The h-principle for the Euler equations

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Incompressible Euler equations

$$\partial_t v + v \cdot \nabla v + \nabla p = 0 \qquad x \in \mathbb{T}^3$$

$$\text{div } v = 0 \qquad t \in [0, T]$$

Some facts:

- To any given suff. smooth initial data there exists, at least for a short time, a unique suff. smooth solution (**Lichtenstein 1930s, Kato 1980s**).
- For any suff. smooth solution, the energy is constant in time (**classical**).
- There exist non-trivial **weak** solutions with compact support in time (**Scheffer 1993**).

Notions of solutions of Euler

Classical	Continuous	Weak	"Very weak"
$v \in C^{1,\alpha}$ or $v \in H^{s>5/2}$ **Lichtenstein, Kato**	$v \in C^0$ $v \in C^\alpha$ **De Lellis-Sz** **Buckmaster** **Isett** **...**	$v \in L^2_{loc}$ $v \in L^\infty$ **Scheffer, Shnirelman,** **De Lellis-Sz** **Sz-Wiedemann** **...**	$v \in L^\infty(\mathcal{M})$ measure-valued **DiPerna-Majda** dissipative **P.L.Lions**
LWP energy conserved blow-up?	"wild behaviour", h-principle non-uniqueness, energy not conserved		existence (via compactness)

Weak solutions and non-uniqueness

Non-uniqueness in L^2

Theorem (Scheffer 93, Shnirelman 97, De Lellis - Sz. 2009)
There exist nontrivial weak solutions of the Euler equations with compact support in space-time.

- [Scheffer] in $\mathbb{R}^2$
- [Shnirelman] in $\mathbf{T}^2$
- [De Lellis-Sz.] works for general domains in any dimension

Non-uniqueness in L^2

Theorem (Scheffer 93, Shnirelman 97, De Lellis - Sz. 2009)

There exist nontrivial weak solutions of the Euler equations with compact support in space-time.

Theorem (De Lellis - Sz. 2010)

Given $e = e(x,t) > 0$, there exist infinitely many weak solutions of the Euler equations with

$$\frac{1}{2}|v(x,t)|^2 = e(x,t)$$

Non-uniqueness in L^2

Theorem (Wiedemann 2011)
For any L^2 initial data there exist infinitely many global weak solutions with bounded energy.

- domain is a torus $n \geq 2$
- **first global existence** result for weak solutions in dimension $n \geq 3$

Non-uniqueness in L^2

Theorem (Wiedemann 2011)
For any L^2 initial data there exist infinitely many global weak solutions with bounded energy.

Theorem
Given v_0 and v_1 with $\int_{\mathbb{T}^d} v_0 \, dx = \int_{\mathbb{T}^d} v_1 \, dx$, there exist infinitely many weak solutions of the Euler equations with

$$v(t=0) = v_0, \quad v(t=1) = v_1$$

Differential Inclusions

~~Differential~~ Inclusions

Toy problem: construct

$$v : [0,1] \to \mathbb{R} \text{ such that } |v| = 1$$

Baire-category approach

(Cellina, Bressan-Flores, Dacorogna-Marcellini, Kirchheim,….)

$\{v : |v| = 1 \text{ a.e.}\}$ residual in $\{v : |v| \leq 1 \text{ a.e.}\}$ $\quad L^\infty \text{ w}^*$

Differential Inclusions: Baire category approach

Theorem **(folklore)**

Any 1-Lipschitz map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ can be uniformly approximated by (weak) Lipschitz isometries, i.e. solutions of

$$Du(x) \in O(n) \qquad \text{a.e. } x$$

Differential Inclusions: Baire category approach

$$Du^T Du = Id$$

$$\Updownarrow$$

isometric $$\begin{cases} \text{curl } A = 0 \\ A^T A = Id \end{cases}$$

Differential Inclusions: Baire category approach

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short
$$\begin{cases} \text{curl } A = 0 \\ A^T A \leq Id \end{cases}$$

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isometric $\begin{cases} \text{curl } A = 0 \\ A^T A = Id \end{cases}$

short $\begin{cases} \text{curl } A = 0 \\ A^T A \le Id \end{cases}$

$$\partial_t v + \text{div } (v \otimes v) + \nabla p = 0$$
$$\text{div } v = 0$$

$$\Updownarrow$$

solution $\begin{cases} \partial_t v + \text{div } \sigma + \nabla p = 0 \\ \text{div } v = 0 \\ v \otimes v = \sigma \end{cases}$

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solution $\begin{cases} \partial_t v + \text{div } \sigma + \nabla p = 0 \\ \text{div } v = 0 \\ v \otimes v = \sigma \end{cases}$

subsolution $\begin{cases} \partial_t v + \text{div } \sigma + \nabla p = 0 \\ \text{div } v = 0 \\ v \otimes v \leq \sigma \end{cases}$

Differential Inclusions: Baire category approach

Theorem: The typical short map is (weakly) isometric.

Theorem: The typical Euler subsolution is a (weak) solution.

Euler subsolution
$$\begin{cases} \partial_t v + \operatorname{div}\,(v \otimes v) + \nabla p = -\operatorname{div}\, R \\ \operatorname{div}\, v = 0 \\ R \geq 0 \end{cases}$$

~~Differential~~ Inclusions

Toy problem: construct

$$v : [0,1] \to \mathbb{R} \text{ such that } |v| = 1$$

Constructive approach: define inductively

$$v_{N+1}(x) = v_N(x) + \underbrace{\tfrac{1}{2}(1 - |v_N(x)|^2)}_{\text{amplitude}} \underbrace{s(\lambda_N x)}_{\text{high-frequency oscillation}}$$

$$s(t) = \operatorname{sign} \sin t$$

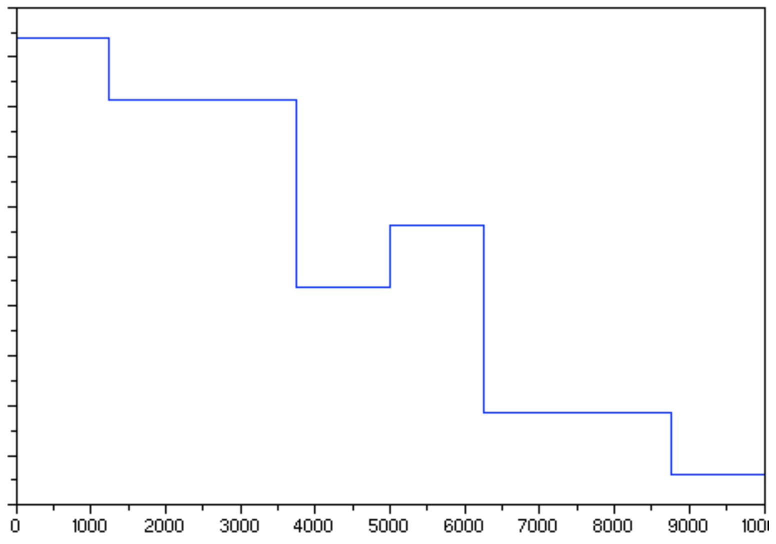
$$|v_N| \leq 1 \;\Rightarrow\; |v_{N+1}| \leq 1$$

