Natural Frames and the Design of Interaction Laws in Three Dimensions

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Funding:
- ARO: “Communicating Networked Control Systems”
- NRL: “Motion Planning and Control of Small Agile Formations”
- AFOSR: “Dynamics and Control of Agile Formations”
- NSF/NIH: “Collaborative Research in Computational Neuroscience”

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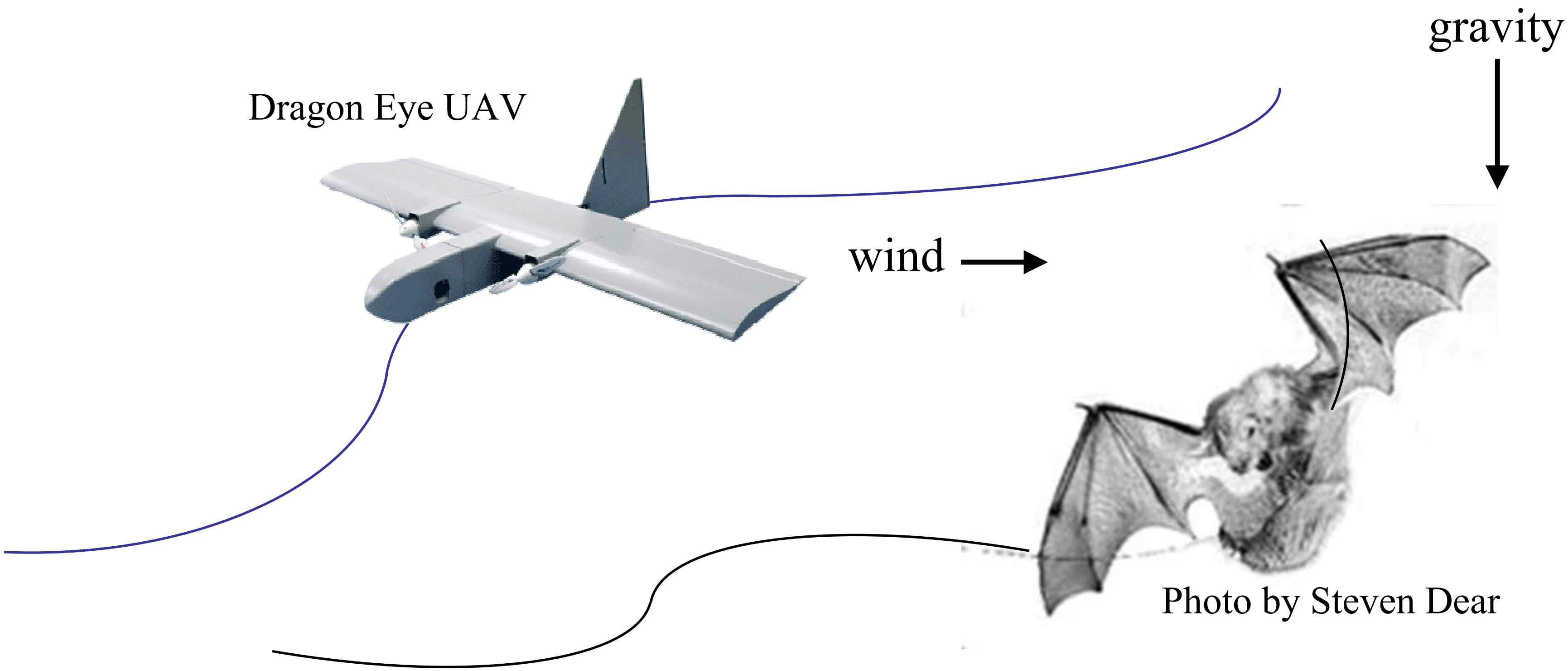
Outline

Message: role of curves and moving frames for vehicle coordination problems in three dimensions and *design* of interaction laws.

- Motivation for unit-speed models
- Curves and moving frames
- Interaction laws and formations
- Boundary following (obstacle avoidance) ← Joint work with Fumin Zhang (Princeton Univ.)
- UAV hardware-in-the-loop testbed (collaboration with NRL)
- Motion camouflage ← See P.S. Krishnaprasad's talk on Friday

Modeling with 3D Trajectories



- Speed is constrained (no stopping) and trajectories are "smooth"
- Want to understand (or design) the trajectories, *which involve sensor feedback*
- An "autopilot" is responsible for trajectory tracking

Framing Trajectories

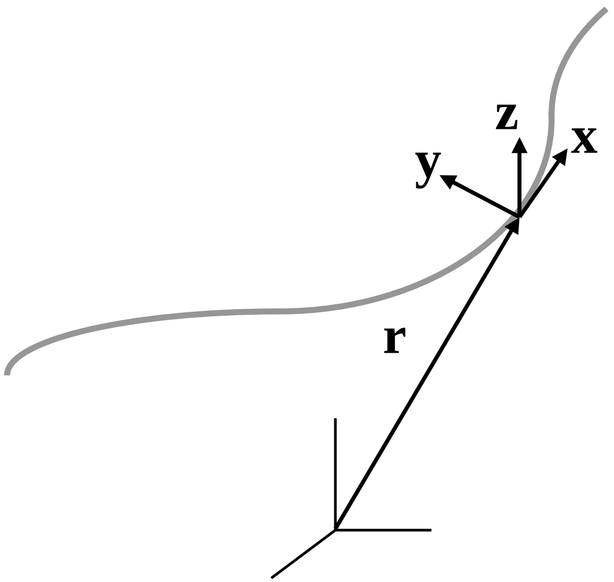
• Natural Frenet Frame equations for arc-length parameterized curve (s = arc-length parameter):

$$\mathbf{r}' = \mathbf{x} \qquad \text{(prime denotes } d/ds\text{)}$$

$$\mathbf{x}' = \mathbf{y}u + \mathbf{z}v$$

$$\mathbf{y}' = -\mathbf{x}u$$

$$\mathbf{z}' = -\mathbf{x}v$$



• $\mathbf{r}$ is the position vector and $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$ is an orthonormal frame (with $\mathbf{x}$ the unit tangent vector to the trajectory).

• The **natural curvatures** are u and v.

• Speed = ν = ds/dt

$$\dot{\mathbf{r}} = \nu\mathbf{x} \qquad \text{(dot denotes } d/dt\text{)}$$

$$\dot{\mathbf{x}} = \nu(\mathbf{y}u + \mathbf{z}v)$$

$$\dot{\mathbf{y}} = -\nu\mathbf{x}u$$

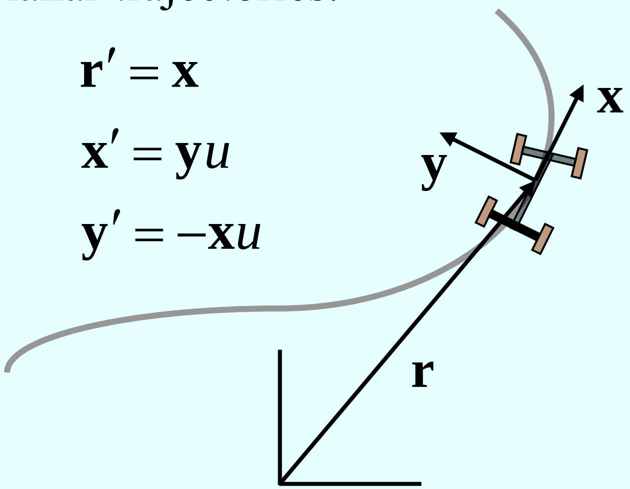
$$\dot{\mathbf{z}} = -\nu\mathbf{x}v$$

Planar trajectories:

