The Bulk-Edge Correspondence via FREDHOLM THEORY

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Theory & Computation for 2D Materials

Based on joint work with: - Graf.
- Fonseca, Sheta, Wang & Yamakawa
$H$ Bulk (\infty space) Hamiltonian

$\text{sgn}(H)$ flat Hamiltonian

$\sigma(\text{sgn}(H))$

Bulk top. invariant

$\Lambda(H)$ truncated, edge, half-\infty Hamiltonian

$E_{\text{EDGE}}$

Edge top. invar.
**Main Message**

- $H$: Bulk (infinite space) Hamiltonian
  - $\text{sgn}(H)$: flat Hamiltonian
  - Bulk top. invariant

- $\Lambda(H)$: truncated, edge, half-\(\infty\) Hamiltonian
  - Edge top. invariant

Bulk-edge correspondence (in the spectral gap regime) amounts to the statement that

$$\text{sgn} \circ \Lambda \sim \Lambda \circ \text{sgn}$$

where $\sim$ is equivalence when computing indices.
TODAY

1. **ID CHIRAL EXAMPLE**,
2. **FREDHOLM BASICS**,
3. **2D IQHE and \(\mathbb{Z}_2\) EXAMPLES**.

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Simplest Example: Chiral ID

1D chain w/ two species of sites

Hamiltonian is chiral iff it only has off-diagonal terms in the +- (chirality) basis.
SIMPLEST EXAMPLE: CHIRAL 1D (SSH e.g.)

1D chain w/ two species of sites

Hamiltonian is chiral if it only has off-diagonal terms in the +− (chirality) basis.

\[ H = \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix} \quad \exists \quad S: \mathcal{H}_+ \rightarrow \mathcal{H}_- . \]

\( S \) need not be self-adjoint. E.g.: \( S = \) shift operator

\[ \Pi = \begin{bmatrix} \Pi \mathcal{H}_+ & 0 \\ 0 & -\Pi \mathcal{H}_- \end{bmatrix} \quad \{ H, \Pi \} = 0 , \quad \Pi^2 = +1 . \]

\( \sigma(H) \) symmetric about 0.
SIMPLEST EXAMPLE: CHIRAL ID (cont.)

\( \sigma(H) \) symmetric about 0. \( \Rightarrow \) \( E_f := 0 \).

Spectral gap for \( H \iff 0 \notin \sigma(H) \).

\( \Rightarrow \) \( \text{sgn}(H) = \frac{H}{|H|}^{-1} \) makes sense.

\( \sigma(H) \) \( \rightarrow \) \( \sigma(\text{sgn}(H)) \).
SIMPLEST EXAMPLE: CHIRAL ID (cont.)

$\sigma(C(H))$ symmetric about $0 \implies E_f := 0$.

Spectral gap for $H \iff 0 \notin \sigma(C(H))$.

$\implies \text{sgn}(C(H)) = H[H]^{-1}$ makes sense.

Bulk index $N := \frac{1}{2} \text{tr} \left( \prod \text{sgn}(H) [\Lambda, \text{sgn}(H)] \right)$

where $\Lambda$ projects onto RHS of space:

$\left( ..., \psi_{-1}, \psi_0, \psi_1, \psi_2, ... \right) \xrightarrow{\Lambda} \left( ..., 0, 0, \psi_1, \psi_2, ... \right)$
After some basic calculation,

\[ N := \frac{1}{2} \text{tr} \left( \text{TT} \, \text{sgn}(H) \, [\Lambda, \text{sgn}(H)] \right) = \ldots = \text{tr} \left( U^* [\Lambda, U] \right) \text{ where } \text{sgn}(H) = \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix} \]

i.e., \( U \) is the polar part in the polar decomp. of \( S \): \( U = S_{11}^{-1} \).
After some basic calculation,

\[ N := \frac{1}{2} \text{tr} ( \text{TT sgn}(\mathbf{H}) [\Lambda, \text{sgn}(\mathbf{H})] ) = \ldots = \text{tr} (U^* \Lambda , U) \]

where \( \text{sgn}(\mathbf{H}) = \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix} \)

i.e., \( U \) is the polar part in the polar decomp. \( \sqrt{S} : \quad U = S1S1^{-1} \).