~~Differential~~ Inclusions

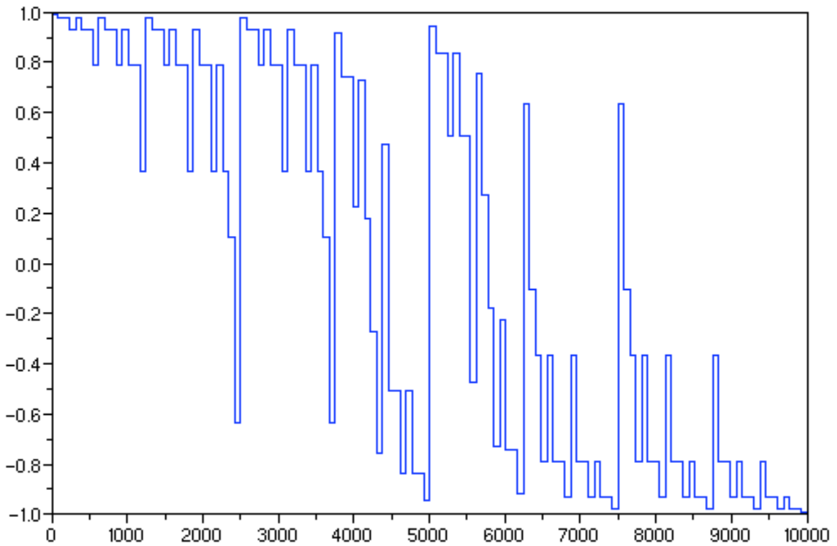
Toy problem: construct

$$v : [0,1] \to \mathbb{R} \text{ such that } |v| = 1$$

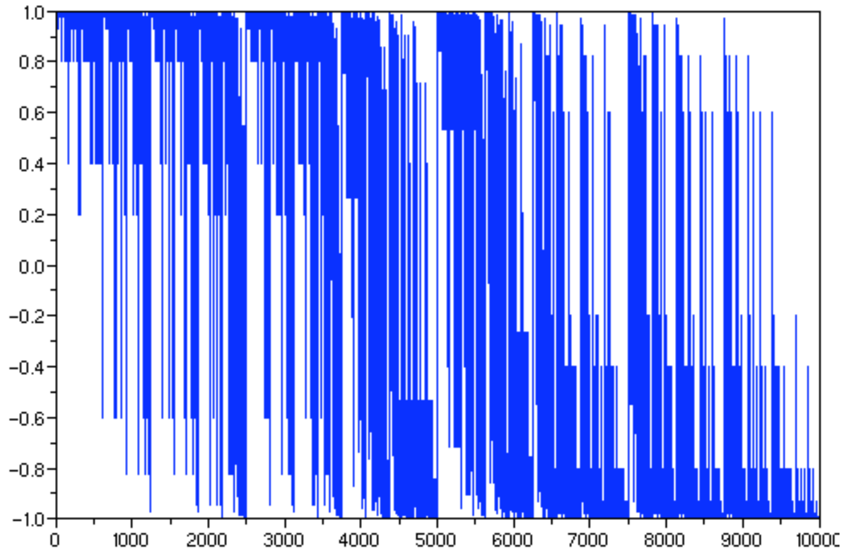
$$v_{N+1}(x) = v_N(x) + \tfrac{1}{2}(1 - |v_N(x)|^2)s(\lambda_N x)$$



v_2

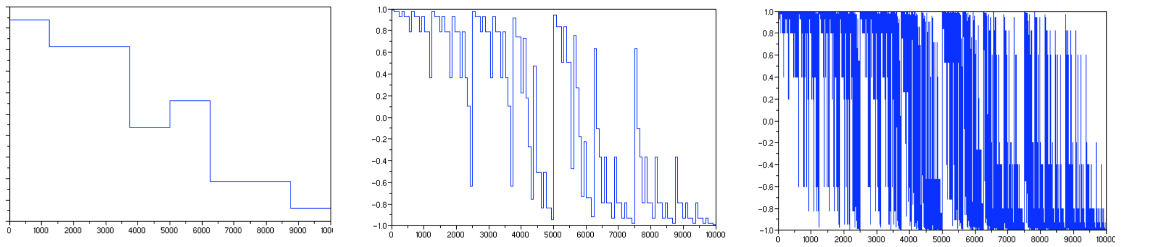


v_6



v_{10}

~~Differential~~ Inclusions



Toy problem: construct

$$v : [0,1] \to \mathbb{R} \text{ such that } |v| = 1$$

$$v_{N+1}(x) = v_N(x) + \tfrac{1}{2}(1 - |v_N(x)|^2)s(\lambda_N x)$$

Lemma

If $\lambda_N = 2^N$, there exists $\alpha > 0$ so that

$$\int_0^1 (1 - |v_N|^2)\, dx \lesssim 2^{-\alpha N}$$

Beyond Lipschitz maps

Theorem **(Nash-Kuiper 1954/55)**

Any short embedding $M^n \hookrightarrow \mathbb{R}^{n+1}$ can be uniformly approximated by C^1 isometric embeddings.

- an example of Gromov's **h-principle**
- method of proof: **convex integration**
- C^2 embeddings are **rigid**
- Lipschitz "version" of theorem is **trivial**

Beyond Lipschitz maps

Theorem **(Nash-Kuiper 1954/55)**

Any short embedding $M^n \hookrightarrow \mathbb{R}^{n+1}$ can be uniformly approximated by C^1 isometric embeddings.

Theorem **(Borisov 1967-2004, Conti-De Lellis-Sz. '09)**

The Nash-Kuiper theorem remains valid for C^1 isometric embeddings with

$$|Du(x) - Du(y)| \leq C|x-y|^\theta \qquad \theta < \frac{1}{1+2n_*}$$

Note: the embedding $S^2 \hookrightarrow \mathbb{R}^3$ is rigid for $\theta > 2/3$

Selection criteria and instabilities

Weak-strong uniqueness

Admissibility:

$$\int |v(x,t)|^2\, dx \leq \int |v_0(x)|^2\, dx$$

Theorem (P.L.Lions 1996)

Given an initial data v_0, if there exists a solution to the IVP with

$$\nabla v + \nabla v^T \in L^\infty$$

then this solution is unique in the class of *admissible* weak solutions.

- The Scheffer-Shnirelman solution clearly **not admissible**
- For the solutions of Wiedemann $E(t)$ has an **instantaneous jump** up at $t = 0$

Wild initial data is generic

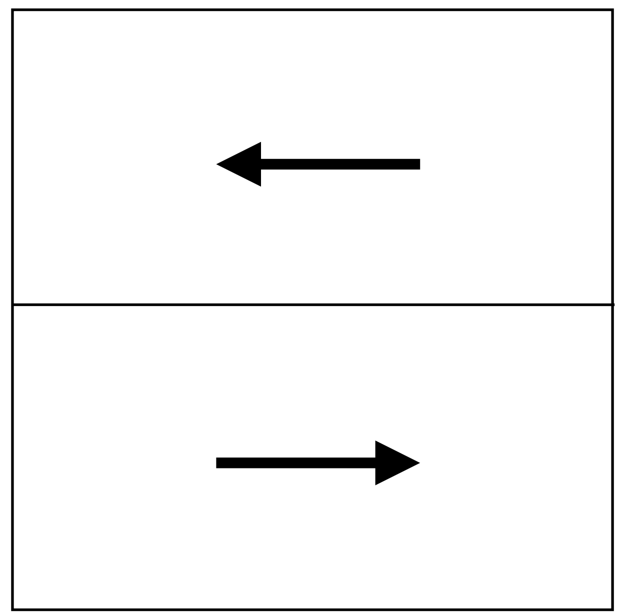
Theorem (De Lellis-Sz. 2010 / Wiedemann-Sz. 2012)
There exists a dense set of initial data $v_0 \in L^2$ for which there exist infinitely many admissible weak solutions.

