Uniqueness and nonuniqueness of turbulent solutions from singular initial data

Alexei Mailybaev

Instituto de Matemática Pura e Aplicada - IMPA, Rio de Janeiro

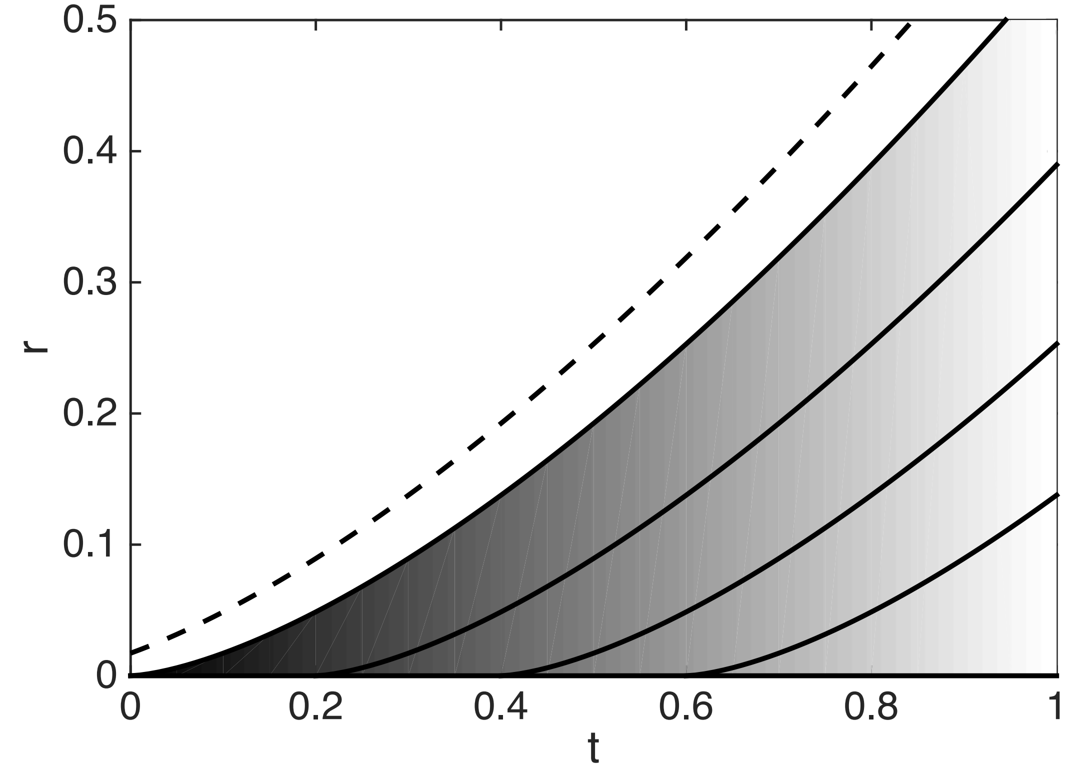
Non-uniqueness and singularities

Non-unique solutions of non-Lipschitz differential equations

$$\dot{r} = r^{1/3} \qquad \text{(Kolmogorov-type singularity)}$$

Solutions starting at the singularity:

$$r(t) = \begin{cases} 0 & t \leq t_s; \\ \left(\frac{2(t-t_s)}{3}\right)^{3/2}, & t > t_s; \end{cases}$$



How to select a solution?

Lagrangian spontaneous stochasticity

Dynamics in the inertial range: inviscid flows in the limit of large Re.

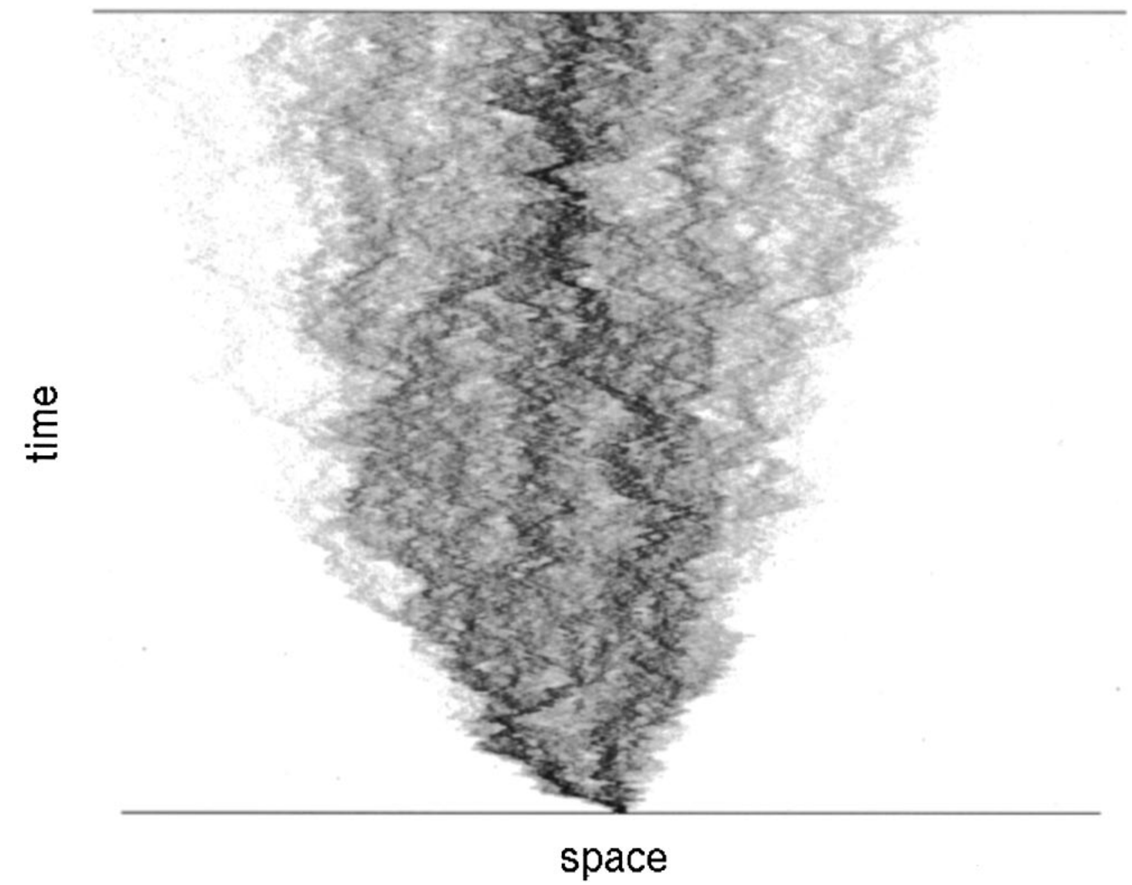
Particle in a singular (non-Lipschitz) velocity field

$$\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R}, t)$$

Solution selection with particle diffusion (Brownian motion):

$$d\mathbf{R} = \mathbf{v}(\mathbf{R}, t)\,dt + \sqrt{2\kappa}\,d\boldsymbol{\beta}(t) \qquad \kappa \to 0$$

Solution remains stochastic in non-diffusive limit.



Falkovich, Gawedzki, Vergassola 2001

Velocity field is also a dynamical variable.

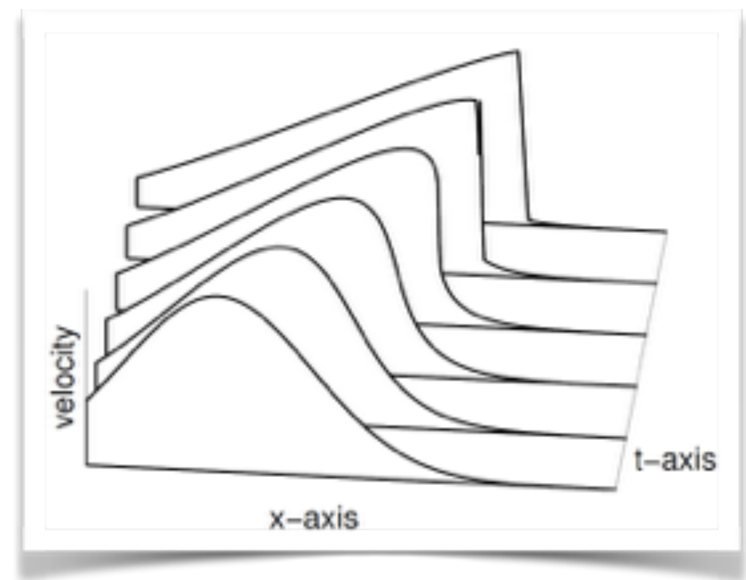
Origin of singularities in inviscid flows

- Turbulent (weak) solutions of Euler equations

- Finite time blowup

Inviscid Burgers equation

$$u_t + uu_x = 0$$



- Discontinuous initial configuration

Kelvin–Helmholtz
or Rayleigh-Taylor
instability

from Wikipedia

Model: nonlocal viscous conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial g}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f, \quad x, t \in \mathbb{R},$$

$$g(x, t) = \frac{1}{2\pi} \int \int K(y - x, z - x) u(y, t) u(z, t) \mathrm{d}y \mathrm{d}z$$

Non-locality “mimics” incompressibility

Extra conditions on the kernel function $K(y,z)$:
energy conservation, Hamiltonian structure. etc.