$$\mathbf{r}' = \mathbf{x}$$

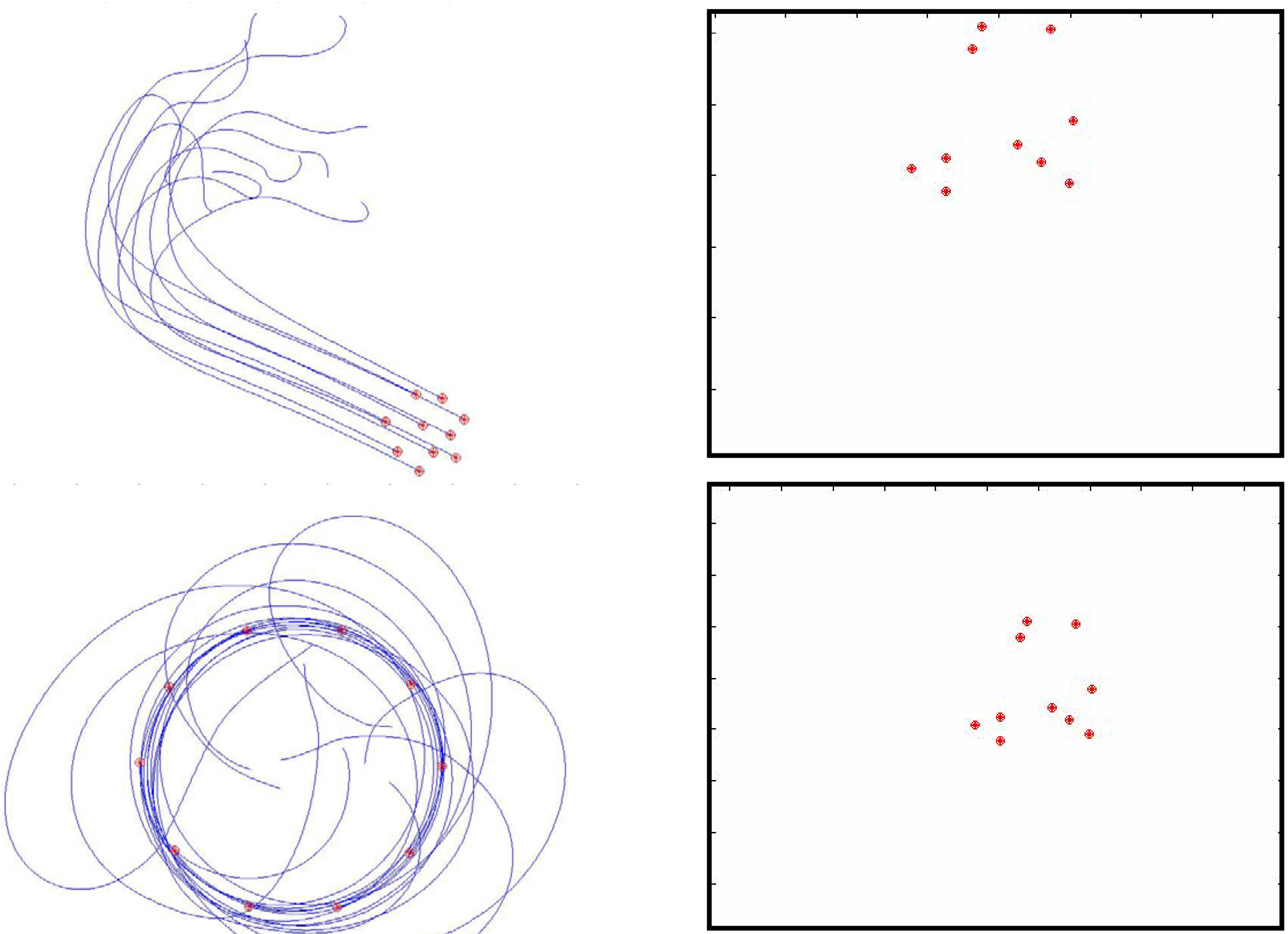
$$\mathbf{x}' = \mathbf{y}u$$

$$\mathbf{y}' = -\mathbf{x}u$$



Planar Unit-Speed Simulations

We take speed $\equiv 1$ for all particles to understand the role of curvature control.



Justh and Krishnaprasad, 2002

Red Arrows

Photos from http://www.sky-flash.com/

Regular Curves and Moving Frames

Frenet-Serret Frame

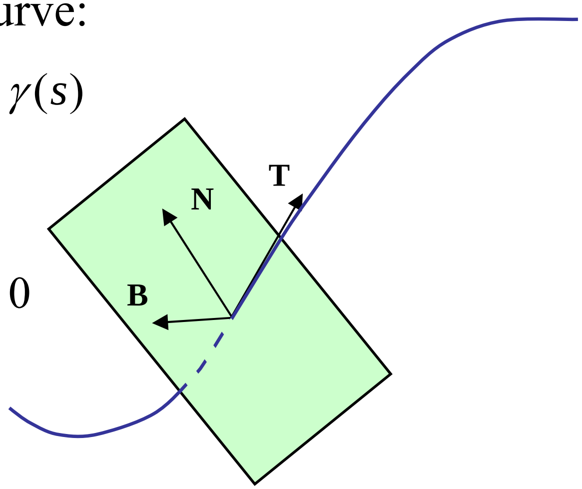
Regular curve:

$$s \mapsto \gamma(s)$$

$$C^3$$

and

$$\gamma'' \neq 0$$



Frenet-Serret equations:

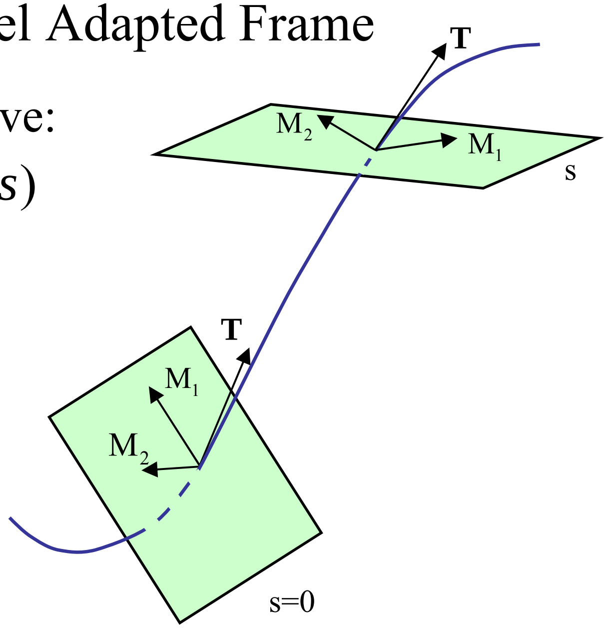
$$\begin{aligned}
\gamma'(s) &= T(s) \\
T'(s) &= \kappa(s)N(s) \\
N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\
B'(s) &= -\tau(s)N(s)
\end{aligned}$$

Natural Frenet or Relatively Parallel Adapted Frame

Regular curve:

$$s \mapsto \gamma(s)$$

$$C^2$$



Equations for relatively parallel adapted frame:

$$\begin{aligned}
\gamma'(s) &= T(s) \\
T'(s) &= k_1(s)M_1(s) + k_2(s)M_2(s) \\
M_1'(s) &= -k_1(s)T(s) \\
M_2'(s) &= -k_2(s)T(s)
\end{aligned}$$

R. L. Bishop, *Amer. Math. Month.* (1975), **82**(3):246-251

Control System on SE(3)

Frenet-Serret frame equations

$$\begin{aligned}
\gamma'(s) &= T(s) \\
T'(s) &= \kappa(s)N(s) \\
N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\
B'(s) &= -\tau(s)N(s)
\end{aligned}$$

Natural Frenet frame equations

$$\begin{aligned}
\gamma'(s) &= T(s) \\
T'(s) &= k_1(s)M_1(s) + k_2(s)M_2(s) \\
M_1'(s) &= -k_1(s)T(s) \\
M_2'(s) &= -k_2(s)T(s)
\end{aligned}$$

$$\frac{d}{ds}\begin{bmatrix} g & \gamma \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} g & \gamma \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \xi & e_1 \\ 0 & 0 \end{bmatrix}$$

where $g = \begin{bmatrix} T & N & B \end{bmatrix}$ or $\begin{bmatrix} T & M_1 & M_2 \end{bmatrix}$

$$\xi = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Inversion

Frenet-Serret:

$$\kappa(s) = \left\| \gamma''(s) \right\|$$

$$\tau(s) = \frac{\gamma'(s) \cdot \left(\gamma''(s) \times \gamma'''(s) \right)}{\left(\kappa(s) \right)^2}$$

Relatively Parallel Adapted Frame:

$$k_i(s) = \gamma''(s) \cdot M_i(0) - \int_0^s k_i(\sigma) \gamma''(s) \cdot \gamma'(\sigma) \; d\sigma, \quad i = 1, 2$$