- solutions also satisfy the **strong** and **local energy inequality**
- a posteriori such initial data needs to be **irregular**

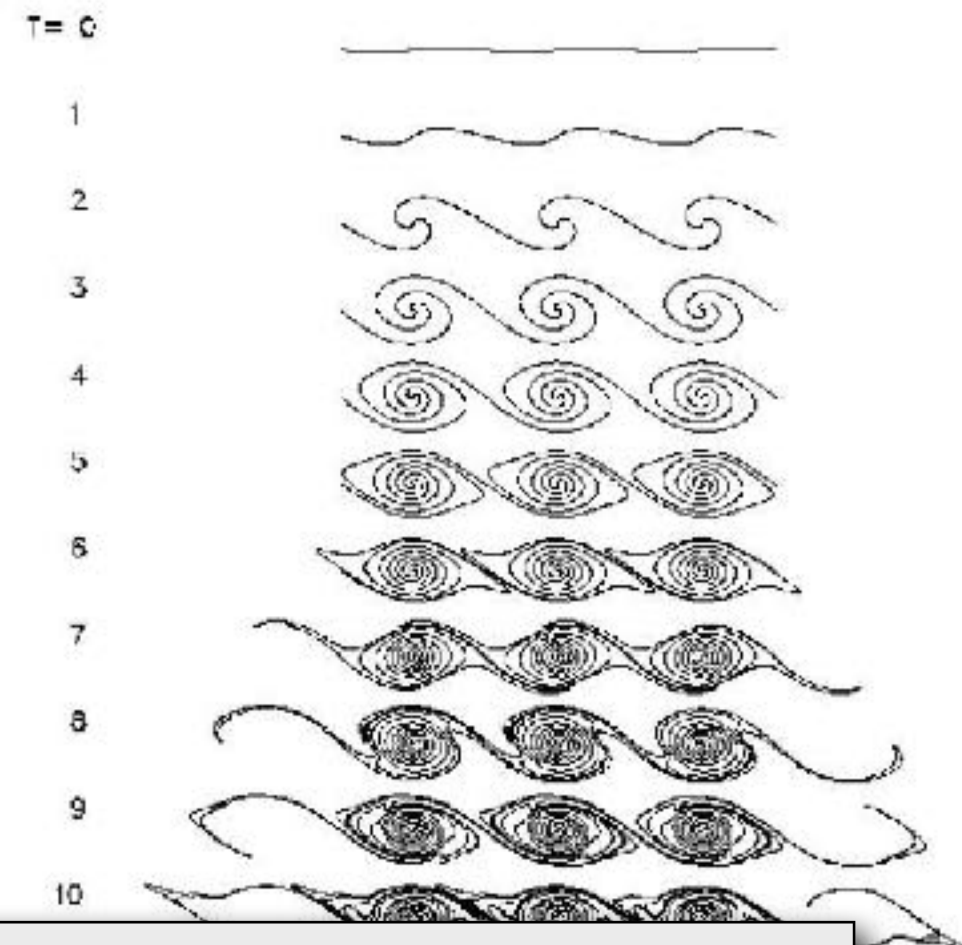
Strong instabilities I: Kelvin-Helmholtz

$\mathbf{T}^2$

$$v_0(x) =$$



Kelvin-Helmholtz instability

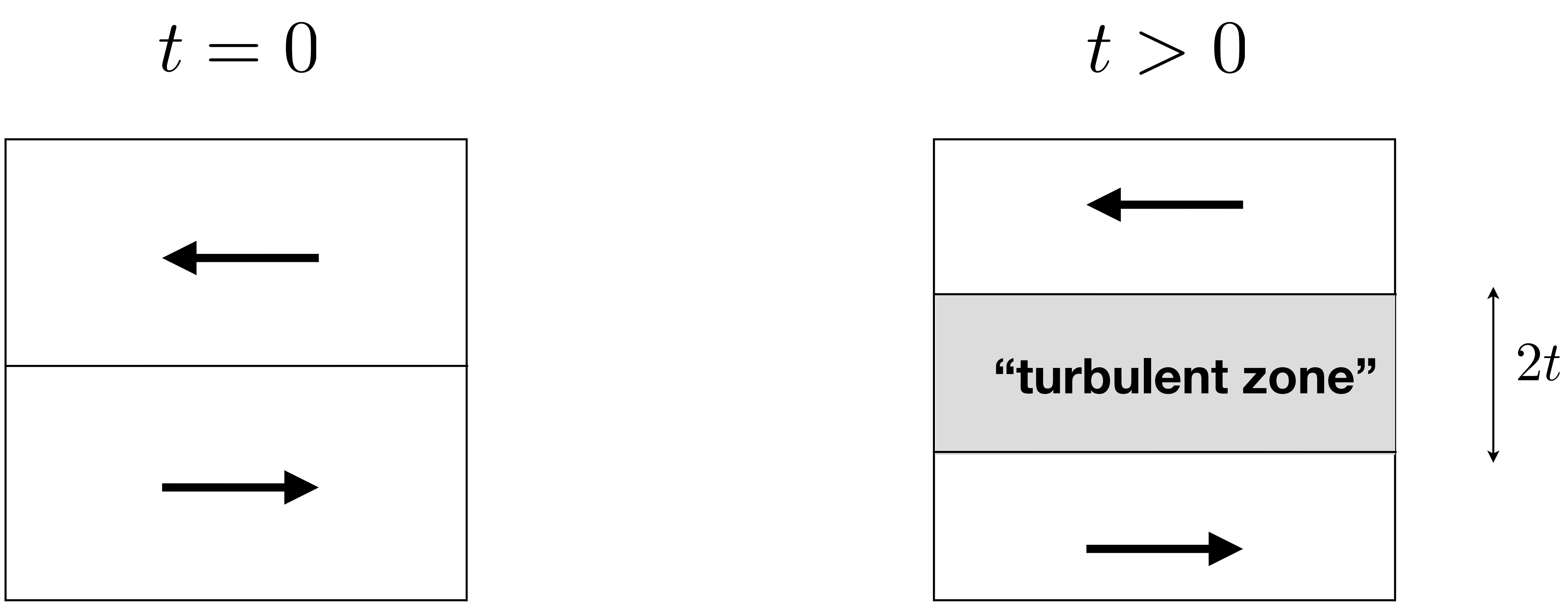


Theorem (Sz. 2011)

There exist infinitely many admissible weak solutions on $\mathbf{T}^2$ with initial data given by v_0 above.

c.f. Delort (1991): there exists a weak solution with $\operatorname{curl} v \in \mathcal{M}_+(\mathbf{T}^2)$

Strong instabilities I: Kelvin-Helmholtz



$$|v| = 1 \ \text{ a.e.}$$

selection principle?

- The solution above is conservative. Strictly **dissipative solutions** also possible.
- There exists a maximal dissipation rate
- **Maximally dissipative** solution is **different from** vanishing viscosity limit $(NS_\nu) \to (E)$
- More realistic limit should include perturbations of the initial condition, i.e.$(NS_{\nu,\varepsilon}) \to (E)$

Strong instabilities II: Rayleigh-Taylor

Incompressible porous medium equation:

$$\partial_t \rho + v \cdot \nabla \rho = 0$$

$$\mathrm{div}\ v = 0$$

$$v + \nabla p = -\rho \mathbf{g} \qquad \text{Darcy's law}$$

Theorem (D. Cordoba - D. Faraco - F. Gancedo 2011)
There exist nontrivial weak solutions with compact support in time.