Example: Constantin–Lax–Majda equation $\omega_t - v_x \omega = 0, \qquad v_x = H\omega$

Special case: Sabra shell model

$$K(y,z) = K_\psi(y,z) + K_\psi(z,y), \quad K_\psi(y,z) = \frac{\sigma}{(\sigma y - z)^2} - \frac{(1+c)\sigma^2}{(\sigma^2 y - z)^2} - \frac{c\sigma}{(\sigma y + z)^2}$$

Solution representation

A.M. 2016, Nonlinearity

$$k_n = k_0 \lambda^n, \quad n \in \mathbb{N}, \qquad 1 \leqslant k_0 < \lambda.$$

$$u_n(t) = k_n^{1/3} \hat{u}\left(k_n^{2/3}, t\right), \quad \hat{u}(k,t) = \int u(x,t) e^{-ikx} dx$$

Sabra shell model

$$\frac{\partial u_n}{\partial t} = \mathrm{i}\left[k_{n+1} u_{n+2} u_{n+1}^* - (1+c) k_n u_{n+1} u_{n-1}^* - c k_{n-1} u_{n-1} u_{n-2}\right] - \nu_n u_n + f_n$$

$$\lambda = \sigma^{3/2} = \sqrt{2 + \sqrt{5}} \approx 2.058$$

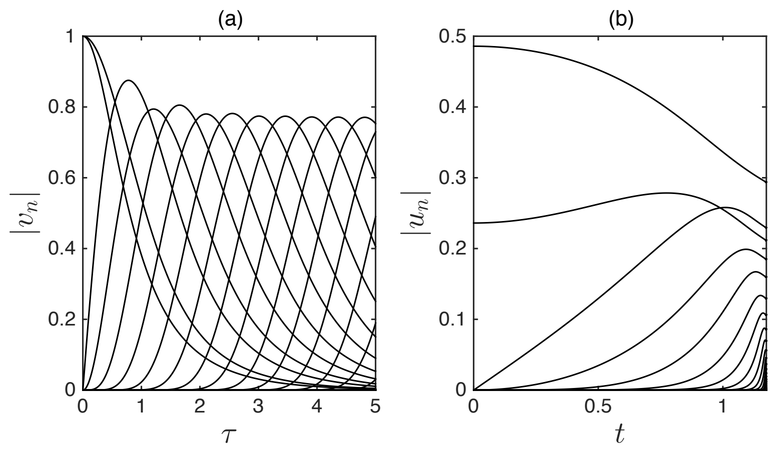
Gledzer-Ohkitani-Yamada (GOY) in 70-80th;
L'vov, Podivilov, Pomyalov, Procaccia, Vandembroucq (Sabra) in 90th

Inviscid invariants: energy, helicity, enstrophy etc. (depending on coefficients)

Blowup and a shock wave

Self-similar blowup

$$u_n(t) = -ie^{i\theta_n} k_n^{z-1} U(k_n^z(t - t_b)), \quad t < t_b,$$

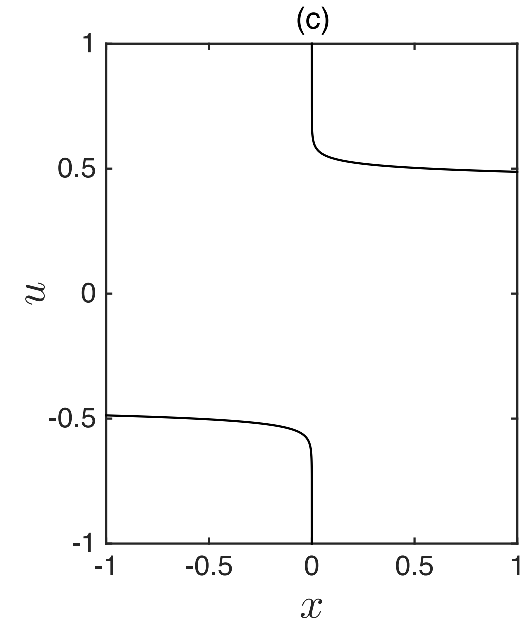


Hombre & Gilson 1988

Continuous representation

$$u(x, t_b) = \frac{\Gamma(1-\beta)}{\pi} \cos\left(\frac{\beta\pi}{2}\right) |x|^{\beta-1} \operatorname{sgn} x$$

$$\beta = 2 - 3z/2 \approx 0.954$$

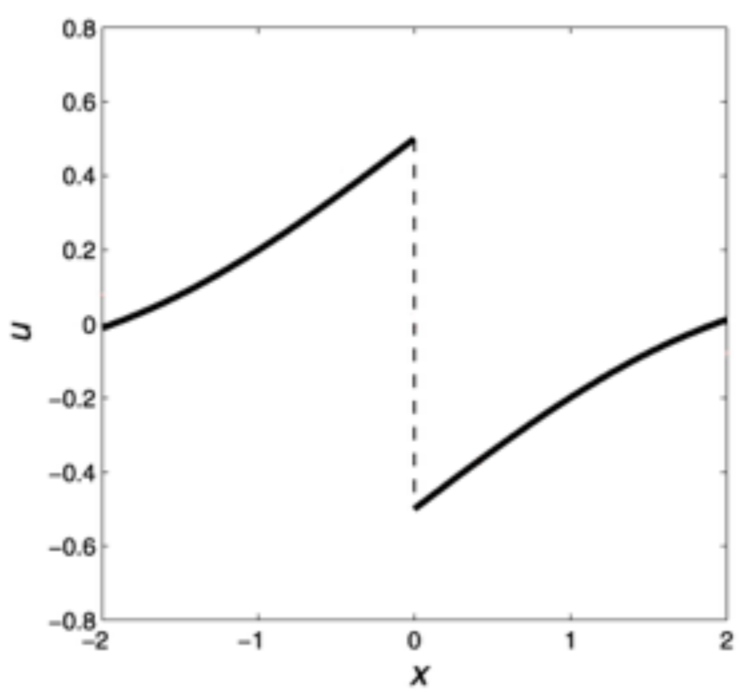


Kolmogorov (inviscid) solution

$$u_n = ik_n^{-1/3}$$

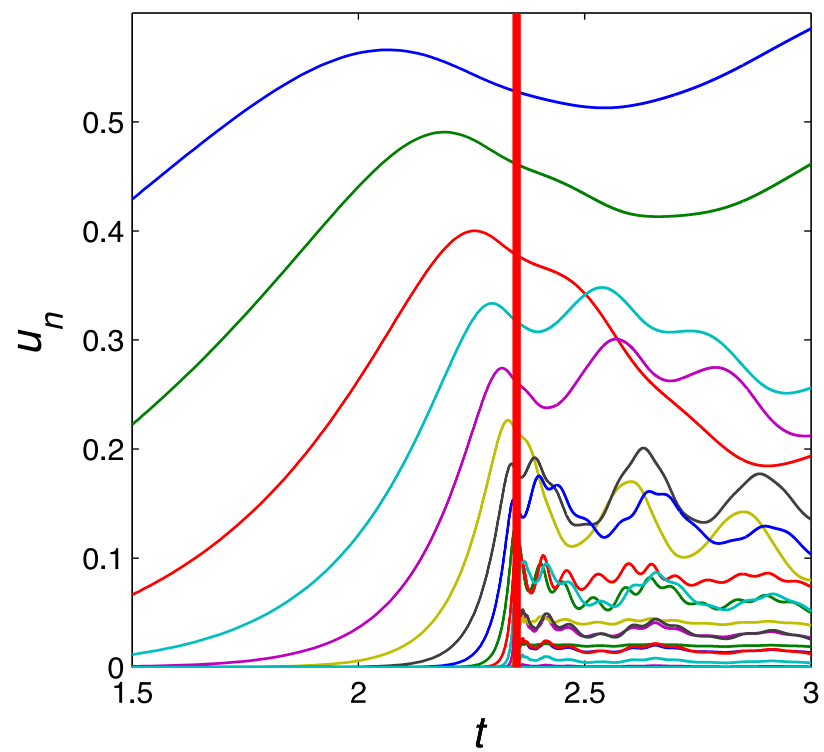
Stationary state: a shock

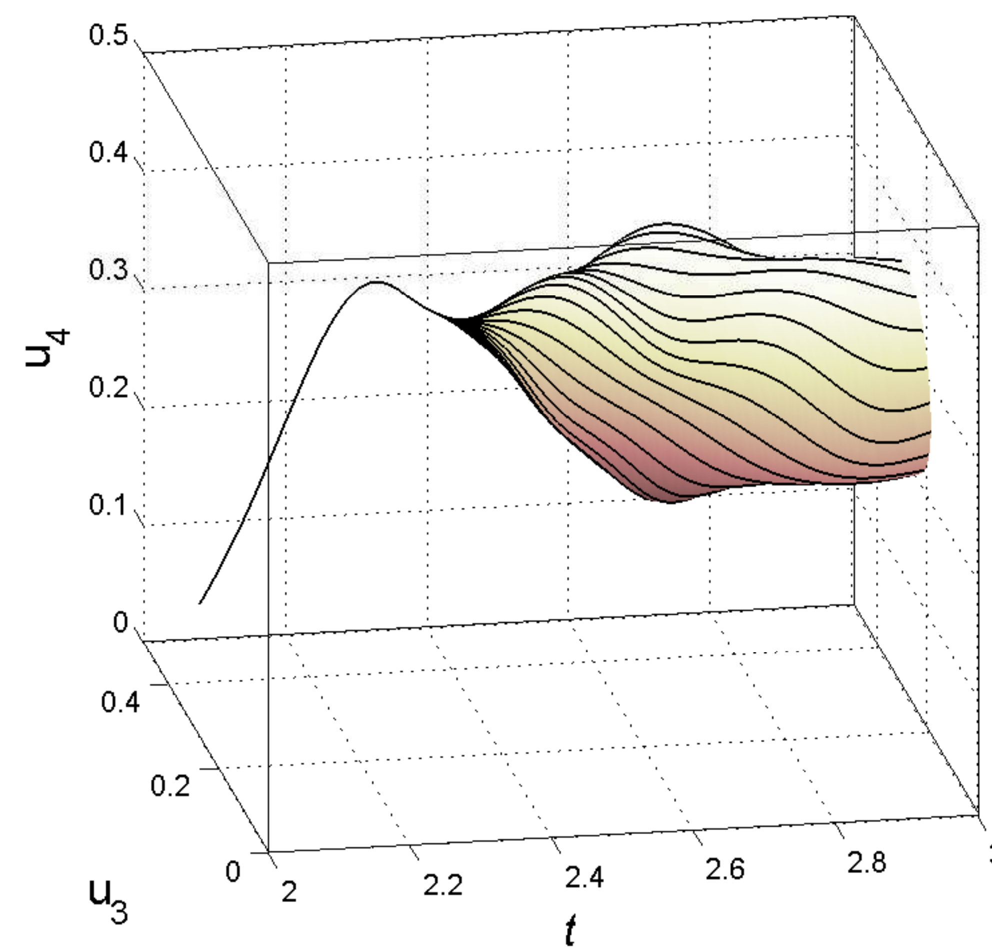
(unstable in Sabra model)