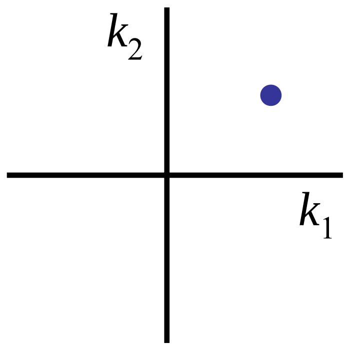
Relationship between (κ, τ) and (k_1, k_2):

$$\kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)}$$

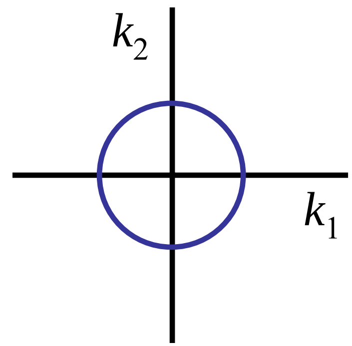
$$\theta'(s) = \tau$$

(where $\theta = \arg(\mathbf{k})$)

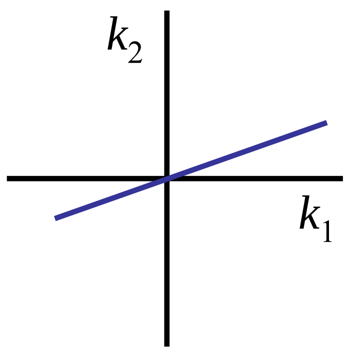
Normal Development of γ



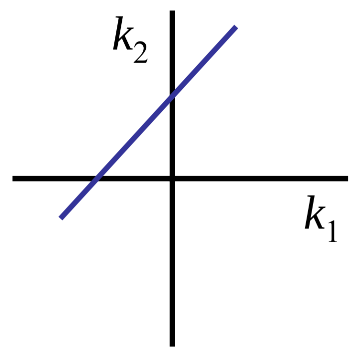
Trajectory is a circle



If traced out at constant speed, trajectory is a helix

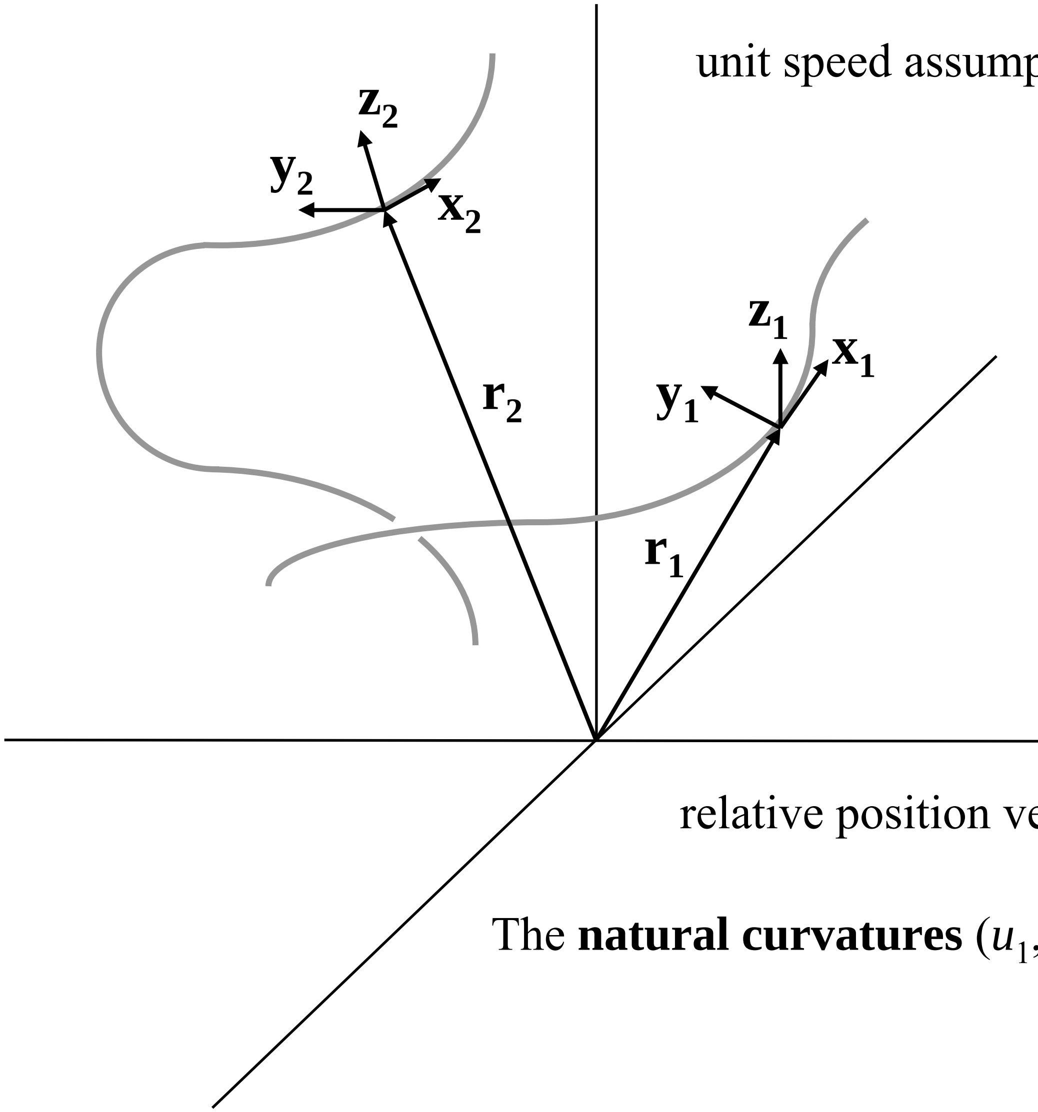


Trajectory is planar



Trajectory lies on a sphere

Two Vehicles Interacting



unit speed assumption

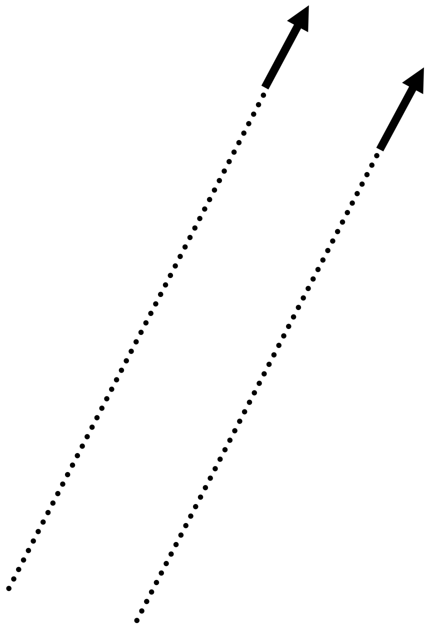
$$\dot{\mathbf{r}}_1 = \mathbf{x}_1$$
$$\dot{\mathbf{x}}_1 = \mathbf{y}_1 u_1 + \mathbf{z}_1 v_1$$
$$\dot{\mathbf{y}}_1 = -\mathbf{x}_1 u_1$$
$$\dot{\mathbf{z}}_1 = -\mathbf{x}_1 v_1$$

$$\dot{\mathbf{r}}_2 = \mathbf{x}_2$$
$$\dot{\mathbf{x}}_2 = \mathbf{y}_2 u_2 + \mathbf{z}_2 v_2$$
$$\dot{\mathbf{y}}_2 = -\mathbf{x}_2 u_2$$
$$\dot{\mathbf{z}}_2 = -\mathbf{x}_2 v_2$$

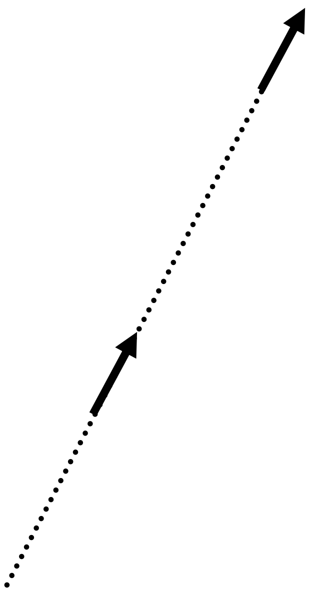
relative position vector $\longrightarrow$ $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$

The **natural curvatures** (u_1,v_1) and (u_2,v_2) are controls.