- R. Shvydkoy 2011: Same result holds for general active scalar equations with even (& non-degenerate) kernel

Strong instabilities II: Rayleigh-Taylor

Incompressible porous medium equation:

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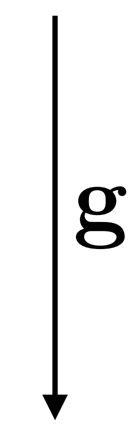
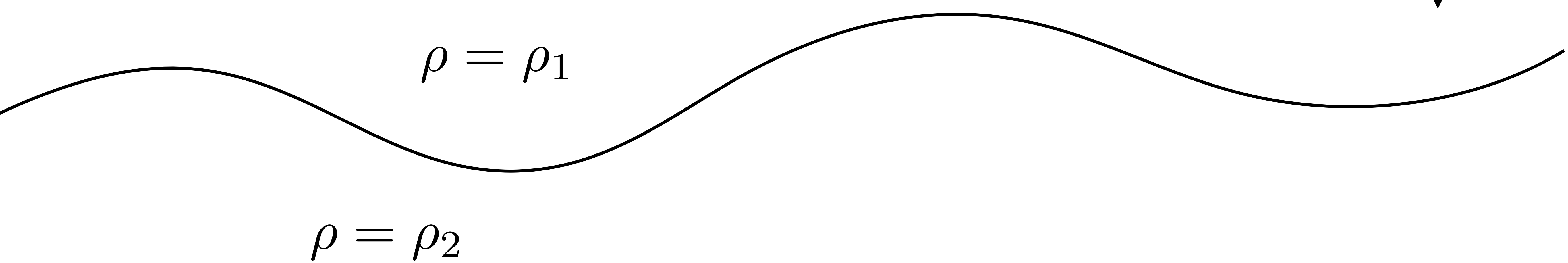
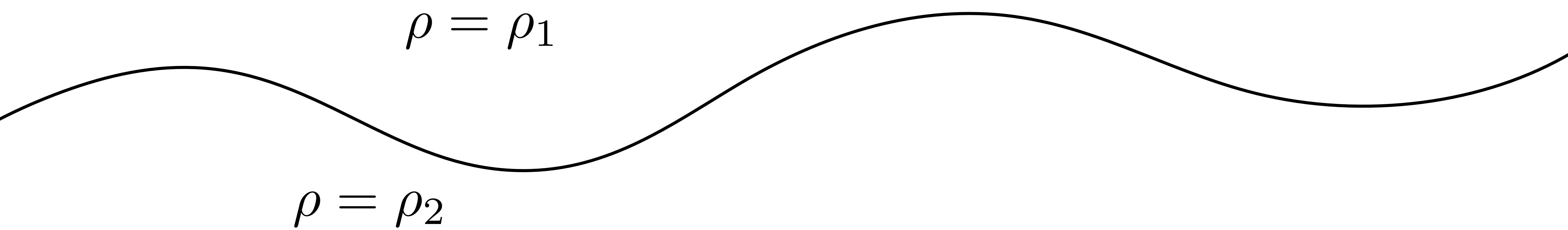




Strong instabilities II: Muskat problem

Muskat curve evolution:

$$\partial_t z(s,t) = \frac{\rho_1 - \rho_2}{2\pi} P.V. \int_{-\infty}^{\infty} \frac{z_1(s,t) - z_1(s',t)}{|z(s,t) - z(s',t)|^2} (\partial_s z(s,t) - \partial_s z(s',t))\, ds'$$



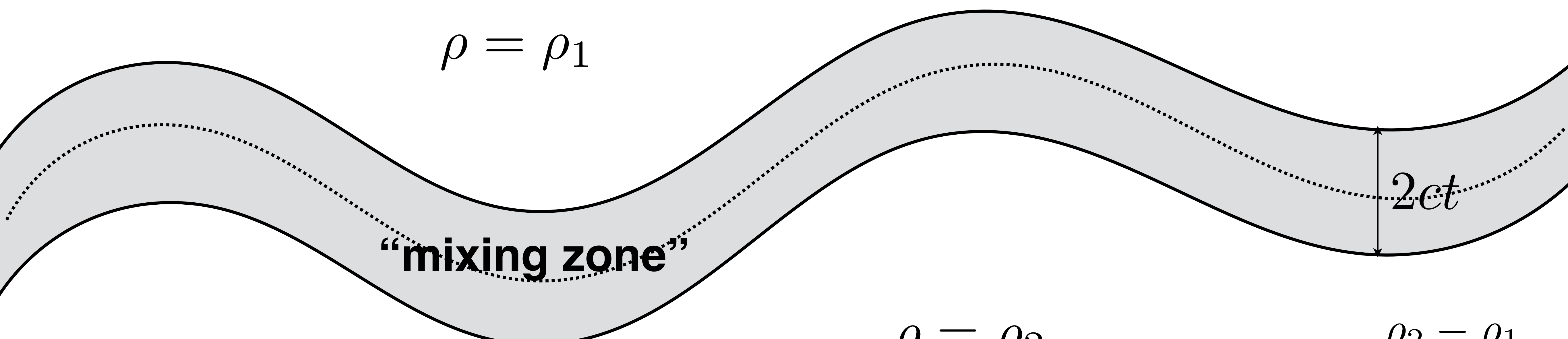
- **Stable case** $\rho_1 < \rho_2$ P.Constantin-F.Gancedo-V.Vicol-R.Shvydkoy 2016
- **Unstable case** $\rho_2 < \rho_1$ A.Castro-D.Cordoba-C.Fefferman-F.Gancedo-M.Lopez-Fernandez 2012

Strong instabilities II: Muskat problem

regularized Muskat curve evolution:

A. Castro - D. Cordoba - D. Faraco 2016

$$\partial_t z(s,t) = \frac{\rho_1-\rho_2}{2\pi}\frac{1}{2ct}\int_{-ct}^{ct}\int_{-\infty}^{\infty}(\partial_s z(s,t)-\partial_s z(s',t))\frac{1}{2ct}\int_{-ct}^{ct}\frac{z_1(s,t)-z_1(s',t)}{|z(s,t)-z(s',t)+(\lambda-\lambda')\mathbf{g}|^2}\,d\lambda' ds' d\lambda$$

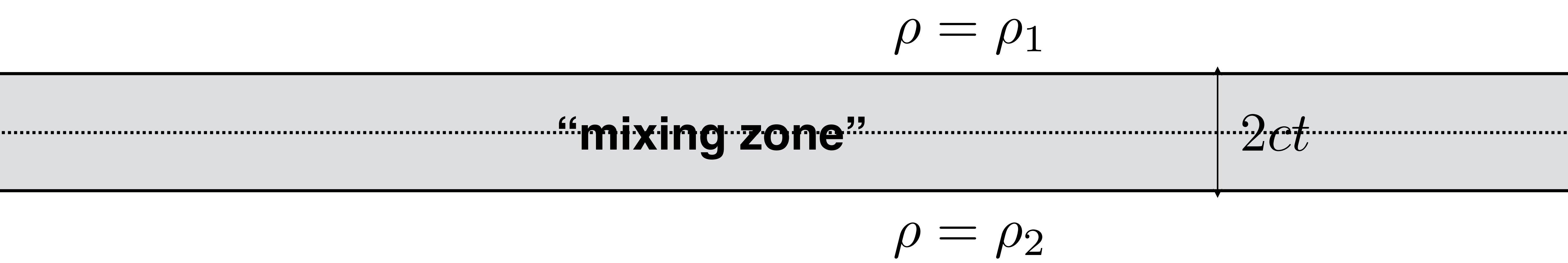


$$c = \frac{\rho_2-\rho_1}{2}$$

Unstable Muskat problem: Selection criteria?