Purely imaginary Sabra (Gledzer) model: Non-unique inviscid limit

$$\nu = 2^{-4(\chi+N)} \xrightarrow[N\to\infty]{} 0$$





Non-unique solutions!
However, a unique solution can be chosen for a given (small) viscosity

A.M. 2016, MMS (SIAM)

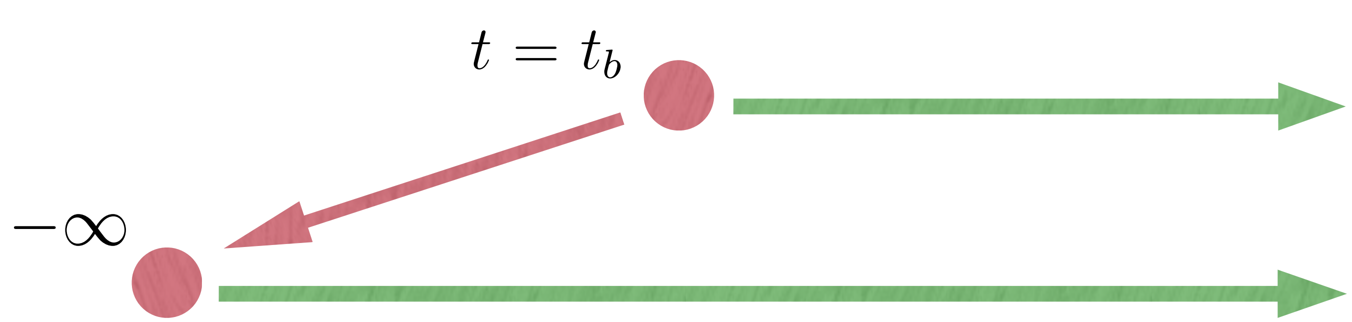
Periodic wave in renormalized system

Renormalized system:

$$\frac{dw_n}{d\tau} = \left(w_n - \frac{1}{\lambda^2} w_{n+2} w^*_{n+1} + \frac{1}{2} w_{n+1} w^*_{n-1} + \frac{\lambda^2}{2} w_{n-1} w_{n-2} \right) \log \lambda.$$

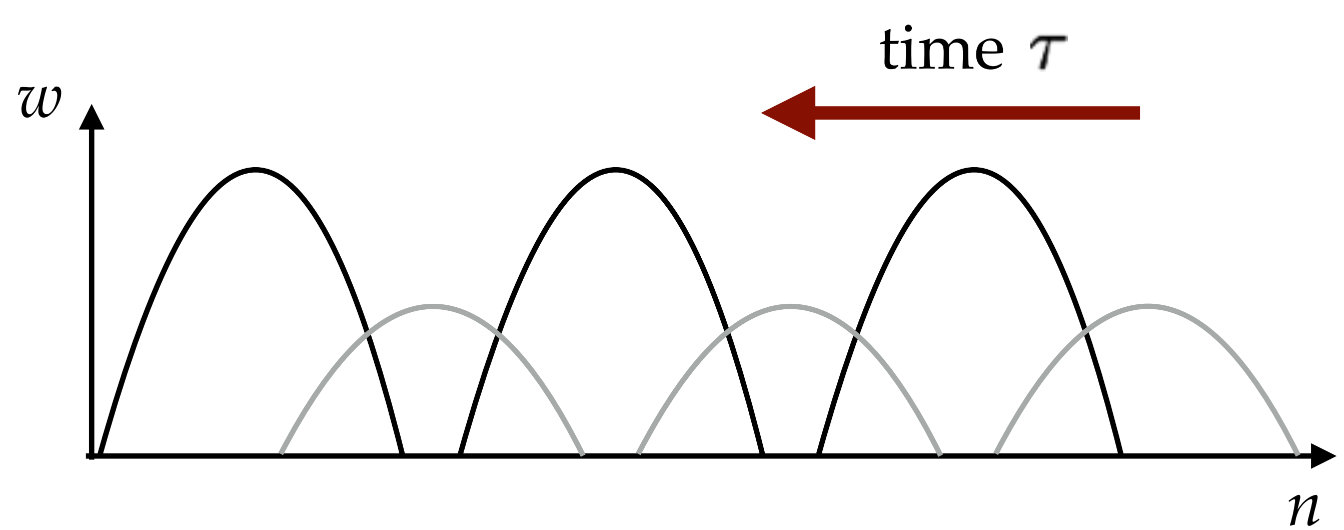
$$t = t_b + \lambda^\tau, \quad u_n = -ik_n^{-1} \lambda^{-\tau} w_n = -ik_0^{-1} \lambda^{-\tau-n} w_n.$$

Logarithmic time: $\tau = \log_\lambda (t - t_b)$ $\quad n = \log_\lambda k_n$



Stable periodic-wave solution:

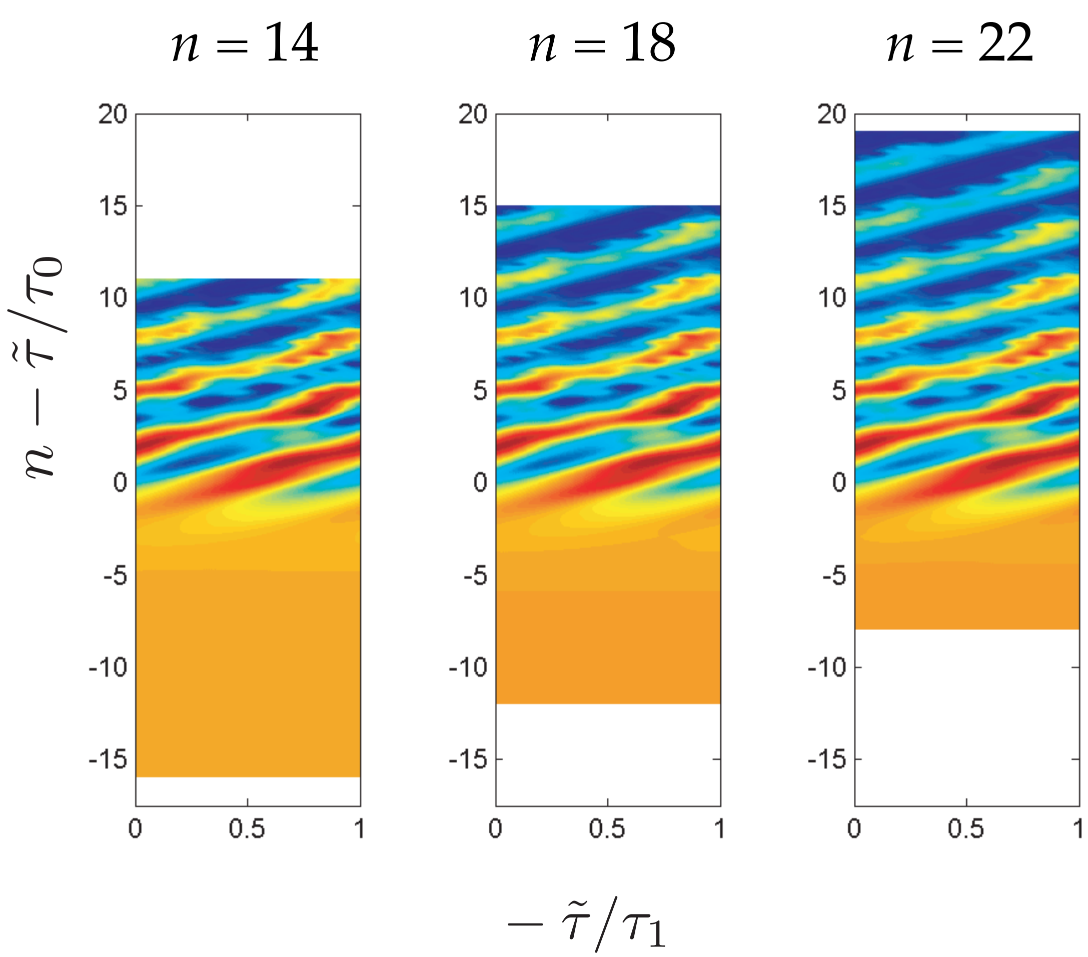
$$w_n(\tau) = W\left(n - \frac{\tau}{\tau_0}, \chi - \frac{\tau}{\tau_1} \right)$$