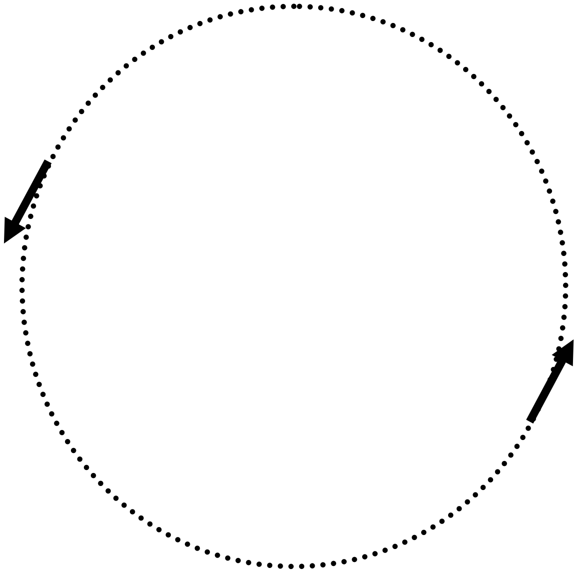
Two-Vehicle Formations



Rectilinear formation (motion perpendicular to the baseline)



Collinear formation



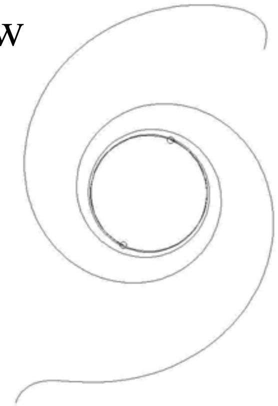
Circling formation (vehicle separation equals the diameter of the orbit)

Two-Vehicle Lyapunov Functions

Rectilinear Law

$$V_{rect} = -\ln(1 + \mathbf{x}_2 \cdot \mathbf{x}_1) + h(|\mathbf{r}|)$$

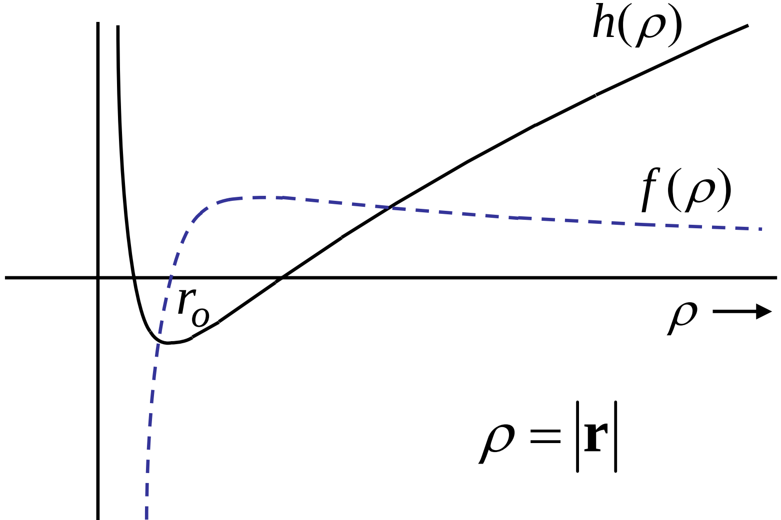
Circling Law



$$V_{circ} = -\ln\left[1 - \mathbf{x}_2 \cdot \mathbf{x}_1 + 2\left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_1\right)\right] + h(|\mathbf{r}|)$$

relative heading term

relative distance term



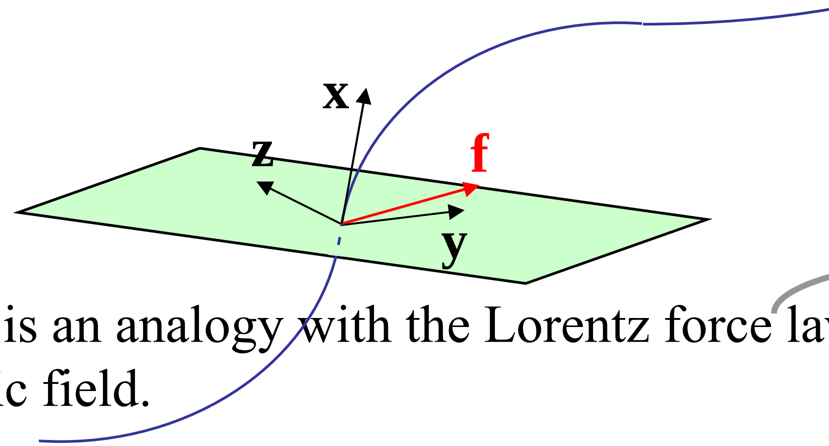
$$\rho = |\mathbf{r}|$$

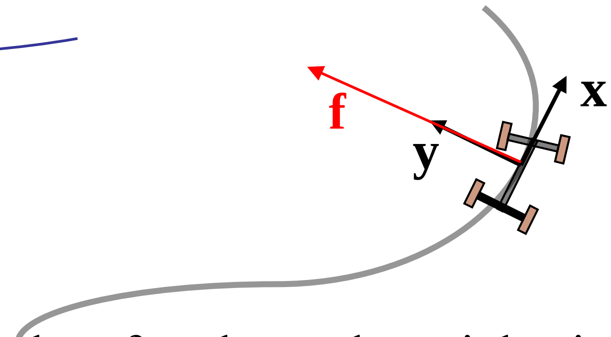
Gyroscopically Interacting Particles

• Note: V is **not** to be thought of as a synthetic potential (commonly used in robotics for directing motion toward a target or away from obstacles).

• V is a Lyapunov function for the **shape dynamics**.

• The kinetic energy of **each particle** is conserved (because they interact via **gyroscopic forces**), and initial conditions are such that they all move at unit speed.





• There is an analogy with the Lorentz force law for charged particles in a magnetic field.

• In mechanics, gyroscopic forces are associated with **vector** potentials.

• References:
- L.-S. Wang and P.S. Krishnaprasad, *J. Nonlin. Sci.*, 1992.
- J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Springer, 1999, (2nd edition)

Lyapunov Function Derivative

- Lyapunov function: $V_{rect} = -\ln(1 + \mathbf{x}_2 \cdot \mathbf{x}_1) + h(|\mathbf{r}|)$

- Derivative of V_{rect} along trajectories:

$$\dot{V}_{rect} = -\frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_1 + \mathbf{x}_2 \cdot \dot{\mathbf{x}}_1}{1 + \mathbf{x}_2 \cdot \mathbf{x}_1} + f(|\mathbf{r}|)\frac{d}{dt}|\mathbf{r}|$$

$$= -\frac{(\mathbf{x}_1 \cdot \mathbf{y}_2)u_2 + (\mathbf{x}_2 \cdot \mathbf{y}_1)u_1 + (\mathbf{x}_1 \cdot \mathbf{z}_2)v_2 + (\mathbf{x}_2 \cdot \mathbf{z}_1)v_1}{1 + \mathbf{x}_2 \cdot \mathbf{x}_1} + f(|\mathbf{r}|)\left[\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \left(\mathbf{x}_2 - \mathbf{x}_1\right)\right]$$

1

$$= -\frac{1}{1 + \mathbf{x}_2 \cdot \mathbf{x}_1}[(\mathbf{x}_1 \cdot \mathbf{y}_2)F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) + (\mathbf{x}_2 \cdot \mathbf{y}_1)F(-\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2)$$

2

$$+ (\mathbf{x}_1 \cdot \mathbf{z}_2)F(\mathbf{r}, \mathbf{x}_2, \mathbf{z}_2, \mathbf{x}_1) + (\mathbf{x}_2 \cdot \mathbf{z}_1)F(-\mathbf{r}, \mathbf{x}_1, \mathbf{z}_1, \mathbf{x}_2)]$$