Flat initial interface

- Lagrangian relaxation (F.Otto 1999): gradient flow formulation
- Eulerian relaxation (Sz. 2012): calculation of “lamination hull”
- Maximal mixing (Sz. 2012): $c \leq (\rho_2 - \rho_1)$



Eulerian relaxation

$$Du^T Du = Id$$

short
$$\begin{cases} \text{curl } A = 0 \\ A^T A \leq Id \end{cases}$$

$$\partial_t v + \text{div } (v \otimes v) + \nabla p = 0$$
$$\text{div } v = 0$$

subsolution
$$\begin{cases} \partial_t v + \text{div } \sigma + \nabla p = 0 \\ \text{div } v = 0 \\ v \otimes v \leq \sigma \end{cases}$$

$$\partial_t \rho + v \cdot \nabla \rho = 0$$
$$\text{div } v = 0$$
$$v + \nabla p = -\rho \mathbf{g}$$

subsolution
$$\begin{cases} \partial_t \rho + \text{div } \sigma = 0 \\ \text{div } v = 0 \\ \text{curl } (v + \rho \mathbf{g}) = 0 \\ \left| \sigma - \rho v + \frac{1}{2}(1 - \rho^2)\mathbf{g} \right| \leq \frac{1}{2}(1 - \rho^2) \end{cases}$$

Onsager's conjecture

Onsager's Conjecture 1949

For (weak) solutions of Euler with

$$|v(x,t) - v(y,t)| \leq C|x-y|^\theta$$

a) If $\theta > 1/3$ energy is conserved.

Onsager 1949, Eyink 1994,
Constantin-E-Titi 1993,
Robert-Duchon 2000,
Cheskidov-Constantin-Friedlander-
-Shvydkoy 2007, …

b) If $\theta < 1/3$ dissipation possible.

Scheffer 1993, Shnirelman 1999
De Lellis - Sz. 2012
Buckmaster-De Lellis-Isett-Sz 2013
Buckmaster 2013
Isett 2016

Two types of statements for b)

Let X be a function space.

Theorem A

There exists a nontrivial weak solution $v \in X$ of the Euler equations with compact support in time.

Theorem B

For any smooth positive function $E(t)$ there exists a weak solution of the Euler equations $v \in X$ such that

$$\frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2 \, dx = E(t) \quad \text{for all } t$$

Theorem A

Theorem A

There exists a nontrivial weak solution $v \in X$ of the Euler equations with compact support in time.

- P. Isett 2013 $X = L^\infty(0,T;C^{1/5-}(\mathbb{T}^3))$
- T. Buckmaster 2013 $v(\cdot,t) \in C^{1/3-}$ a.e. t
- T. Buckmaster-De Lellis-Sz. 2015 $X = L^1(0,T;C^{1/3-}(\mathbb{T}^3))$
- P. Isett 2016 $X = L^\infty(0,T;C^{1/3-}(\mathbb{T}^3))$

Theorem B

Theorem B

For any smooth positive function $E(t)$ there exists a weak solution of the Euler equations $v \in X$ such that

$$\frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2\, dx = E(t) \quad \text{for all } t$$

- De Lellis-Sz. 2012 $X = L^\infty(0,T;C^{1/10-}(\mathbb{T}^3))$
- A. Choffrut 2013 $X = L^\infty(0,T;C^{1/10-}(\mathbb{T}^2))$
- T. Buckmaster-De Lellis-Isett-Sz. 2015 $X = L^\infty(0,T;C^{1/5-}(\mathbb{T}^3))$
- T. Buckmaster-De Lellis-Sz.-Vicol **work in progress:**
  $X = L^\infty(0,T;C^{1/3-}(\mathbb{T}^3))$

H-principle and Closure: Subsolutions

Reynolds decomposition: $v = \bar{v} + w$

“average” “fluctuation”

Euler-Reynolds system:

$$\partial_t \bar{v} + \mathrm{div}(\bar{v} \otimes \bar{v}) + \nabla \bar{p} = -\mathrm{div} \bar{R}$$
$$\mathrm{div}\ \bar{v} = 0$$

where

$$\bar{R} = \overline{v \otimes v} - \bar{v} \otimes \bar{v} = \overline{w \otimes w}$$

Closure problem: equation for $\bar{R}$?

We know: $\bar{R} \geq 0$.

H-principle and Closure: Subsolutions

Deterministic turbulence via weak convergence (following P.D.Lax)

Assume

$$v_k \overset{*}{\rightharpoonup} \bar{v} \quad \text{in} \quad L^\infty$$

with

$$\begin{aligned} \partial_t v_k + \operatorname{div}(v_k \otimes v_k) + \nabla p_k &= \nu_k \Delta v_k \\ \operatorname{div}\, v_k &= 0 \end{aligned} \qquad \nu_k \to 0$$

Then:

$$\left.\begin{aligned} \partial_t \bar{v} + \operatorname{div}(\bar{v} \otimes \bar{v}) + \nabla \bar{p} &= -\operatorname{div} \bar{R} \\ \operatorname{div}\, \bar{v} &= 0 \end{aligned}\right\}$$

with $\bar{R} = w - \lim_{k \to \infty} (v_k - \bar{v}) \otimes (v_k - \bar{v}) \geq 0$

Energy:

$$E(t) = \frac{1}{2} \int |\bar{v}|^2 + \operatorname{tr}\, \bar{R}\, dx$$

H-principle and Closure

Theorem B (S. Daneri - Sz '16)

Let $(\bar{v}, \bar{p}, \bar{R})$ be a smooth **strict** subsolution. For any $\theta < 1/5$ there exist a sequence (v_k, p_k) of weak solutions of Euler such that

$$|v_k(x,t) - v_k(y,t)| \leq C|x-y|^\theta$$

moreover

$$v_k \overset{*}{\rightharpoonup} \bar{v} \quad \text{and} \quad v_k \otimes v_k \overset{*}{\rightharpoonup} \bar{v} \otimes \bar{v} + \bar{R} \quad \text{in} \quad L^\infty$$

and

$$\frac{1}{2}\int_{\mathbb{T}^3} |v_k|^2\, dx = \frac{1}{2}\int_{\mathbb{T}^3} |\bar{v}|^2 + \operatorname{tr} \bar{R}\, dx \qquad \forall\, t$$

Expect same result for all $\theta < 1/3$ ....

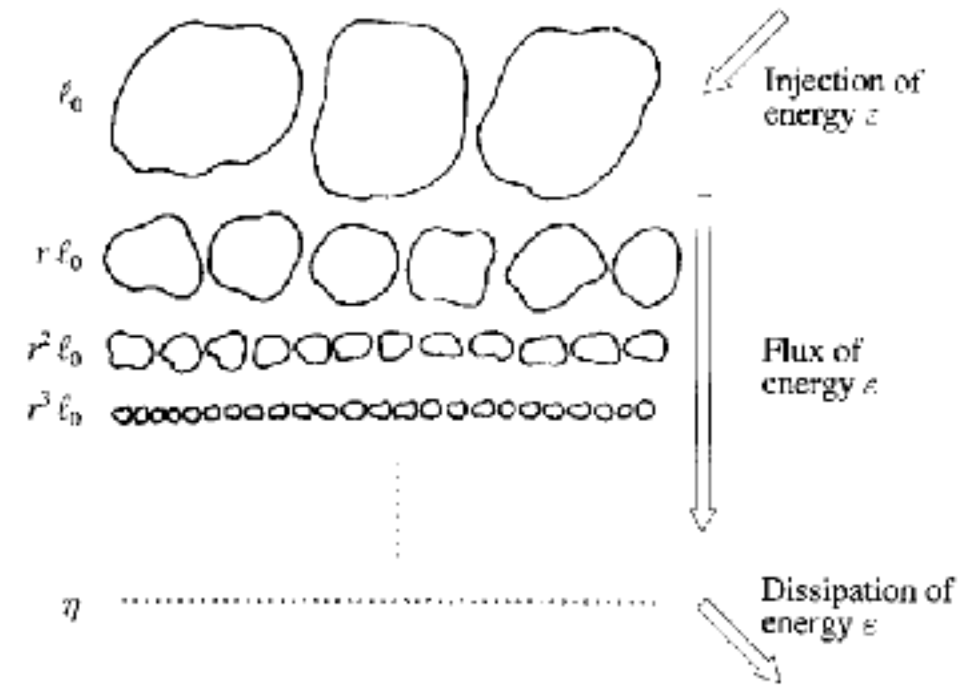
Some ideas of the construction

(based on De Lellis-Sz. 2012)