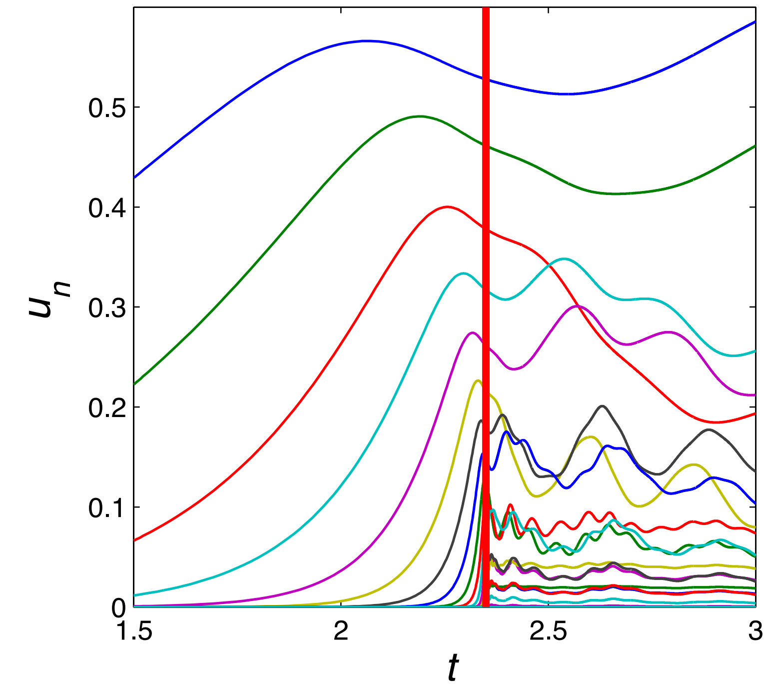


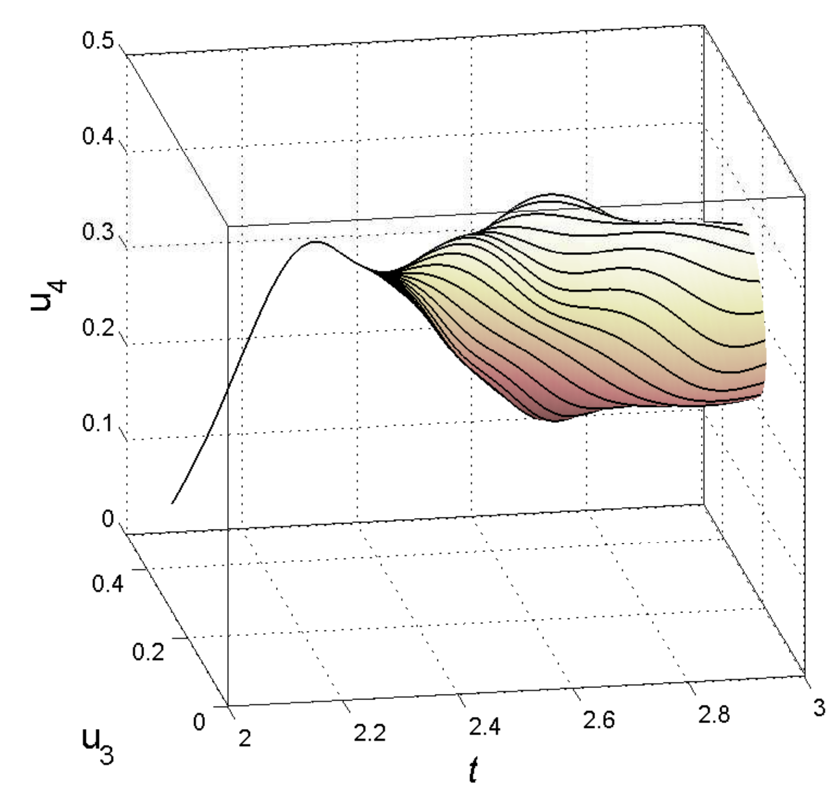
Periodic wave: numerical simulations

$$\nu = 2^{-4(\chi+N)} \xrightarrow[N\to\infty]{} 0$$

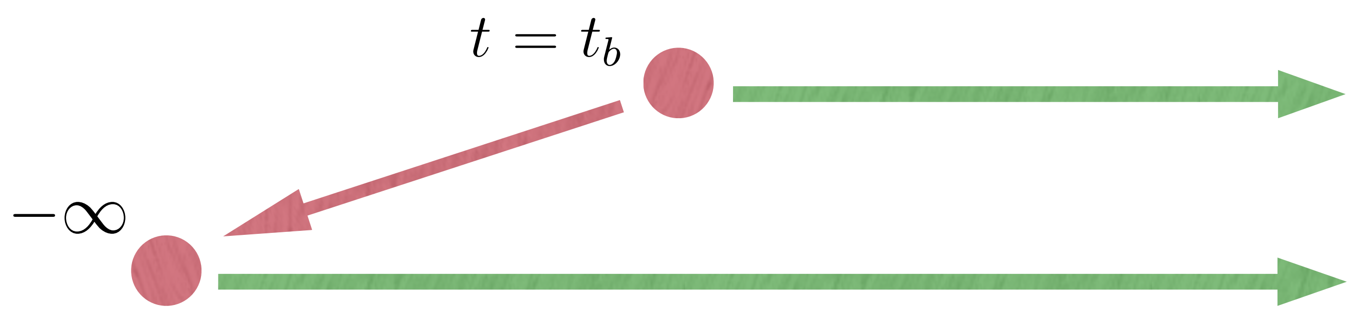
$$w_n(\tau) = W\left(n - \frac{\tau}{\tau_0}, \chi - \frac{\tau}{\tau_1}\right)$$



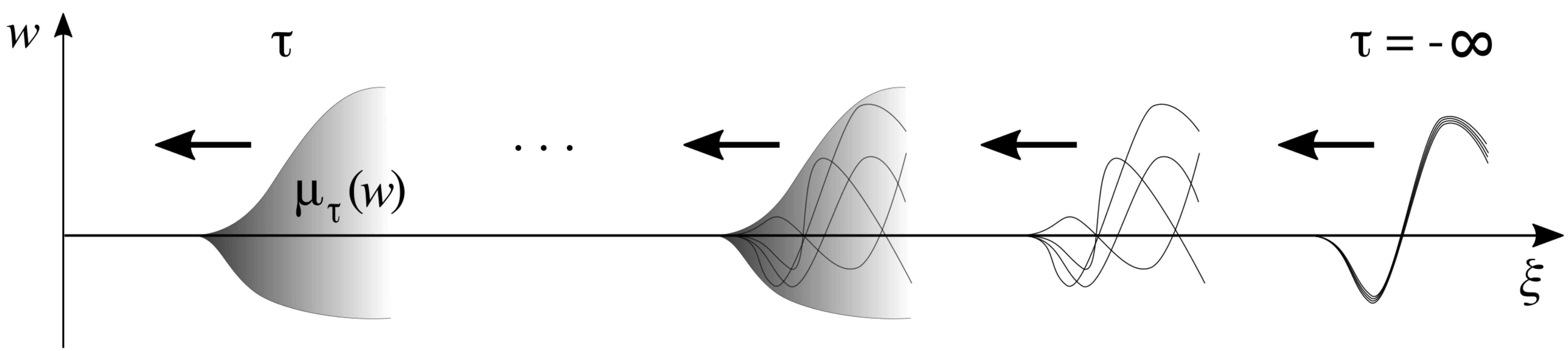




Complex Sabra model
Chaotic wave in renormalized system: spontaneous stochasticity



Dynamics in renormalized time:



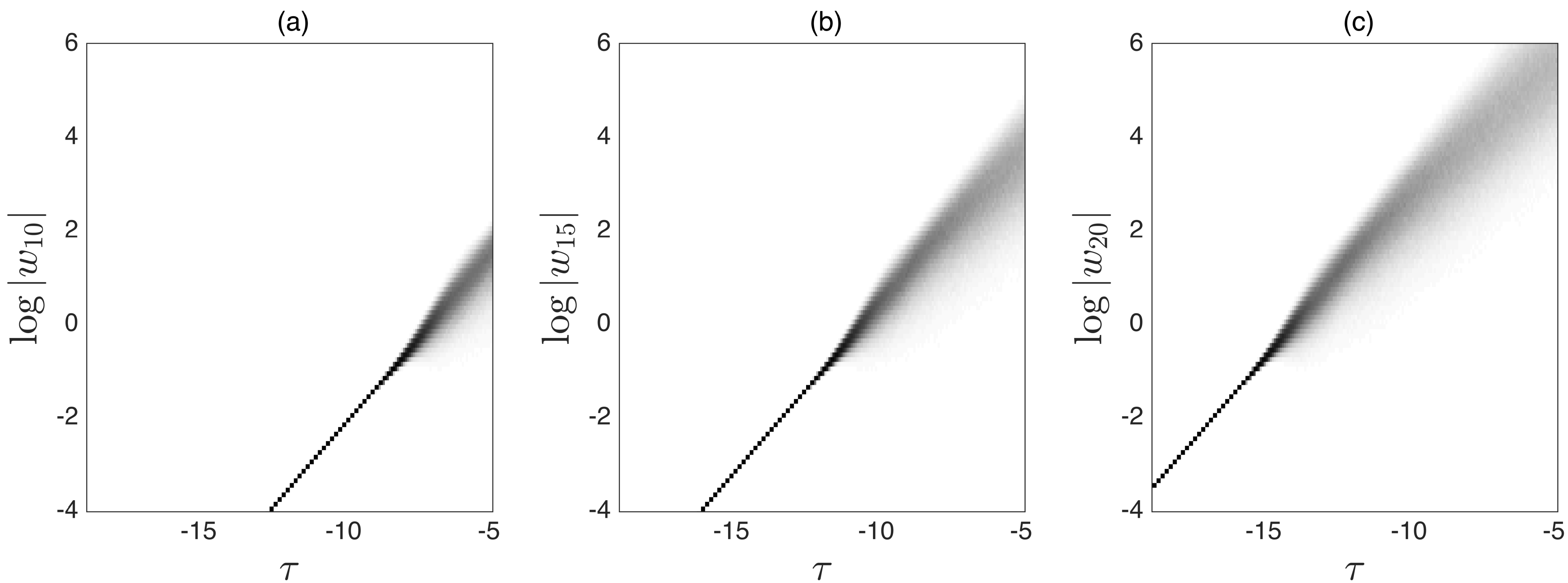
Implications:

