_k} = \sigma_k k \ (g_{s_p}^{s_k} b_p^{\sigma_p} u_q^{s_q} - g_{s_k}^{s_k} u_p^{s_p} b_q^{\sigma_q}) \]

\[ \partial_t b_p^{\sigma_p} = \sigma_p p \ (g_{s_k}^{s_q} b_q^{s_q} u_k^{s_k} - g_{s_q}^{s_p} u_k^{s_p} b_k^{s_k}) \]

\[ \partial_t b_q^{\sigma_q} = \sigma_q q \ (g_{s_k}^{s_p} b_k^{s_k} u_p^{s_p} - g_{s_k}^{s_q} u_k^{s_q} b_p^{s_p}) \]
Large-scale dynamo: DNS - laminar flow ($Re_\lambda = 15$)
Large-scale dynamo: DNS - turbulent flow ($Re_\lambda = 140$)
Inverse cascade of magnetic helicity: DNS

Magnetic field

Velocity field

$u = u^+$
CONCLUSIONS

ROLE OF HELICITY IN THE FORWARD/BACWARD 3D ENERGY TRANSFER (FOURIER)

ROLE OF HOMO-CHIRAL TRIADS VISIBLE ALSO IN ROTATING TURBULENCE

EXISTENCE OF A SHARP PHASE-TRANSITION BAKWARD/FORWARD IF SOME NON-LINEAR INTERACTION ARE REWEIGHTED

HETERO-CHIRAL TRIADS PLAY A SINGULAR ROLE FOR INTERMITTENCY IF PARTICIPATING WITH THE CORRECT PREFACTOR

IMPLICATION FOR REGULARITY OF SOLUTIONS

IMPLICATION FOR SMALL AND LARGE SCALE DYNAMO

IMPLICATION FOR REAL-SPACE INTERMITTENCY AND ENERGY BACKSCATTER?