The Euler-Reynolds equations

Euler-Reynolds system: $q \in \mathbb{N}$

$$\partial_t v_q + \nabla \cdot (v_q \otimes v_q) + \nabla p_q = -\nabla \cdot R_q$$
$$\nabla \cdot v_q = 0$$

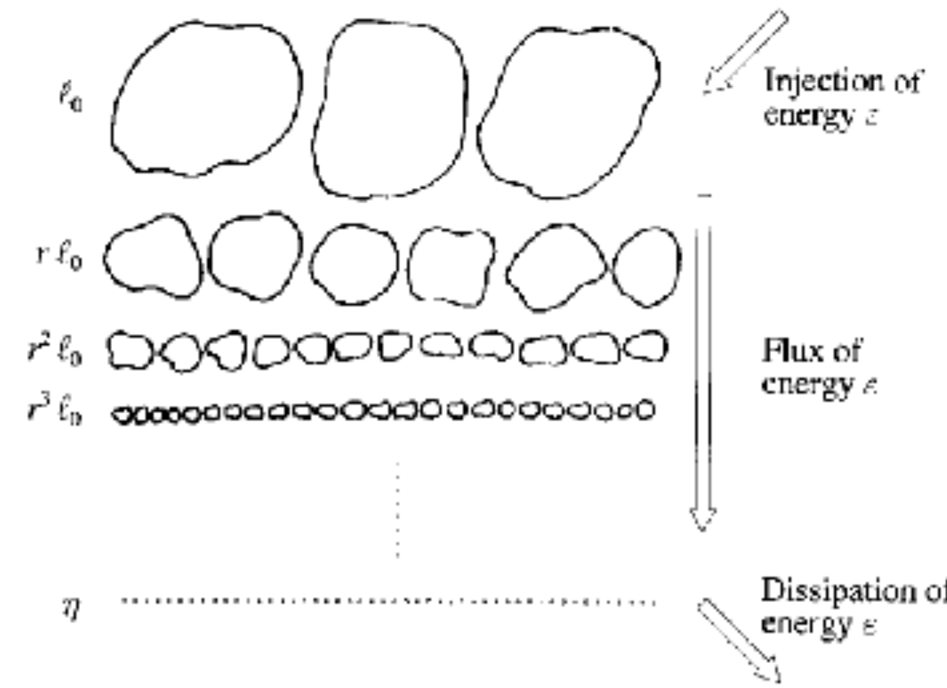


$$v_{q+1}(x,t) = v_q(x,t) + W(\underbrace{x,t}_{\text{slow}}, \underbrace{\lambda_{q+1}x, \lambda_{q+1}t}_{\text{fast}})$$

"**fluctuation**" - analogue of Nash twist

The Euler-Reynolds equations

Euler-Reynolds system: $q \in \mathbb{N}$



$$\partial_t v_q + \nabla \cdot (v_q \otimes v_q) + \nabla p_q = -\nabla \cdot R_q$$

$$\nabla \cdot v_q = 0$$

more precisely:

$$v_{q+1}(x,t) = v_q(x,t) + W\big(v_q(x,t), R_q(x,t), \lambda_{q+1}x, \lambda_{q+1}t\big)$$

explicit dependence of previous “state”

Conditions on the "fluctuation" $W = W(v, R, \xi, \tau)$

(H1) $\xi \mapsto W(v, R, \xi, \tau)$ periodic with average zero:

$$\langle W \rangle = \int_{\mathbf{T}^3} W(v, R, \xi, \tau)\, d\xi = 0$$

(H2) $\langle W \otimes W \rangle = R$ prescribed average stress

(H3)
$$\partial_\tau W + v \cdot \nabla_\xi W + W \cdot \nabla_\xi W + \nabla_\xi P = 0$$
$$\mathrm{div}_\xi W = 0$$
in $\mathbb{T}^3 \times \mathbb{R}$

(H4) $|W| \lesssim |R|^{1/2}$ $\quad |\partial_v W| \lesssim |R|^{1/2}$ $\quad |\partial_R W| \lesssim |R|^{-1/2}$

Estimating new Reynolds stress R_{q+1}

$$R_{q+1} = -\mathrm{div}^{-1}\Big[\partial_t v_{q+1} + v_{q+1}\cdot\nabla v_{q+1} + \nabla p_{q+1}\Big]$$

(I) $$= -\mathrm{div}^{-1}\Big[\partial_t w_{q+1} + v_q \cdot \nabla w_{q+1}\Big]$$

(II) $$-\mathrm{div}^{-1}\Big[\nabla\cdot\big(w_{q+1}\otimes w_{q+1} - R_q\big) + \nabla(p_{q+1} - p_q)\Big]$$

(III) $$-\mathrm{div}^{-1}\Big[w_{q+1}\cdot\nabla v_q\Big]$$

Estimating new Reynolds stress

$$(III) = -\mathrm{div}^{-1}\left[w_{q+1}\cdot\nabla v_q\right] = \mathrm{div}^{-1}\left[\sum_{k\neq 0} a_k(x,t)e^{i\lambda_{q+1}k\cdot x}\right]$$

(H1)

“stationary phase” + (H4)

$$\|(III)\|_0 \lesssim \frac{\sum_k \|a_k\|_0}{\lambda_{q+1}}$$

$$\lesssim \frac{\|R_q\|_0^{1/2}\|\nabla v_q\|_0}{\lambda_{q+1}}$$

Estimating new Reynolds stress

$$(II) = -\mathrm{div}^{-1}\Big[\nabla \cdot (W \otimes W - R_q) + \nabla P\Big] = \mathrm{div}^{-1}\Big[\sum_{k \neq 0} b_k(x,t) e^{i\lambda_{q+1} k \cdot x}\Big]$$

slow

(H3)

(H2) $\langle W \otimes W \rangle = R$

“stationary phase” + (H4)

$$\|(II)\|_0 = O(\frac{1}{\lambda_{q+1}})$$

...and similarly $\|(I)\|_0 = O(\frac{1}{\lambda_{q+1}})$

Estimating new Reynolds stress

Summarizing:

$$v_{q+1} = v_q + W(v_q, R_q, \lambda_{q+1}x, \lambda_{q+1}t) + corrector$$

1) $$\|v_{q+1} - v_q\|_0 \lesssim \|R_q\|_0^{1/2}$$

…leads to convergence in C^0

2) $$\|R_{q+1}\|_0 \lesssim O(\frac{1}{\lambda_{q+1}})$$

… more careful estimates lead in the **ideal case**:

2') $$\|R_{q+1}\|_0 \lesssim \frac{\|R_q\|_0^{1/2}\|\nabla v_q\|_0}{\lambda_{q+1}}$$