- physically relevant solution is a (**spontaneous!**) probability distribution
- **unique** probabilistic description in inviscid limit
  in the form of a **steady-state traveling stochastic wave**

Probability distribution as a steady-state traveling wave

$$\mu_{\tau+\tau_0}(w) = \mu_\tau(Tw), \qquad \tau_0 = 1/a$$

$$T : (w_1, w_2, \ldots) \mapsto (w_2, w_3, \ldots)$$



$\nu = 10^{-15}$

small-scale noise:

$$u_{36}(0) = (-i + 0.01x)k_{36}^{z-1}$$

no dependence on size of small noise: this is not chaos!

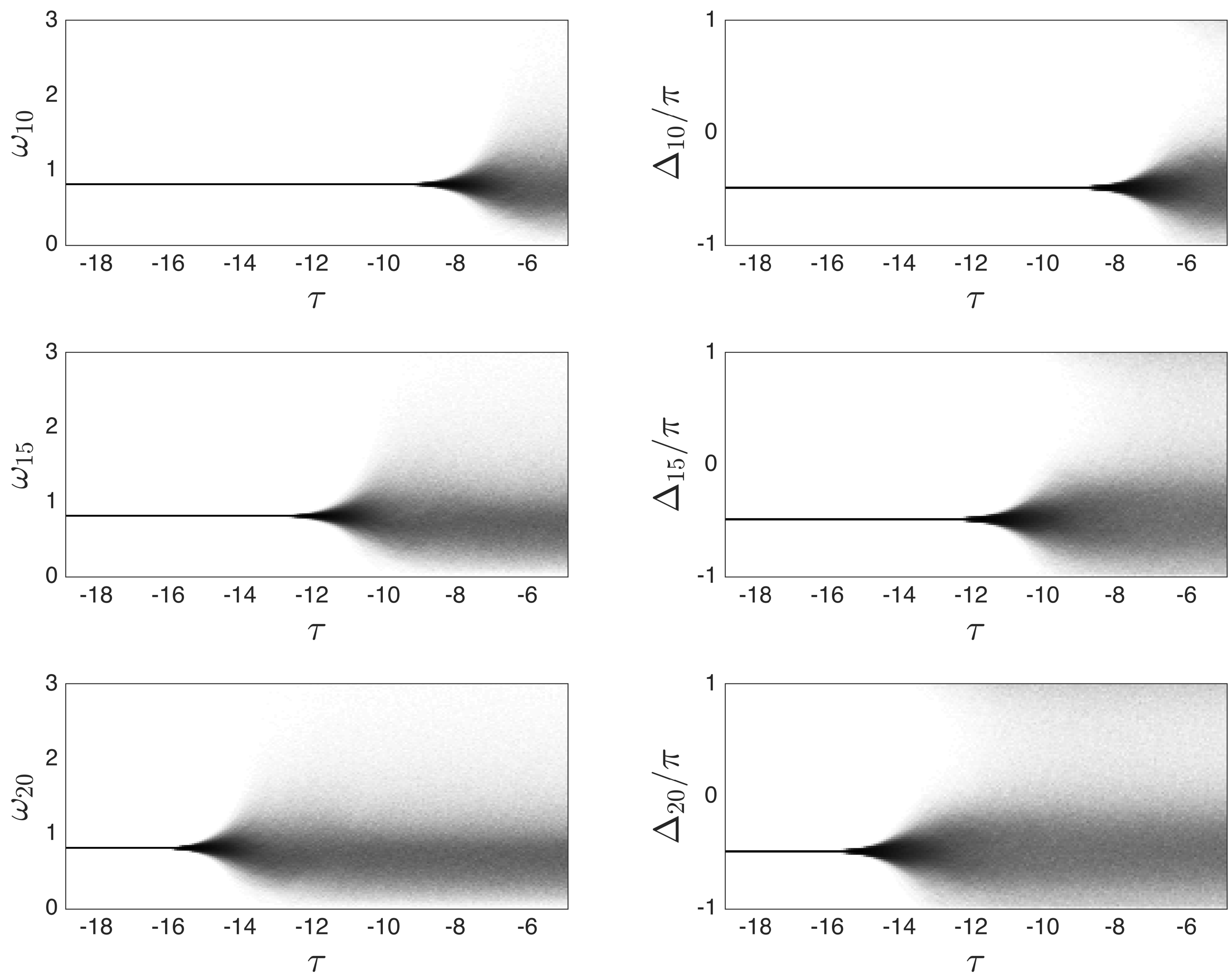
A.M. 2016, Nonlinearity

Traveling probability measure with constant limiting states

Kolmogorov hypothesis on universality of velocity increments
(Kolmogorov 62; Benzi, Biferale & Parisi 93; Eyink 2003):

$$\omega_n = |u_n/u_{n-1}|, \quad \Delta_n = \arg(u_{n-2}u_{n-1}u_n^*)$$

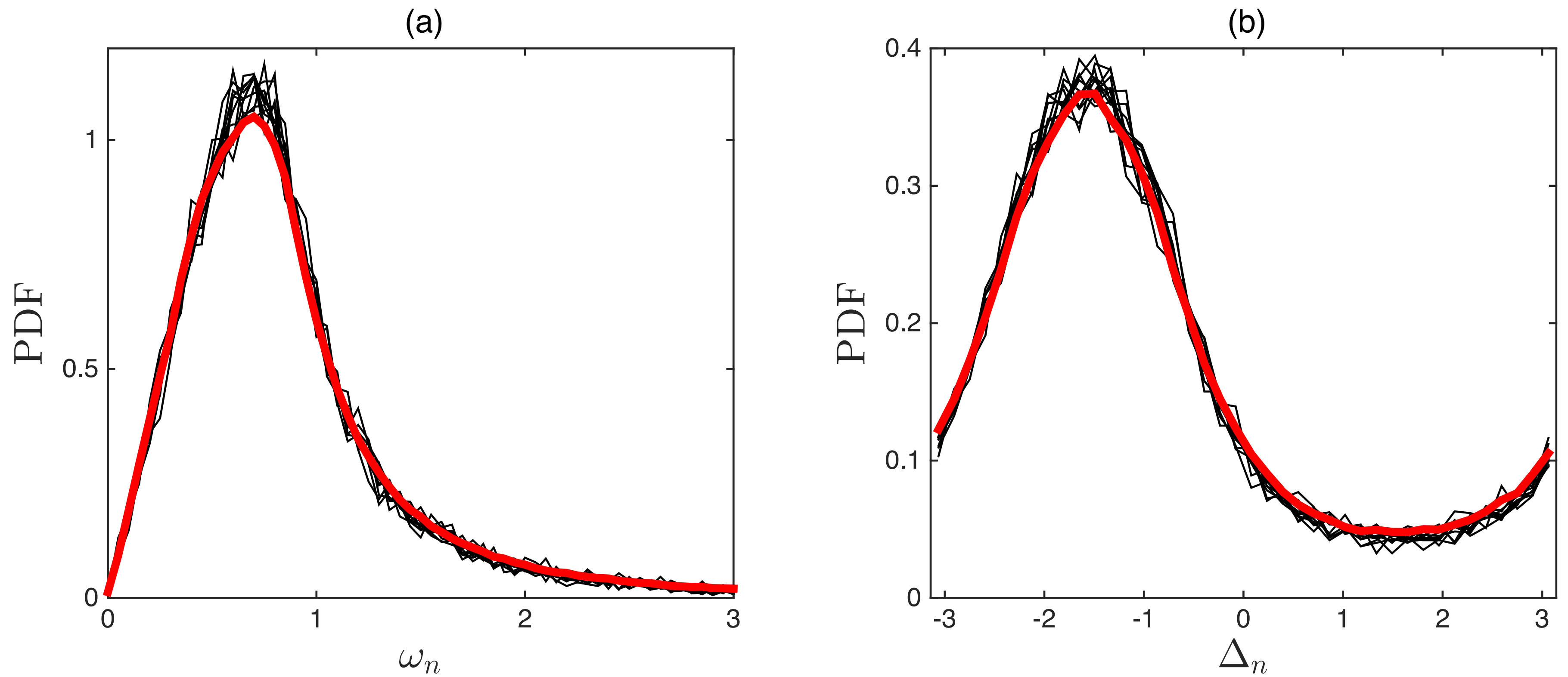
deterministic state



stochastic state

stable traveling wave: universal route to spontaneous stochasticity

Stochastic constant state describes the equilibrium turbulent statistics



PDFs at stochastic constant state of the traveling wave ($n = 15,\ldots,25$)
vs. PDFs of turbulent dynamics in inertial interval for the statistical equilibrium