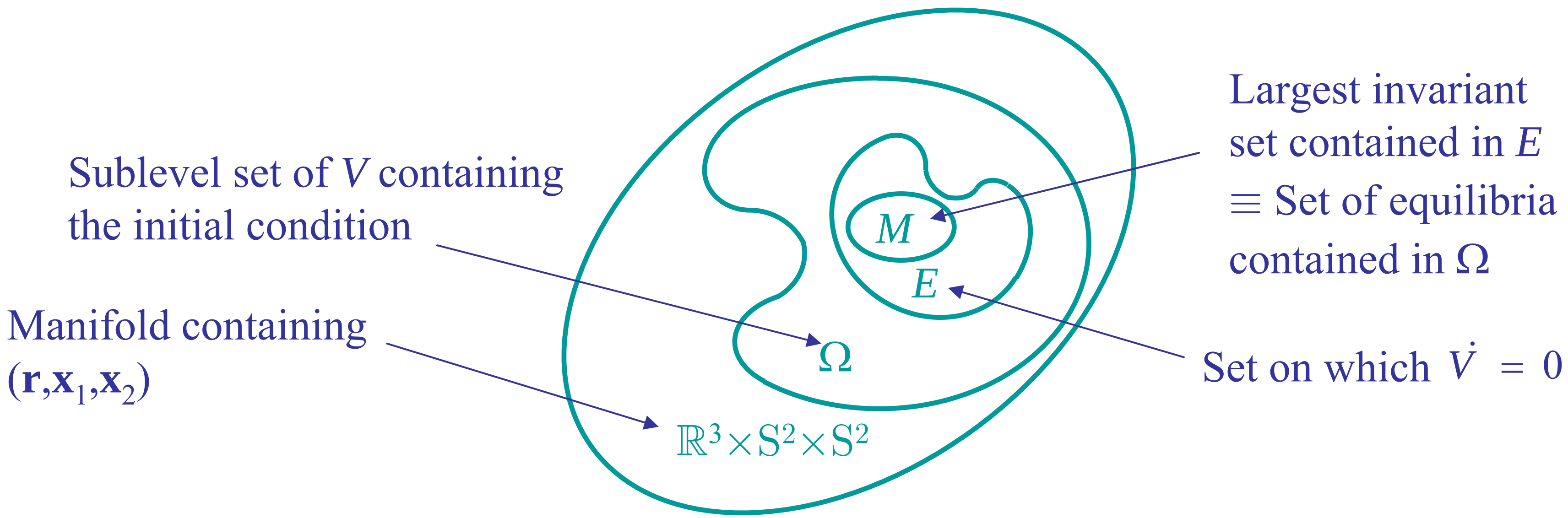
$$\leq 0$$

Find a control law which cancels the term $f(|\mathbf{r}|)\left[\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \left(\mathbf{x}_2 - \mathbf{x}_1\right)\right]$

and ensures $\dot{V}_{rect} \leq 0$

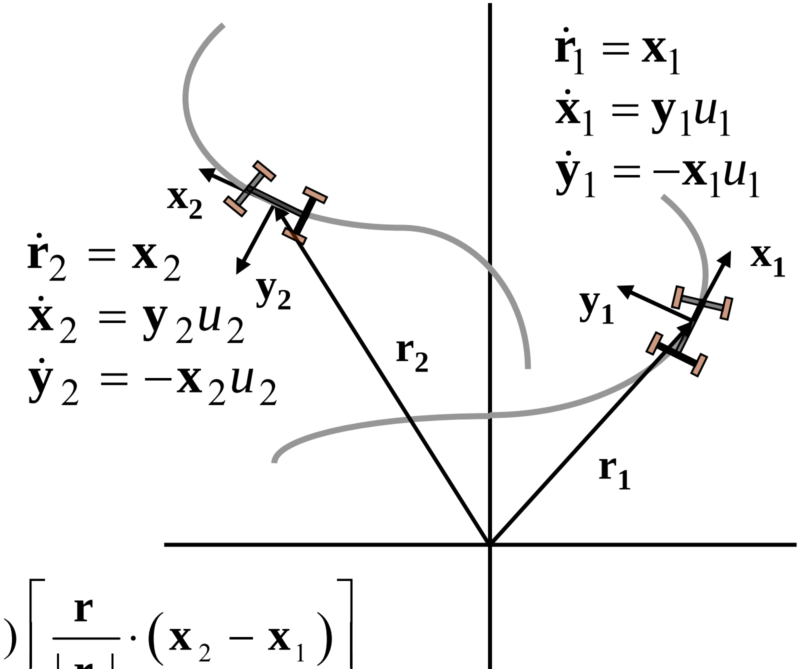
Outline of Convergence Proof

- LaSalle's Invariance Principle used to prove convergence of the $(\mathbf{r},\mathbf{x}_1,\mathbf{x}_2)$-dynamics.



- Linear analysis used to assess stability of particular equilibria.

Systematically Deriving a 3D Law



- Restrict to the planar setting.

$$V_{rect} = -\ln(1 + \mathbf{x}_2 \cdot \mathbf{x}_1) + h(|\mathbf{r}|)$$

(V_{rect} has the same form in 2D as in 3D)

- Find a suitable planar control law.

$$\dot{V}_{rect} = -\frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_1 + \mathbf{x}_2 \cdot \dot{\mathbf{x}}_1}{1 + \mathbf{x}_2 \cdot \mathbf{x}_1} + f(|\mathbf{r}|)\frac{d}{dt}|\mathbf{r}|$$

$$= -\frac{(\mathbf{x}_1 \cdot \mathbf{y}_2)u_2 + (\mathbf{x}_2 \cdot \mathbf{y}_1)u_1}{1 + \mathbf{x}_2 \cdot \mathbf{x}_1} + f(|\mathbf{r}|)\left[\frac{\mathbf{r}}{|\mathbf{r}|} \cdot (\mathbf{x}_2 - \mathbf{x}_1)\right]$$

$$= -\frac{1}{1 + \mathbf{x}_2 \cdot \mathbf{x}_1}[(\mathbf{x}_1 \cdot \mathbf{y}_2)F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) + (\mathbf{x}_2 \cdot \mathbf{y}_1)F(-\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2)]$$

$$\leq 0$$

- Return to the 3D setting, and determine how to generalize the planar law so that the calculation of $\dot{V}_{rect}$ works out analogously to the planar setting.
- Interpret the resulting 3D control law (to understand why it takes the form derived).

Planar Law and Interpretation

(Justh and Krishnaprasad, 2002)

Steering controls:

$$u_1 = F(-\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2) - f(|\mathbf{r}|)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right)$$

$$u_2 = F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right)$$

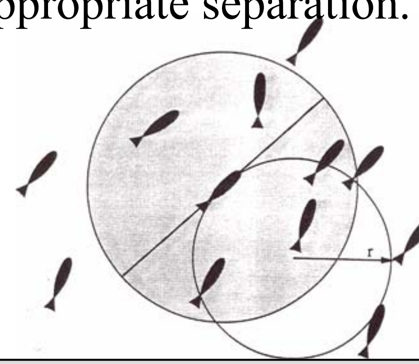
$$= -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) + \mu\mathbf{x}_1\cdot\mathbf{y}_2 - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right)$$

Align each vehicle perpendicular to the baseline between the vehicles.

Align with the other vehicle's heading.

Steer toward or away from the other vehicle to maintain appropriate separation.

- Biological analogy (swarming, schooling):
  - Decreasing responsiveness at large separation distances.
  - Switch from attraction to repulsion based on separation distance or density.
  - Mechanism for alignment of headings.

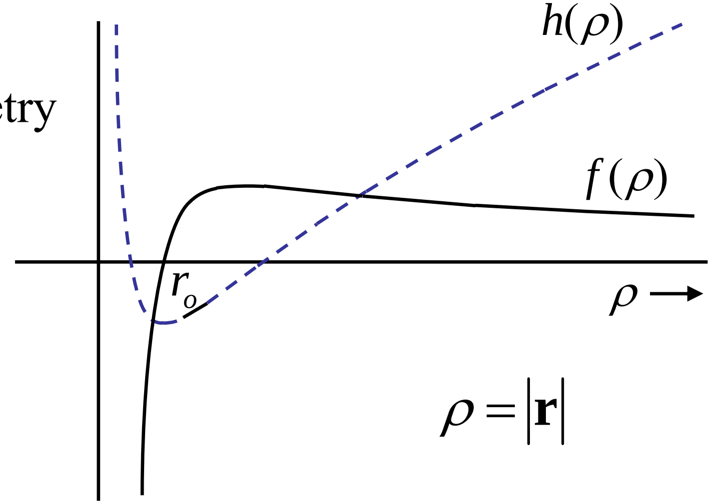


D. Grünbaum, "Schooling as a strategy for taxis in a noisy environment," in *Animal Groups in Three Dimensions*, J.K. Parrish and W.M. Hamner, eds., Cambridge University Press, 1997.