\( S \) invertible \( \Rightarrow U \) is unitary.

\( \Lambda = \Lambda^* = \Lambda^2 \) is a proj.

\[ \Rightarrow N = \text{tr} (U^* \Lambda , U) = \text{tr} (U^* \Lambda U - \Lambda) \]

\[ = \ldots = \text{index}(\Lambda U \Lambda + \Lambda^2) \in \mathbb{Z} \]

FEDOSOV

\[ \text{Fredholm index} \]

\[ =: \text{index}(\Lambda(U)) \]

\( \Lambda(U) = \Lambda U \Lambda + \Lambda^\perp \).
Fredholm index ?
Fredholm Theory

Operator $A$ is Fredholm iff:

1. $\dim \ker A < \infty$
2. $\dim \ker A^* < \infty$
3. $\text{range}(A)$ is closed $\iff \exists \varepsilon > 0 : \|A\phi\| \geq \varepsilon \|\phi\| \forall \phi \in (\ker A)^\perp$.

(outside the kernel inverse bdd.) $\ker(A^*) \subseteq \text{coker}(A)$ if $\text{range}(A)$ closed.
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Then we def. 

$\text{index } A := \dim \ker A - \dim \ker A^* \in \mathbb{Z}$
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(outside the kernel inverse bdd.)

Then we define:

$\text{index } A := \dim \ker A - \dim \ker A^*$

$\in \mathbb{Z}$

(degree of non-invertibility)

Facts:

1. Index stable under cpt. perturbations.
2. Index stable under norm-small perturbations.
Fredholm Theory Examples

1. \( L \) is Fredholm w/ index 0.
FREDHOLM THEORY EXAMPLES

1. $I$ is Fredholm w/ index $0$.
2. $O$ is not Fredholm on $n$-dim. Hilbert sp.
Fredholm Theory Examples

1. \( \downarrow \) is Fredholm w/ index 0.
2. 0 is not Fredholm on \( \mathbb{N} \)-dim Hilbert sp.
3. Right shift is Fredholm on:
   \( \ell^2(\mathbb{N}) \) w/ index = -1,
   \( \ell^2(\mathbb{Z}) \) w/ index = 0.
FREDHOLM THEORY EXAMPLES

1. \( f \) is Fredholm w/ index 0.
2. \( 0 \) is not Fredholm on \( \mathbb{R} \) dim. Hilbert sp.
3. Right shift is Fredholm on:
   - \( l^2(\mathbb{N}) \) w/ index = -1,
   - \( l^2(\mathbb{Z}) \) w/ index = 0.
4. \( X^{-1} \) is not Fredholm on \( l^2(\mathbb{N}) \) though it has finite kernel & is self-adjoint.

\[
(X^{-1} \psi)_n = \frac{1}{n} \psi_n \quad (n \in \mathbb{N})
\]
Fredholm Theory Examples

1. $\lambda$ is Fredholm w/ index 0.
2. $0$ is not Fredholm on $\mathbb{R}^{dim}$ Hilbert sp.
3. Right shift is Fredholm on:
   - $\ell^2(\mathbb{N})$ w/ index = $-1$,
   - $\ell^2(\mathbb{Z})$ w/ index = $0$.
4. $X^{-1}$ is not Fredholm on $\ell^2(\mathbb{N})$ though it has finite kernel & it is self-adjoint.
   $$(X^{-1}y)_n = \frac{1}{n} y_n \quad (n \in \mathbb{N})$$
5. $\Lambda \Lambda^\dagger + \Lambda^\dagger$ for chiral 1D bulk system is Fredholm because:
   - $H$ is local: $\|H_{xy}\| \leq ce^{-|x-y|}$
   - $H$ has gap.
For half-\(\infty\) geometry, we truncate \(H \rightarrow \hat{H}\).