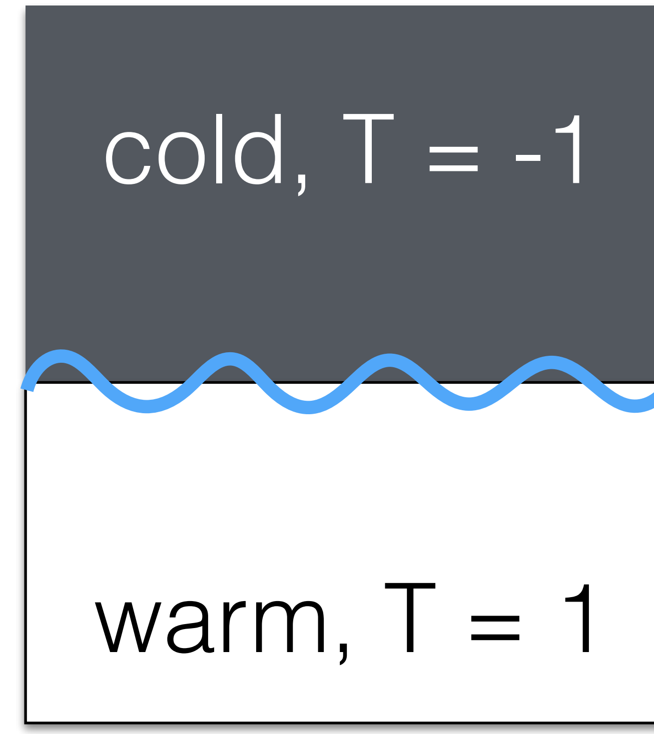
Spontaneously stochastic solution is a traveling wave separating
the two constant states: deterministic blowup state and developed turbulent state.

Similar behavior from different singular initial conditions, not related to blowup.

Rayleigh–Taylor instability

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \beta g \mathbf{e}_z T,$$ (2D or 3D)

$$\partial_t T + \mathbf{u}\cdot\nabla T = \kappa\nabla^2 T, \quad \nabla\cdot\mathbf{u} = 0,$$



Linear analysis for ideal fluid:

$$\lambda = \pm\sqrt{\sigma k} \qquad \text{growth rate} \propto e^{\lambda t}$$

ill-conditioned problem: explosive growth of small-scale perturbations

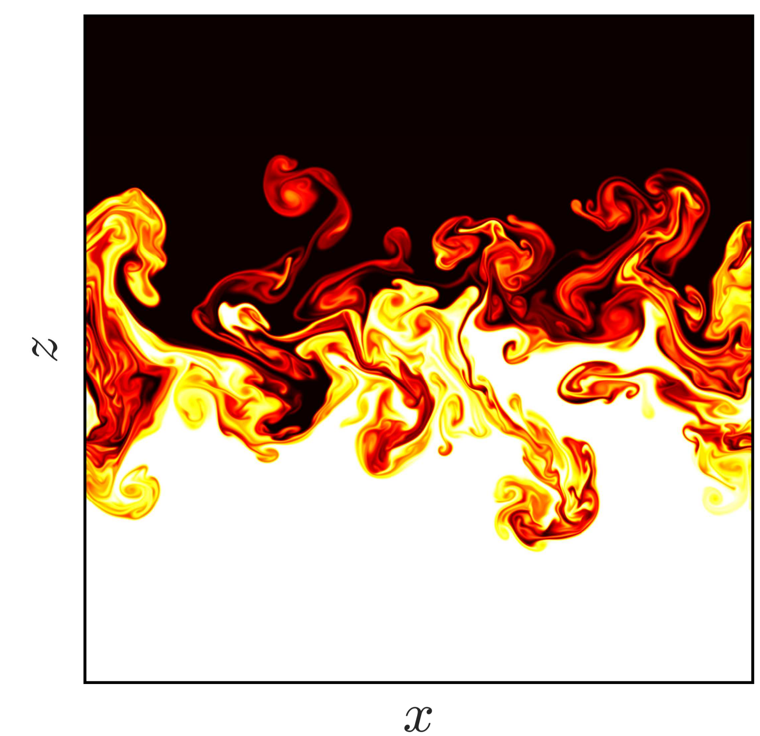
Nonlinear (turbulent) dynamics: phenomenological theory by Chertkov 2003

Shell model:

$$\dot{\omega}_n + \nu k_n^2 \omega_n = \left[\omega_{n-1}^2 - c\omega_n\omega_{n+1} + 0.1(\omega_{n-1}\omega_n - c\omega_{n+1}^2)\right] + k_n R_n,$$

$$\dot{R}_n + \kappa k_n^2 R_n = \omega_n R_{n+1} - \omega_{n-1} R_{n-1} + \gamma\omega_n T_n,$$

$$\dot{T}_n + \kappa k_n^2 T_n = \omega_n T_{n+1} - \omega_{n-1} T_{n-1} - \gamma\omega_n R_n,$$



models: stationary state, stability, dispersion relation,
phenomenology of turbulent dynamics, intermittency

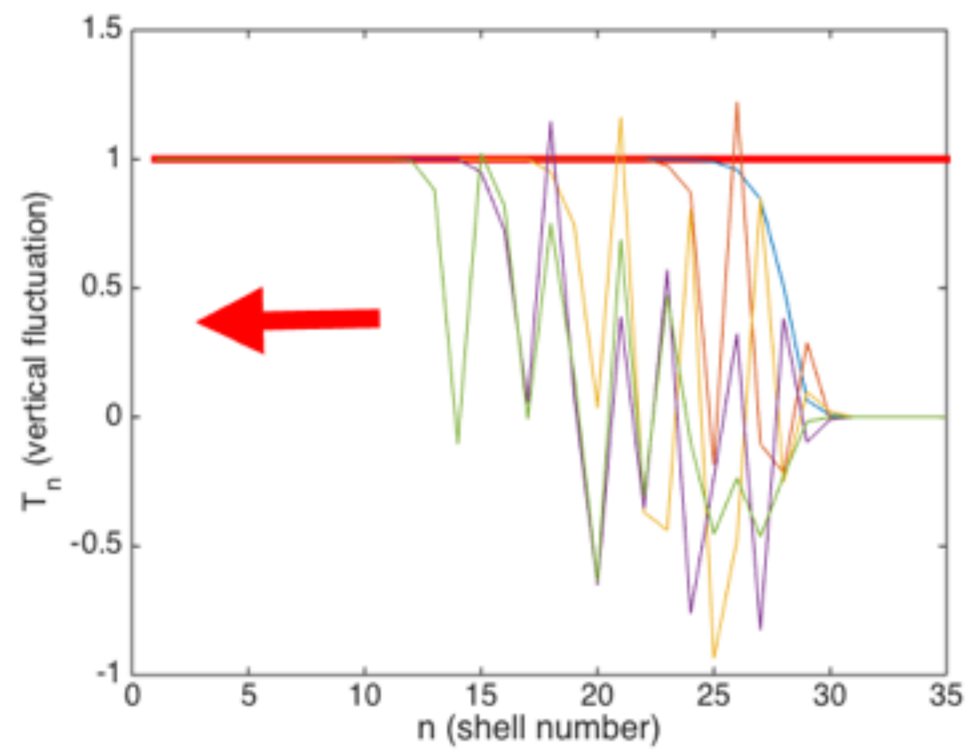
A.M. 2016 ArXiv: 1610.03181

Turbulent dynamics of a shell model

Initial condition:

$$t = 0: \quad \omega_n = 0, \quad R_n = 0, \quad T_n = 1, \quad n = 1, 2, 3, \ldots$$

+ small perturbation



Mixing layer:

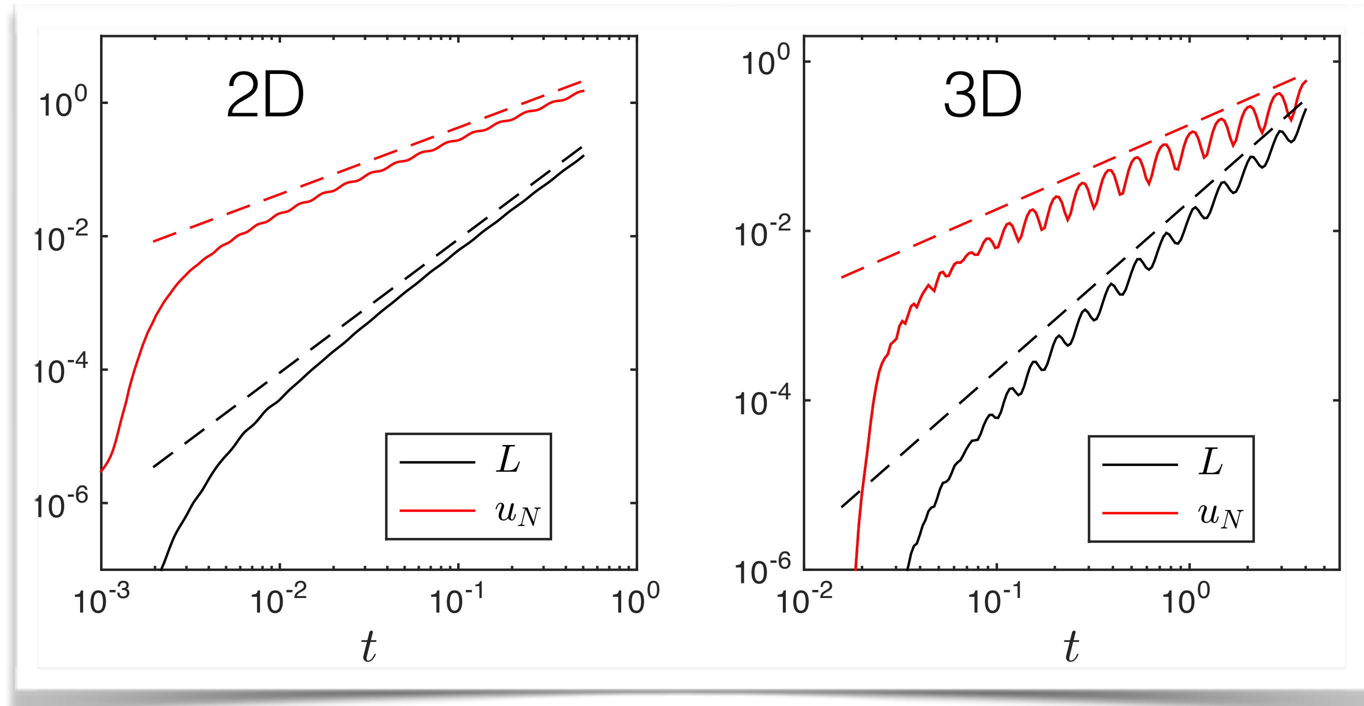
$$L = \int (1 - |T|) dz \qquad \text{(continuous version)}$$

$$L(t) = \sum_n \langle 1 - T_n(t) \rangle r_n, \quad u_N(t) = \left\langle \sum_n u_n^2(t) \right\rangle^{1/2}$$

$h = 2, \ \nu = \kappa = 10^{-14}$

Phenomenology of turbulent dynamics:

$$L(t) = h^{-N(t)} \sim t^2, \quad u_N(t) \sim t$$



average over 100000 simulations
with a small random initial perturbation

Renormalization

Renormalized variables:

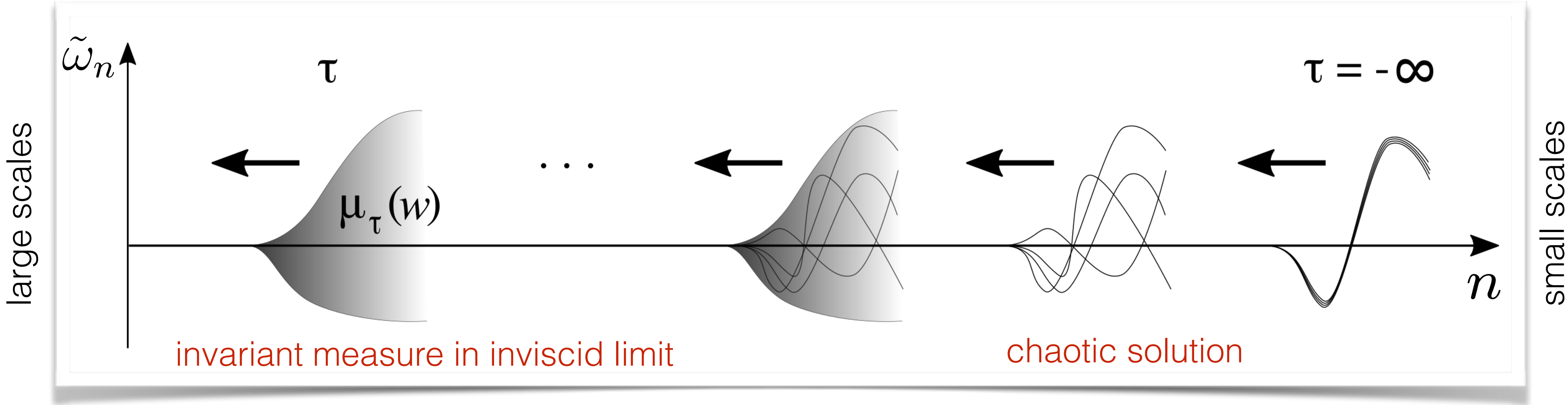
$$t = h^{\tau}, \quad \omega_n = h^{-\tau}\tilde{\omega}_n, \quad R_n = h^{-n-2\tau}\tilde{R}_n, \quad T_n = h^{-n-2\tau}\tilde{T}_n$$

Renormalized inviscid system (translation invariant in logarithmic time and scale):

$$\alpha\frac{d\tilde{\omega}_n}{d\tau} = \tilde{\omega}_n + \tilde{\omega}_{n-1}^2 - c\tilde{\omega}_n\tilde{\omega}_{n+1} + 0.1(\tilde{\omega}_{n-1}\tilde{\omega}_n - c\tilde{\omega}_{n+1}^2) + \tilde{R}_n,$$
$$\alpha\frac{d\tilde{R}_n}{d\tau} = 2\tilde{R}_n + h^{-1}\tilde{\omega}_n\tilde{R}_{n+1} - h\tilde{\omega}_{n-1}\tilde{R}_{n-1} + \gamma\tilde{\omega}_n\tilde{T}_n,$$
$$\alpha\frac{d\tilde{T}_n}{d\tau} = 2\tilde{T}_n + h^{-1}\tilde{\omega}_n\tilde{T}_{n+1} - h\tilde{\omega}_{n-1}\tilde{T}_{n-1} - \gamma\tilde{\omega}_n\tilde{R}_n,$$

initial time $t = 0$ corresponds $\tau \to -\infty$

Stochastic traveling wave solution (depending on the single variable $\xi = n - v\tau$)



RT instability for 2D

Convenient variables (multipliers):

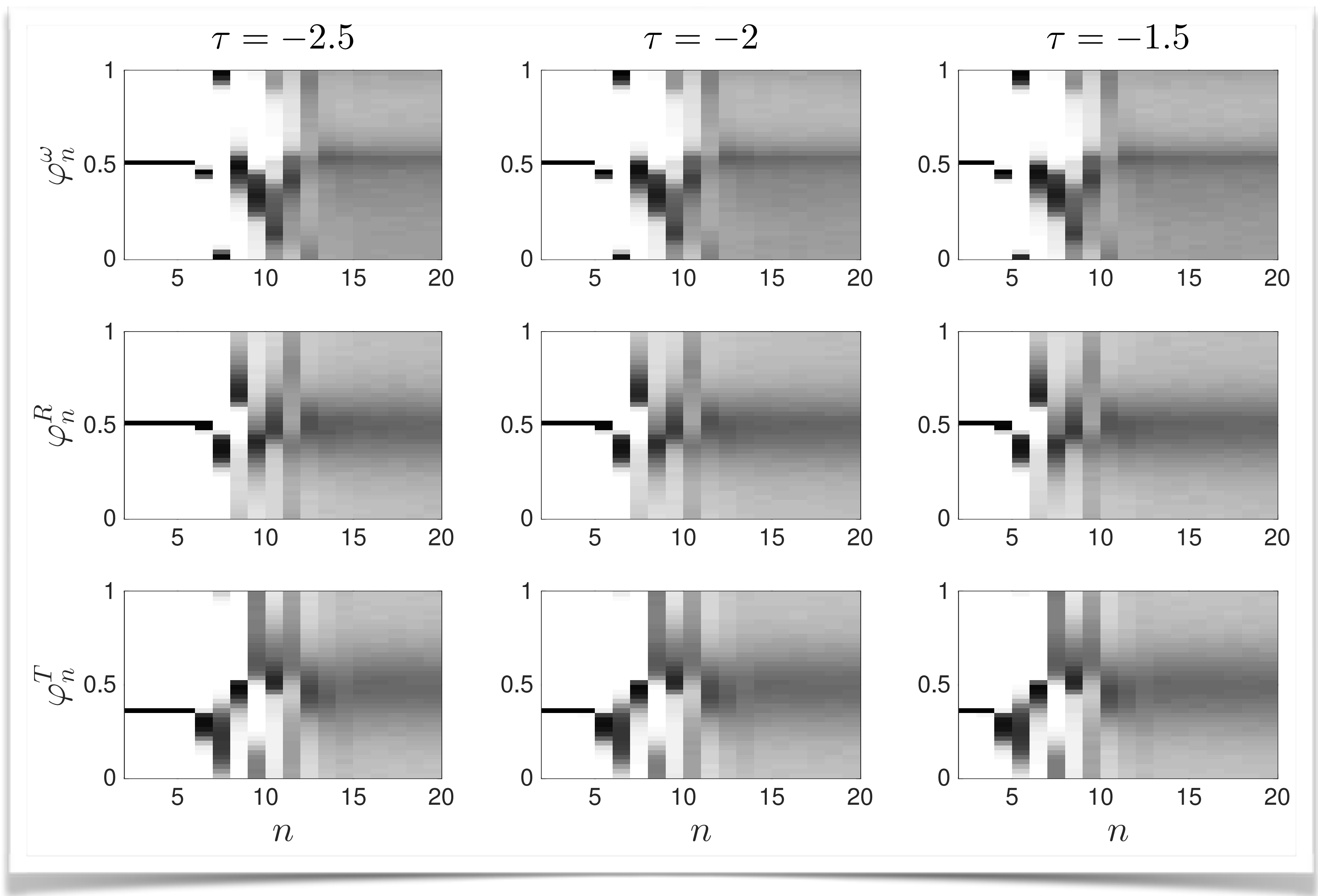
$$\rho_n^\omega = \frac{\tilde{\omega}_n}{\tilde{\omega}_{n-1}} = \frac{\omega_n}{\omega_{n-1}}, \quad \rho_n^R = \frac{\tilde{R}_n}{\tilde{R}_{n-1}} = \frac{hR_n}{R_{n-1}}, \quad \rho_n^T = \frac{\tilde{T}_n}{\tilde{T}_{n-1}} = \frac{hT_n}{T_{n-1}}$$