Control Laws for 3D

- Symmetries:
  - Depend only on *shape* quantities: SE(3) symmetry
  - Their *effect* depends only on $\mathbf{r}$, $\mathbf{x}_1$, and $\mathbf{x}_2$
  - Both vehicles "run the same algorithm"

$$u_2 = F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) - \underbrace{f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{y}_2\right)}_{\text{Collision avoidance}}$$



$$\rho = |\mathbf{r}|$$

- Rectilinear Law:

$$F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) = \underbrace{-\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{y}_2\right)}_{\text{Baseline alignment}} + \underbrace{\mu(\mathbf{x}_1 \cdot \mathbf{y}_2)}_{\text{Heading alignment}}$$

$$\mu > \tfrac{1}{2}\eta > 0$$

- Circling Law:

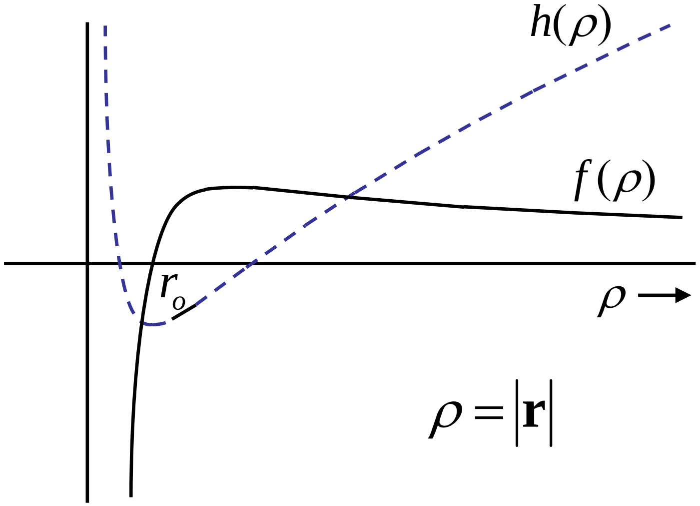
$$F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{y}_2\right) + \mu\left[-\mathbf{x}_1 \cdot \mathbf{y}_2 + 2\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{x}_1\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{y}_2\right)\right]$$

- Frame evolution:

$$\begin{aligned} \dot{\mathbf{r}}_1 &= \mathbf{x}_1 & \dot{\mathbf{r}}_2 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_1 &= \mathbf{y}_1 u_1 + \mathbf{z}_1 v_1 & \dot{\mathbf{x}}_2 &= \mathbf{y}_2 u_2 + \mathbf{z}_2 v_2 \\ \dot{\mathbf{y}}_1 &= -\mathbf{x}_1 u_1 & \dot{\mathbf{y}}_2 &= -\mathbf{x}_2 u_2 \\ \dot{\mathbf{z}}_1 &= -\mathbf{x}_1 v_1 & \dot{\mathbf{z}}_2 &= -\mathbf{x}_2 v_2 \end{aligned}$$

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

Rectilinear Control Law for 3D



- Natural curvatures for first particle:

$$u_1 = F(-\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2) - f(|\mathbf{r}|)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{y}_1\right)$$

$$v_1 = F(-\mathbf{r}, \mathbf{x}_1, \mathbf{z}_1, \mathbf{x}_2) - f(|\mathbf{r}|)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{z}_1\right)$$

- Natural curvatures for second particle:

$$u_2 = F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{y}_2\right)$$

$$v_2 = F(\mathbf{r}, \mathbf{x}_2, \mathbf{z}_2, \mathbf{x}_1) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{z}_2\right)$$

- Alignment term:

$$F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot \mathbf{y}_2\right) + \mu\left(\mathbf{x}_1 \cdot \mathbf{y}_2\right) \qquad \mu > \tfrac{1}{2}\eta > 0$$

- Frame evolution:

$$\begin{aligned} \dot{\mathbf{r}}_1 &= \mathbf{x}_1 & \dot{\mathbf{r}}_2 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_1 &= \mathbf{y}_1 u_1 + \mathbf{z}_1 v_1 & \dot{\mathbf{x}}_2 &= \mathbf{y}_2 u_2 + \mathbf{z}_2 v_2 \\ \dot{\mathbf{y}}_1 &= -\mathbf{x}_1 u_1 & \dot{\mathbf{y}}_2 &= -\mathbf{x}_2 u_2 \\ \dot{\mathbf{z}}_1 &= -\mathbf{x}_1 v_1 & \dot{\mathbf{z}}_2 &= -\mathbf{x}_2 v_2 \end{aligned}$$

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

Interpretation of Rectilinear Law

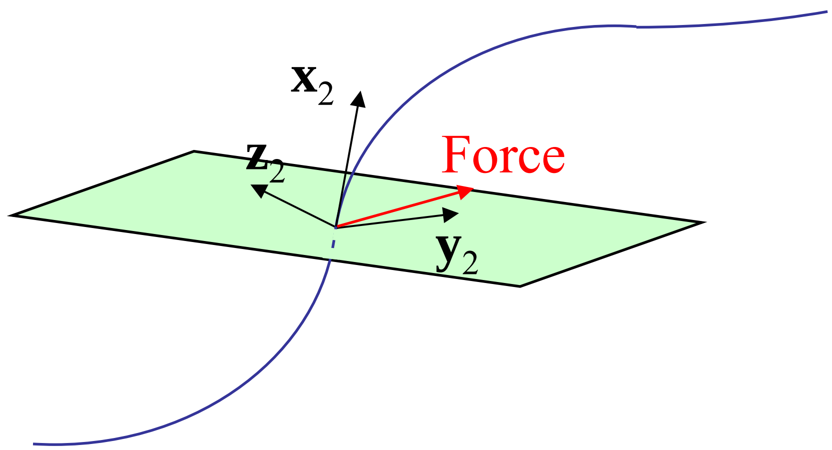
- For particle #2, can express the control law as:

$$\mathbf{a}(\mathbf{r}, \mathbf{x}_2, \mathbf{x}_1) = -\eta \left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_2 \right) \frac{\mathbf{r}}{|\mathbf{r}|} + \mu \mathbf{x}_1 - f(|\mathbf{r}|) \frac{\mathbf{r}}{|\mathbf{r}|}$$