Since \([\Pi, X] = 0\), \(\hat{H} = \begin{bmatrix} 0 & \hat{S}^* \end{bmatrix}\) and \(\hat{S}\) is the truncation of \(S\). By locality of \(H\), \(S\) is also local (and invertible, so Fredholm).

\[
\hat{S} = \hat{S} \oplus \hat{S} + \text{cpt.} \quad \Rightarrow \quad \hat{S} \text{ also Fredholm.}
\]
\[ \hat{S} \text{ is Fredholm} \Rightarrow \text{Def. } \hat{N} = \text{index}(\hat{S}) \in \mathbb{Z}. \]
\[ \mathcal{N} \equiv \text{index}(\Lambda \Lambda + \Lambda^\perp) \overset{?}{=} \text{index}(\hat{\Lambda}) \equiv \hat{\mathcal{N}} \]

\[ \hat{\mathcal{N}} \equiv \Lambda S (\Lambda^\perp + \Lambda^\perp \Lambda^\perp) =: \Lambda (S) \]

Fact: \( \text{index}(A) = \text{index}(VV^* - V^*V) \) where \( V := \text{polar}(A) \).

\[ \Rightarrow \text{index}(A) = \text{index(\text{polar}(A)))} \]

So, BEC reduces to the question:

\[ \text{index}(\Lambda(\text{polar}(S))) = \text{index(\text{polar}(\Lambda(S))))?} \]

Do truncation and flattening commute on the index?
Answer: Yes! By homotopy: \( S_t := tS + (1-t)\text{polar}(S) \)

Spectral gap and locality of \( S \) guarantee
\( \Lambda(S_t) \) is Fredholm \( \forall \ t \in [0,1] \)

\[ t \mapsto \text{index}(\Lambda(S_t)) \]

is constant \( \text{by cont. of index}. \)

\[ N = \text{index}(\Lambda(S_0)) \]

\[ \hat{N} = \text{index}(\Lambda(S_1)) \]

\[ \Rightarrow \boxed{N = \hat{N}} \]
2D BULK SYSTEMS w/ or w/o TRI

$H$ is gapped @ 0 energy (WLOG), $E_p := 0$, $p := \mathcal{X}_{(-\infty,0),CH}$ Fermi projection

Fermionic ground state.

Note: $\rho = \frac{1}{2}(1 - \text{sgn}(H)) \rightarrow \rho$ related to flattening.

Let $U := \exp(i\text{arg}(X_1+iX_2))$ (Laughlin's flux insertion).

Then $PUP + P^\perp$ is Fredholm (by gap of $H$)

$\|Px\| \leq C e^{-\mu x y^\nu}$.

$p$ still -synch.
For the IQHE, the top invar. is the Hall conductivity: 

$$N = 2\pi i \operatorname{tr}(P \left[ [\lambda_1, P], [\lambda_2, P] \right]),$$

($$\lambda_1, \lambda_2$$ proj. onto right/upper half space); trace class bcs. $$P$$ is local.

According to Bellissard et al., after some work, $$N = \cdots = \text{index}(PUP + P\perp)$$

$$\Rightarrow N \in \mathbb{Z}.$$
Operator $\Theta : \mathcal{H} \rightarrow \mathcal{H}$ s.t.

1. anti-unitary, anti-linear:
   $$\langle \Theta \psi, \Theta \phi \rangle = \langle \phi, \psi \rangle,$$
2. $\Theta^2 = -\mathbb{1}$,
3. $[\Theta, X] = 0$.  

TIME REVERSAL
Operator \( \Theta : \mathcal{H} \to \mathcal{H} \) s.t.: ① anti-unitary, anti-linear:
\[
\langle \Theta \psi, \Theta \phi \rangle = \langle \phi, \psi \rangle ,
\]
② \( \Theta^2 = -1 \) , \text{ Fermions}
③ \( [\Theta, X] = 0 \).

\( H \) \text{ TR1} iff \[ [\Theta, H] = 0 . \]

By anti-C-linearity, \( U = -\Theta U^* \Theta \) , as \( U = e^{i\arg(x_1 + ix_2)} \).
\( P \Theta = \Theta P \to P = -\Theta P \Theta \)

So \( \boxed{PUP + P^\perp = -\Theta (PUP + P^\perp)^* \Theta} \) for TR1 \( H \) .
Fact: ① $\text{index}(AB) = \text{index}(A) + \text{index}(B)$. (log rule)
② $\text{index}(A^*) = -\text{index}(A)$.