Conditions on the "fluctuation" $W = W(v, R, \xi, \tau)$

(H1) $\xi \mapsto W(v, R, \xi, \tau)$ periodic with average zero:

$$\langle W \rangle = \int_{\mathbf{T}^3} W(v, R, \xi, \tau)\, d\xi = 0$$

(H2) $\langle W \otimes W \rangle = R$

convection of microstructure

(H3)
$$\partial_\tau W + v \cdot \nabla_\xi W + W \cdot \nabla_\xi W + \nabla_\xi P = 0 \qquad \text{in} \quad \mathbb{T}^3 \times \mathbb{R}$$
$$\mathrm{div}_\xi W = 0$$

(family of) stationary solutions: Beltrami flows

(H4) $|W| \lesssim |R|^{1/2}$ $\quad |\partial_v W| \lesssim |R|^{1/2}$ $\quad |\partial_R W| \lesssim |R|^{-1/2}$

Convection of microstructure

Better *Ansatz*:

$$v_{q+1}(x,t) = v_q(x,t) + W\big(R_q(x,t), \lambda_{q+1}\Phi_q(x,t)\big)$$

inverse flow of v_q

P. Isett 2013, PhD Thesis (1/5-Hölder, Theorem A)

c.f. also **D.W.McLaughlin-G.C.Papanicolaou-O.R.Pironneau** 1985

(H1) $\langle W \rangle = 0$

(H3') $W \cdot \nabla_\xi W + \nabla_\xi P = 0$

$\mathrm{div}_\xi W = 0$

(H2) $\langle W \otimes W \rangle = R$

(H4) $|W| \lesssim |R|^{1/2}$

$|\partial_v W| \lesssim |R|^{1/2} \qquad |\partial_R W| \lesssim |R|^{-1/2}$

Convection of microstructure

Better *Ansatz*:

$$v_{q+1}(x,t) = v_q(x,t) + W\big(R_q(x,t), \lambda_{q+1}\Phi_q(x,t)\big)$$

inverse flow of v_q

Problem: Flow only controlled for very short times

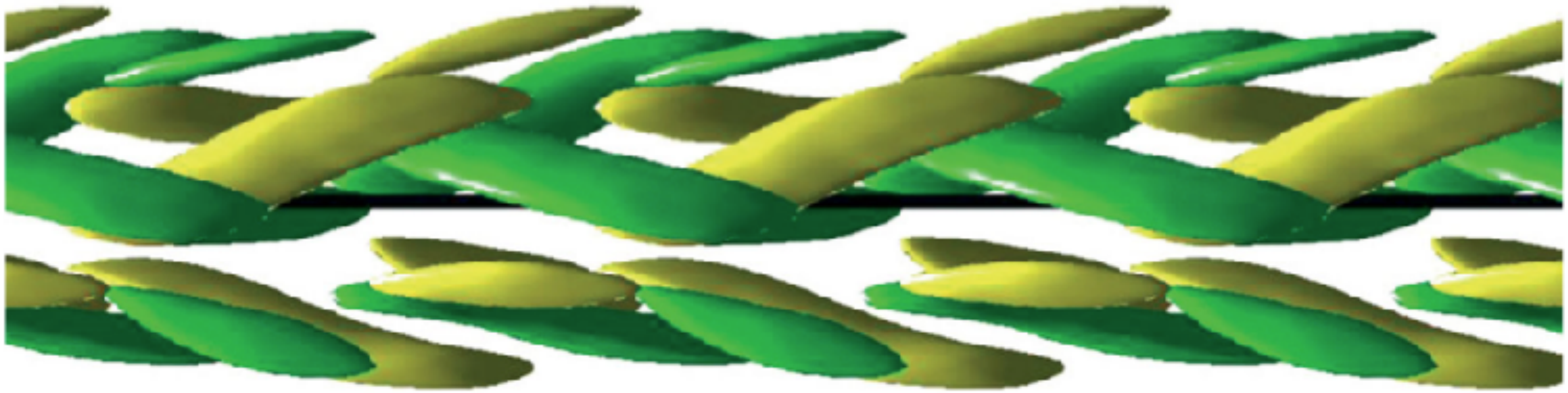
$$v_{q+1}(x,t) = v_q(x,t) + \sum_i \chi_i(t) W_i\big(R_q(x,t), \lambda_{q+1}\Phi_{q,i}(x,t)\big)$$

time cut-off
partition of unity

inverse flow restarted at time t_i

Convection of microstructure

B. Eckhardt - B. Hof - H.Faisst 2006



“turbulent puffs”

Problem: Flow only controlled for very short times

$$v_{q+1}(x,t) = v_q(x,t) + \sum_i \chi_i(t) W_i\big(R_q(x,t), \lambda_{q+1}\Phi_{q,i}(x,t)\big)$$

P. Isett 2013, PhD Thesis (1/5-Hölder, Theorem B)

Buckmaster - Isett - De Lellis - Sz. 2014, (1/5-Hölder, Theorem A)

Buckmaster - De Lellis - Sz. 2015, (1/3-Hölder L1 in time, Theorem B)

P. Isett 2016,(1/3-Hölder, Theorem B)

Convection of microstructure

Main problems

- **cell-problem:** Beltrami modes not exact any more
- **gluing:** orthogonality of Beltrami modes