$$u_2 = \mathbf{a} \cdot \mathbf{y}_2$$

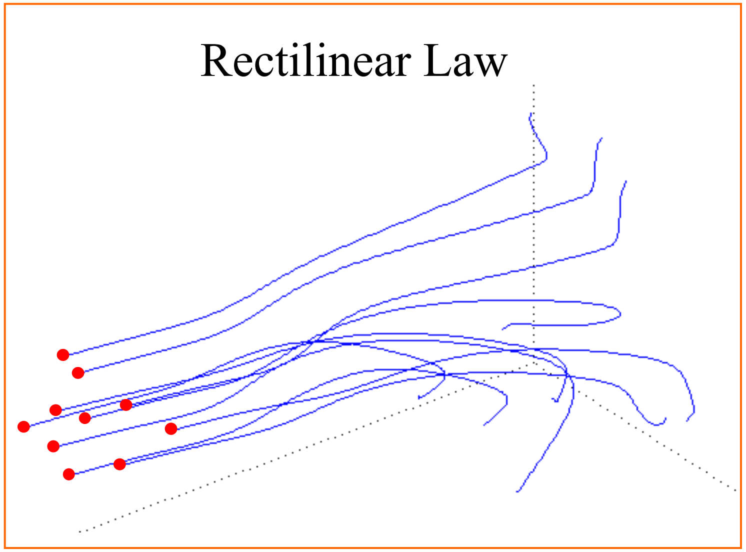
$$v_2 = \mathbf{a} \cdot \mathbf{z}_2$$

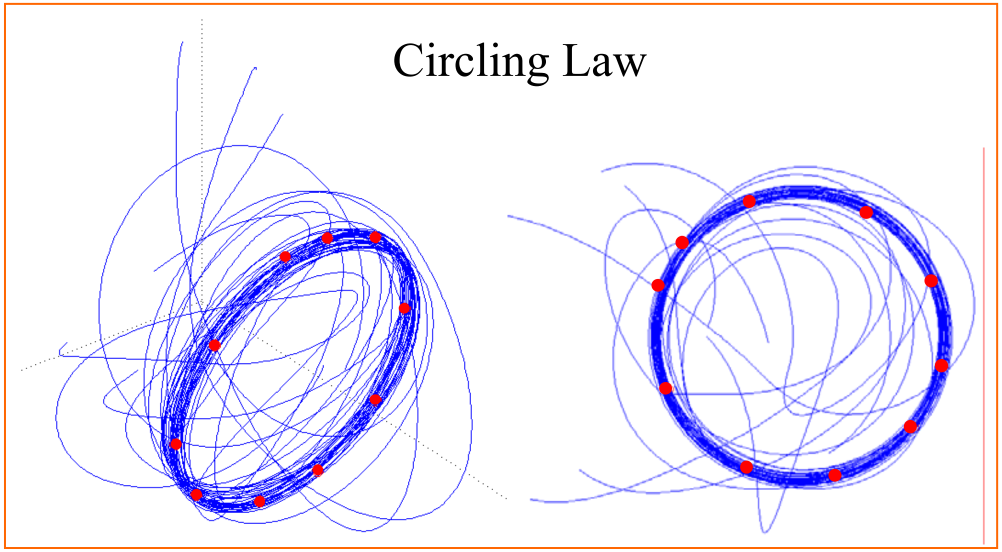
- Force = projection of **a** onto the normal plane (to $\mathbf{x}_2$).



- Similarly for particle #1.

Generalization to *n* vehicles





- One copy of the natural Frenet frame equations for each vehicle:

$$\dot{\mathbf{r}}_j = \mathbf{x}_j$$
$$\dot{\mathbf{x}}_j = \mathbf{y}_j u_j + \mathbf{z}_j v_j \qquad j = 1, 2, \ldots, n$$
$$\dot{\mathbf{y}}_j = -\mathbf{x}_j u_j$$
$$\dot{\mathbf{z}}_j = -\mathbf{x}_j v_j$$

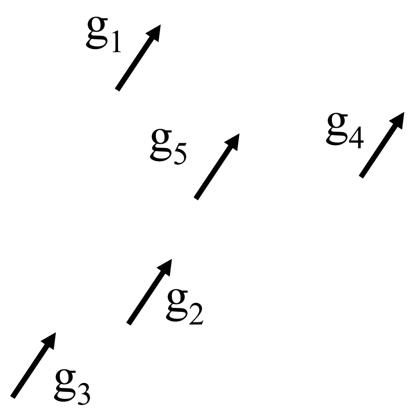
- Average of pair-wise interactions used in the two-vehicle law:

$$u_j = \frac{1}{n}\sum_{k \neq j}\left[F(\mathbf{r}_j - \mathbf{r}_k, \mathbf{x}_j, \mathbf{y}_j, \mathbf{x}_k) - f(|\mathbf{r}_j - \mathbf{r}_k|)\left(\frac{\mathbf{r}_j - \mathbf{r}_k}{|\mathbf{r}_j - \mathbf{r}_k|}\cdot \mathbf{y}_j\right)\right]$$

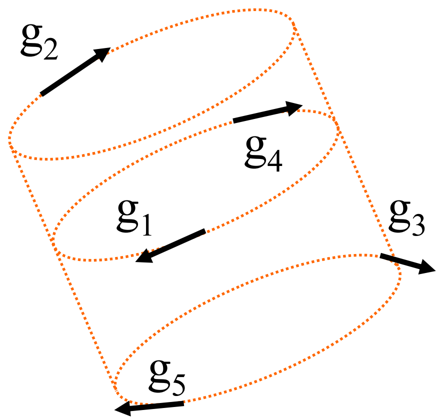
- Convergence **conjectured**: we are led to consider a new class of *n*-body problems.

3-D Equilibrium Shapes (Formations)

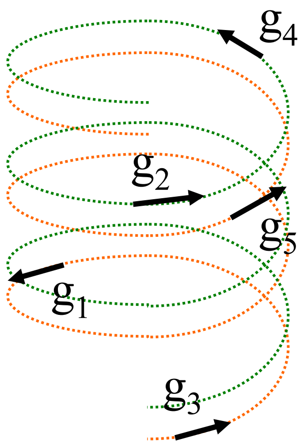
- Control laws are assumed to be invariant under rigid motions.
- Shape variables capture relative distances and angles between vehicles.
- Shape equilibria correspond to steady-state formations.
- Three possibilities for particles moving at unit speed:



rectilinear formation



circling formation



helical formation

Lie Group Setting

- Represent each vehicle trajectory as a function on the Lie group SE(3) of rigid motions:

$$g_1 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & \mathbf{r}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} \mathbf{x}_2 & \mathbf{y}_2 & \mathbf{z}_2 & \mathbf{r}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3)$$

- Define the shape variable:

$$g = g_1^{-1} g_2 = \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{x}_2 & \mathbf{x}_1 \cdot \mathbf{y}_2 & \mathbf{x}_1 \cdot \mathbf{z}_2 & (\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{x}_1 \\ \mathbf{y}_1 \cdot \mathbf{x}_2 & \mathbf{y}_1 \cdot \mathbf{y}_2 & \mathbf{y}_1 \cdot \mathbf{z}_2 & (\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{y}_1 \\ \mathbf{z}_1 \cdot \mathbf{x}_2 & \mathbf{z}_1 \cdot \mathbf{y}_2 & \mathbf{z}_1 \cdot \mathbf{z}_2 & (\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{z}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Left-invariant dynamics on SE(3):

$$\dot{g}_1 = g_1 \begin{bmatrix} 0 & -u_1 & -v_1 & 1 \\ u_1 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} u_1 &= u_1(g) \\ v_1 &= v_1(g) \end{aligned}$$

$$\dot{g}_2 = g_2 \begin{bmatrix} 0 & -u_2 & -v_2 & 1 \\ u_2 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} u_2 &= u_2\left(g^{-1}\right) \\ v_2 &= v_2\left(g^{-1}\right) \end{aligned}$$

Lie Group Setting (cont.)

- Lyapunov function:

$$V_{rect} = -\ln(1 + g_{11}) + h(r)$$

- Control law:

$$u_1 = -\eta\left(\frac{g_{14}g_{24}}{r^2}\right) + \mu g_{21} + f(r)\left(\frac{g_{24}}{r}\right)$$

$$v_1 = -\eta\left(\frac{g_{14}g_{34}}{r^2}\right) + \mu g_{31} + f(r)\left(\frac{g_{34}}{r}\right)$$

$$u_2 = -\eta\left(\frac{g^{14}g^{24}}{r^2}\right) + \mu g^{21} + f(r)\left(\frac{g^{24}}{r}\right)$$