Mapping to a unit interval for representation purpose:

$$\varphi_n^\omega = \frac{1}{\pi}\arctan \rho_n^\omega, \quad \varphi_n^R = \frac{1}{\pi}\arctan \rho_n^R, \quad \varphi_n^T = \frac{1}{\pi}\arctan \rho_n^T$$

Traveling stochastic wave separating two constant states:

deterministic state (small n) and developed turbulent state (large n)

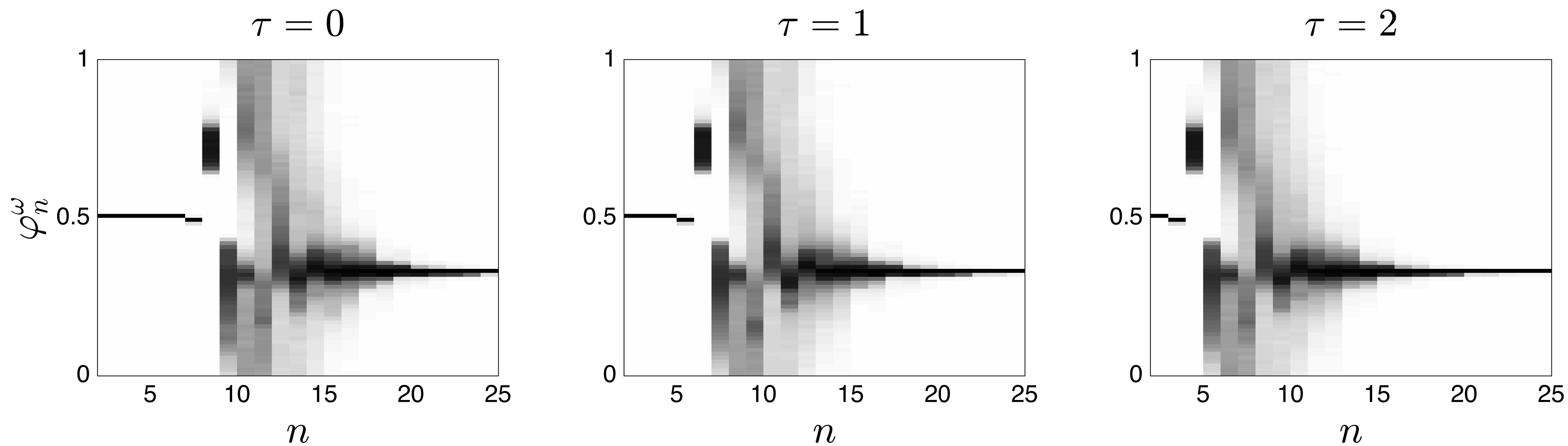


RT instability for 3D

Kolmogorov (non-intermittent) solution at small scales:

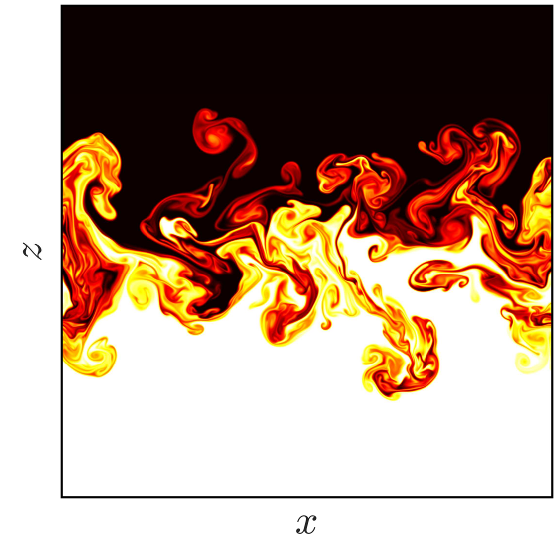
$$\omega_n = k_n u_n = \alpha_1 k_n^{2/3}, \quad R_n + iT_n = \alpha_2 \zeta^n + \alpha_3 \bar{\zeta}^n, \quad \zeta = \frac{i\gamma}{2} \pm \sqrt{-\frac{\gamma^2}{4} + h^{-2/3}},$$

Stochastic wave for the RT instability between two regular constant states:



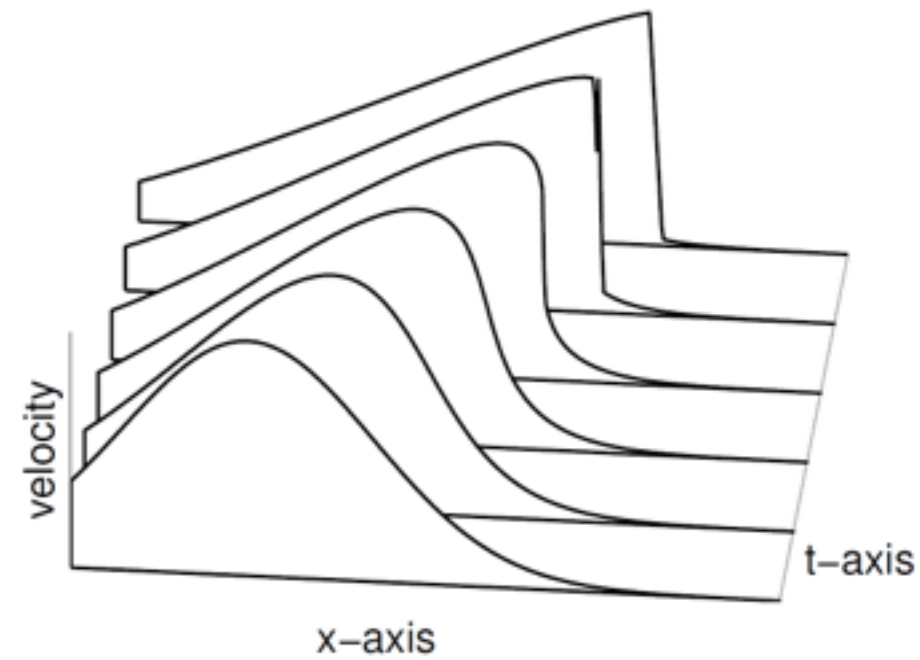
Stochastic wave is generated by an attractor
for finite-dimensional chaos in renormalized system,

generating the spontaneously stochastic solution
in original variables



Summary

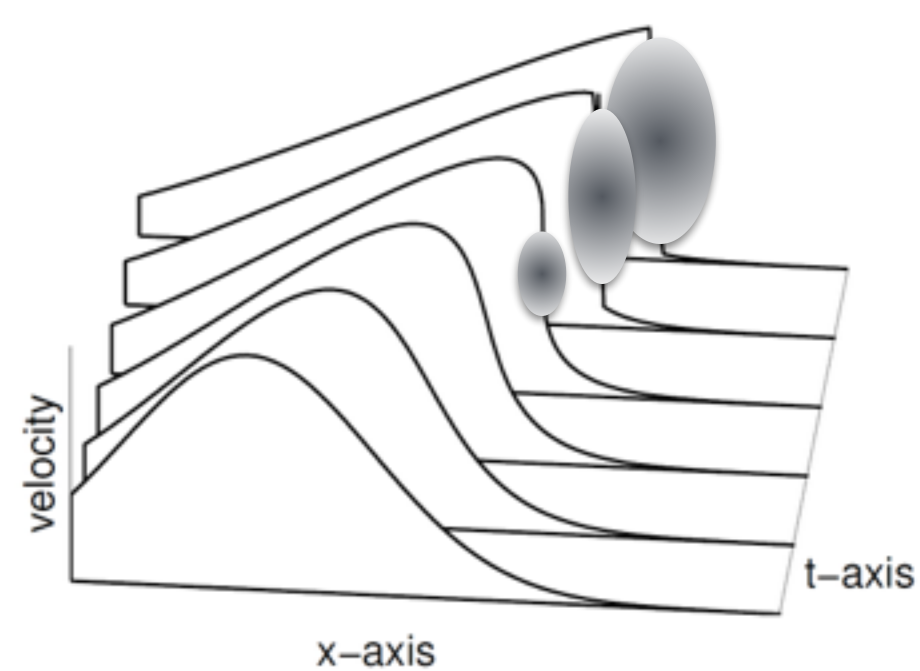
Inviscid Burgers equation
(compressible gas dynamics)



$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x, t \in \mathbb{R}, \quad \nu \to 0^+ \quad f = u^2/2.$$

A notion of weak solution, entropy condition, extended functional spaces, etc. yields a **unique weak solution**

Nonlocal flux term
("incompressible" flow?)



$$f(x,t) = \frac{1}{2\pi} \int \int K(y-x, z-x) u(y,t) u(z,t) \mathrm{d}y \mathrm{d}z$$

Renormalization, viscous regularization with infinitesimal noise, etc. yields a **unique stochastic solution**

Implications:

- concept of regularization and inviscid limit: vanishing viscosity and noise
- concept of weak solution needs to be extended to a weak stochastic solution
- spontaneous stochasticity (infinite dim) vs. deterministic chaos (finite dim)