So for $\text{TRI} \equiv 1$, 

$$N = \text{index}(PUP + P^\perp) = \text{index}(- \oplus (PUP + P^\perp) \oplus) = \text{index}(- \oplus) + \text{index}(PUP + P^\perp)^* \oplus \text{index}(\oplus) = 0$$

as $\oplus$ bijective

$$= - \text{index}(PUP + P^\perp) = - N \Rightarrow \boxed{N = 0}$$

For Fredholm operators $A$ which obey

$$A = - \oplus A^* \oplus$$

($\oplus$-odd Fredholm op.)

$\text{index}(A) = 0$ so define instead:

$\text{index}_2(A) := \dim \ker A \mod 2 \in \mathbb{Z}_2$. 

Odd Fredholm theory?
Since \( N = \text{index}(PUP + p \perp) = 0 \) always, define \( N_2 := \text{index}_2(PUP + p \perp) \in \mathbb{Z}_2 \).

This is the FU-KANE-MELE Pfaffian index for TRI vector bundles when \( \exists \) translation invariance.
In the IQHE, invariant is edge Hall conductivity:

$$ \hat{\mathcal{N}} = i \text{ tr}(g'(\hat{H}) [\hat{H}, \Lambda]) $$

where $g'$ is a smooth version of $X(\alpha, 0)$:

$\text{supp}(g')$ within bulk gap $\Rightarrow g'(\hat{H})$ "projects" onto edge states.
Thm. (Kellendonk-Richter-Schulz-Baldes)
\[ \hat{N} = \text{index} \left( \Lambda_1 \exp(-2\pi i g(\hat{A})) \Lambda_1^+ \right) \in \mathbb{Z}. \]

As before, \([\hat{A}, \Theta] = 0\) for TRI, so \([g(\hat{A}), \Theta] = 0\).

\[ \Rightarrow \hat{N} = \text{index} \left( \Lambda_1 \exp(-2\pi i g(\hat{A})) \Lambda_1^+ \right) = 0. \]

Instead, for TR1 edge systems take
\[ \hat{N}_2 := \text{index}_2 \left( \Lambda_1 \exp(-2\pi i g(\hat{A})) \Lambda_1^+ \right). \]
2D BEC (TRI or not)

(Can add subscript 2 everywhere)

\[ N = \text{index}(\text{PUP} + \rho \pm) \overset{?}{=} \text{index}(\Lambda_1 e^{2\pi i gCH} \Lambda_1 + \Lambda_1^\perp) = \hat{N} \]

\[ \text{index}(\Lambda_1 e^{2\pi i \rho_2 \Lambda_2} \Lambda_1 + \Lambda_1^\perp) \quad (V, it a.e.) \]

- By homotopy, \( P R_2 R \rightarrow \Lambda_2 \rho \Lambda_2 \), truncation of \( P \).
- \( \rho \rightarrow g(CH) \) in spec. gap regime.
- \( \hat{H} \approx \Lambda_2 H \Lambda_2 \)

So,

\[ \text{index}(\Lambda_1 e^{2\pi i \Lambda_2 g(CH) \Lambda_2} \Lambda_1 + \Lambda_1^\perp) \overset{?}{=} \text{index}(\Lambda_1 e^{2\pi i \Lambda_2 g(CH) \Lambda_2} \Lambda_1 + \Lambda_1^\perp) \]

Yes, by \( \Lambda_t := t \Lambda_2 g(CH) \Lambda_2 + (1-t) \Lambda_2 g(CH \Lambda_2) \Lambda_2 \quad t \in \mathbb{C} \).
Main Ingredient for the 2D Homotopies

Fact: \( \exp(2\pi i Q) = 1 \) for Q projection.

Lemma: If \( A^2 - A \) decays into the bulk, \( \exp(2\pi i A) - 1 \) decays into the bulk.

(decay into the bulk means:

\( |B_{xy}\| \) decays in \( x_1, y_2 \).)