Convection of microstructure

Main problems

- **cell-problem:** Beltrami modes not exact any more
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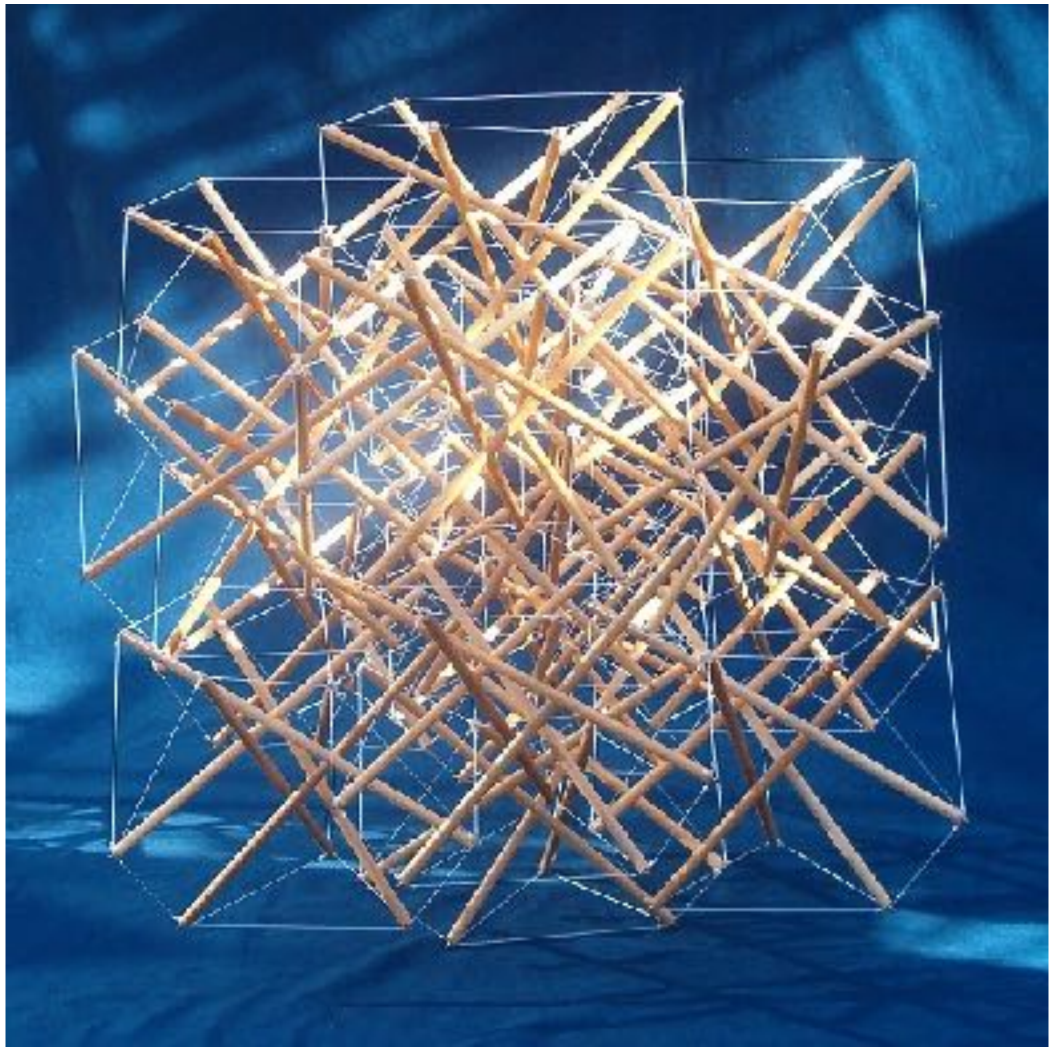
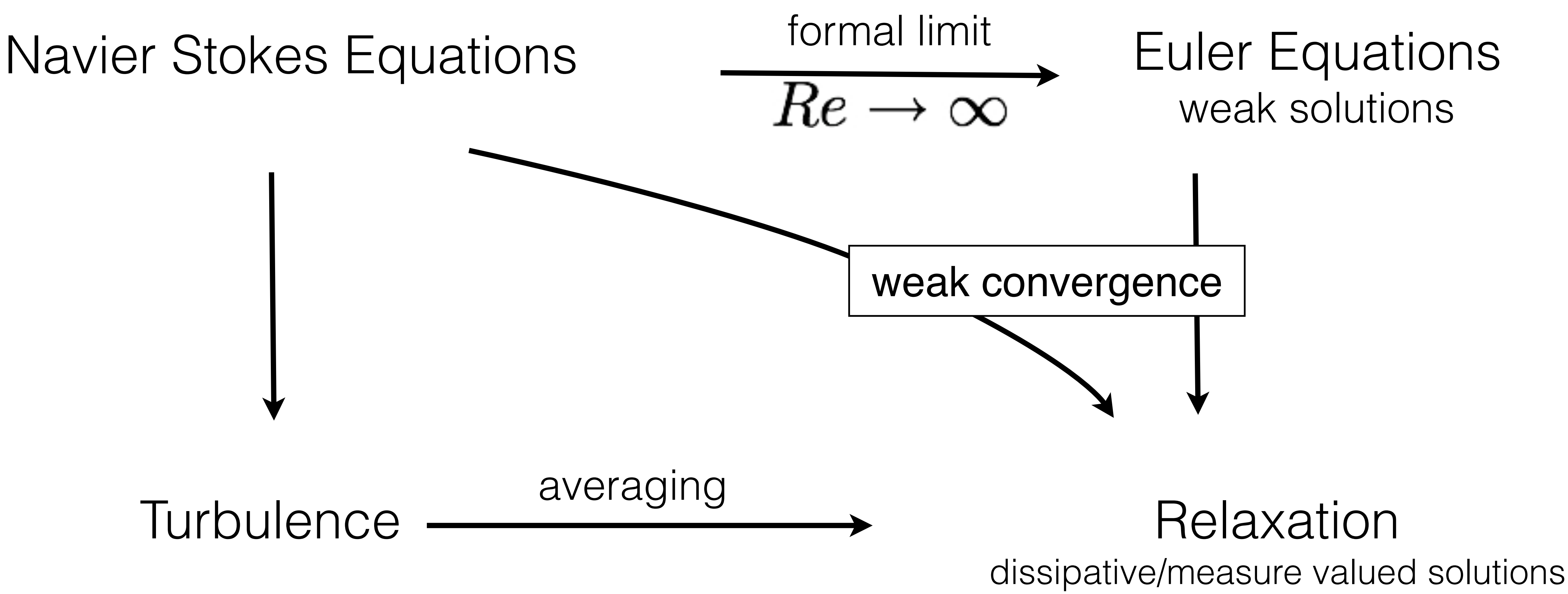
Alternative to Beltrami-flows: “**Mikado flows**”

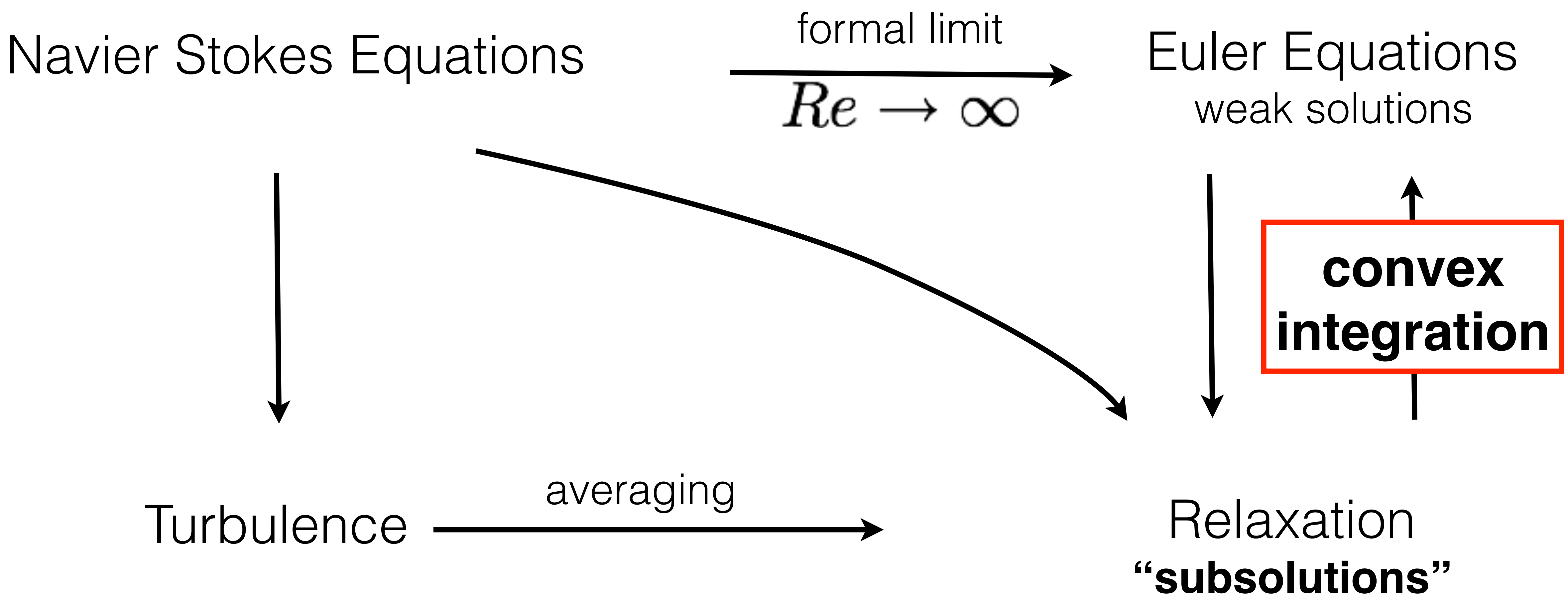


“tensegrity”

Deterministic approach



Deterministic approach



Meta-Theorem

Any Euler subsolution can be approximated by weak solutions.

Thank you

for your attention