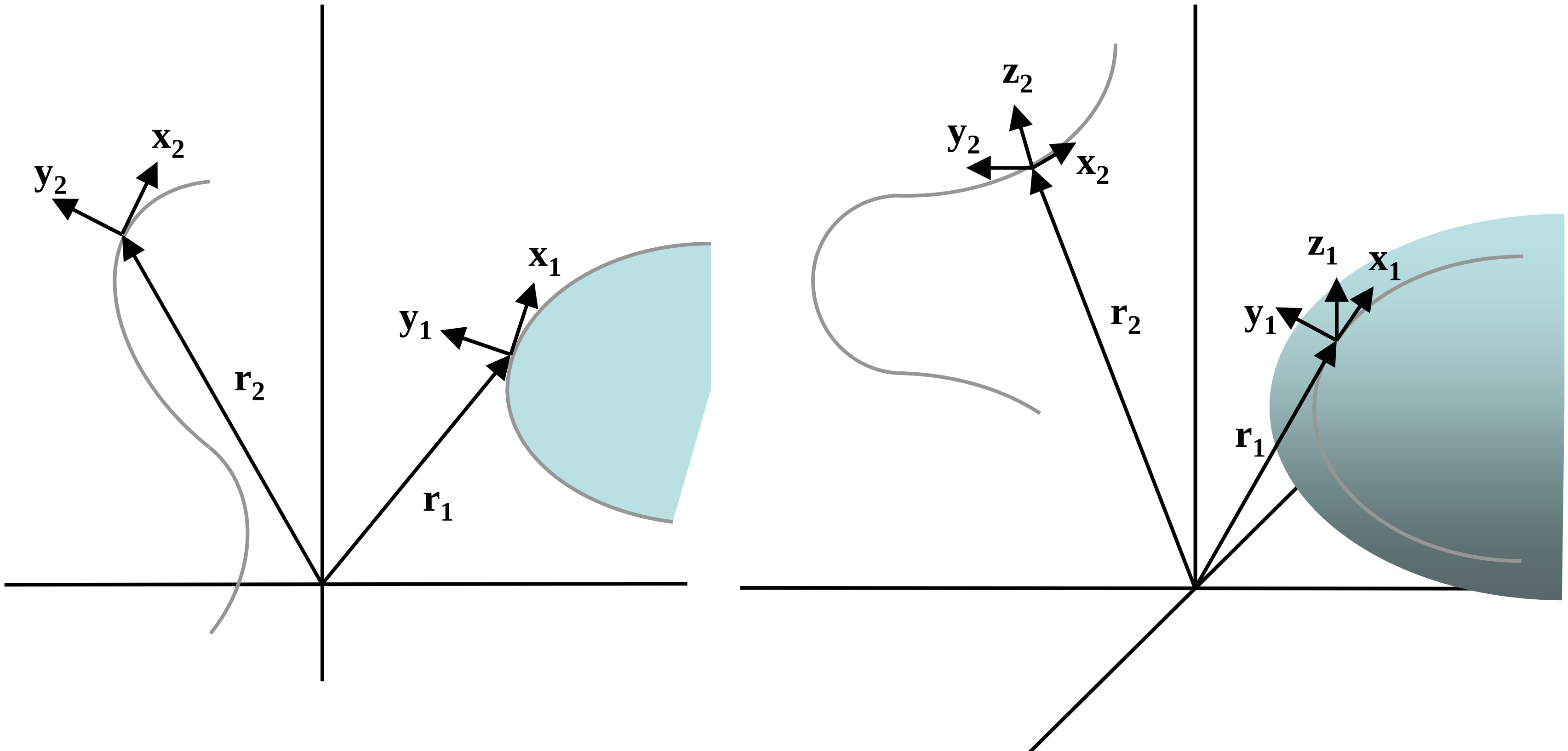
$$v_2 = -\eta\left(\frac{g^{14}g^{34}}{r^2}\right) + \mu g^{31} + f(r)\left(\frac{g^{34}}{r}\right)$$

$$g = \left[g_{ij}\right], \quad g^{-1} = \left[g^{ij}\right], \quad r = \sqrt{g_{14}^2 + g_{24}^2 + g_{34}^2}$$

Obstacle Avoidance

• Idea: control inputs for the moving vehicle are determined by the trajectory of the closest point on the obstacle surface.

• Applications: obstacle avoidance and boundary following with non-collision.



F. Zhang, E.W. Justh, and P.S. Krishnaprasad, CDC 2004.

Control Law for Boundary Following

Lyapunov function: $V_{\text{obst}} = \underbrace{-\ln\left(\mathbf{x}_2 \cdot \mathbf{x}_1\right)} + \underbrace{h(|\,\mathbf{r}\,|)}$

Favor motion parallel to the boundary curve

Penalize distances from the boundary curve which are too large or small

2-D law: $u_2 = \underbrace{\mu\left(\mathbf{x}_1 \cdot \mathbf{y}_2\right)} - \underbrace{f(|\,\mathbf{r}\,|)\left(\frac{\mathbf{r}}{|\,\mathbf{r}\,|} \cdot \mathbf{y}_2\right)} + \underbrace{\left(\frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{1 \pm |\kappa_1||\mathbf{r}|}\right)\kappa_1}$

Align parallel to the boundary curve.

Steer toward or away from the boundary curve to maintain appropriate separation.

Respond to the nonzero curvature of the boundary curve.

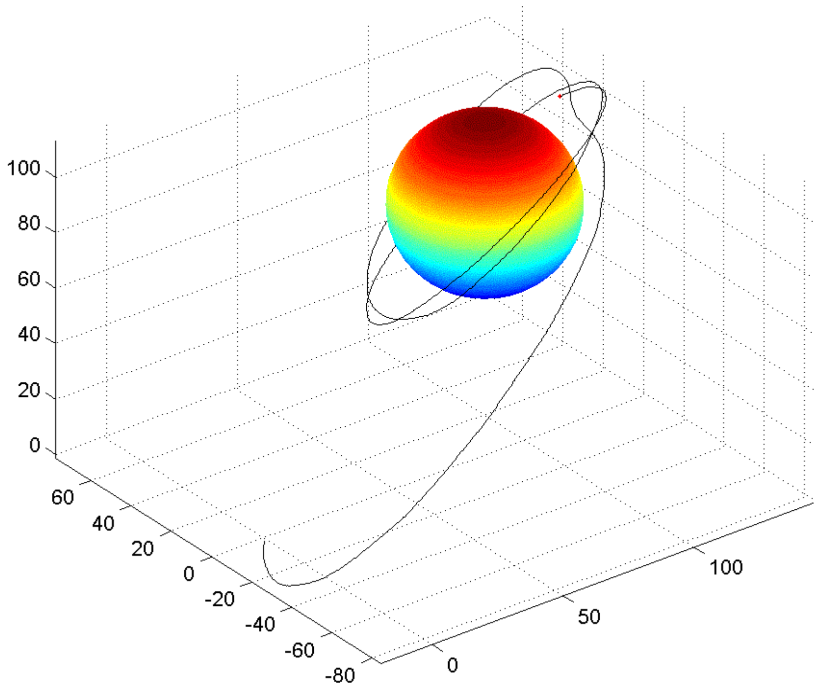
3-D law:

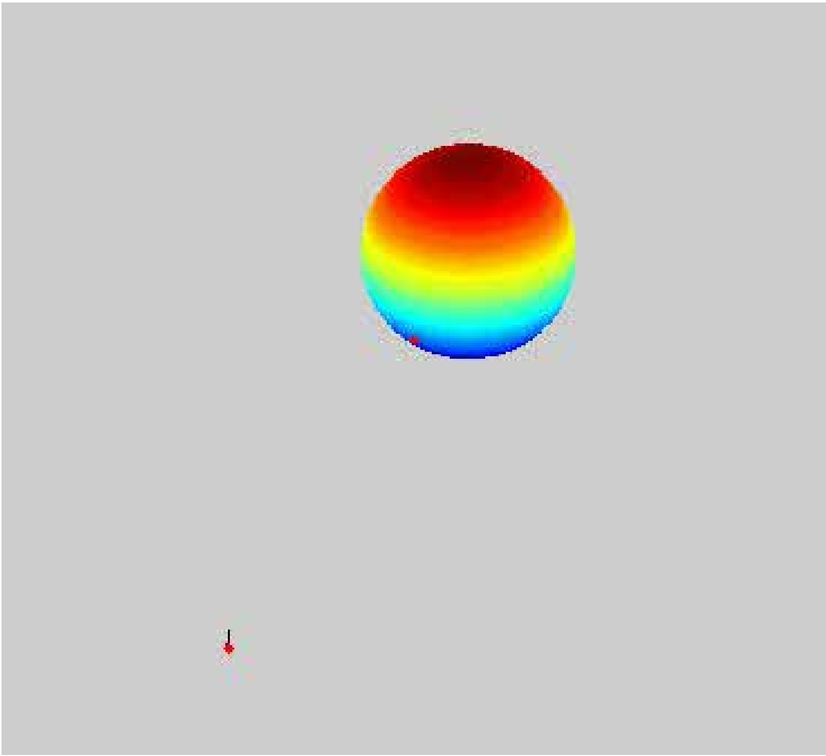
$$u_2 = \mu\left(\mathbf{x}_1 \cdot \mathbf{y}_2\right) - f(|\,\mathbf{r}\,|)\left(\frac{\mathbf{r}}{|\,\mathbf{r}\,|} \cdot \mathbf{y}_2\right) + \text{term involving } (u_1, v_1)$$

$$v_2 = \mu\left(\mathbf{x}_1 \cdot \mathbf{z}_2\right) - f(|\,\mathbf{r}\,|)\left(\frac{\mathbf{r}}{|\,\mathbf{r}\,|} \cdot \mathbf{z}_2\right) + \text{term involving } (u_1, v_1)$$

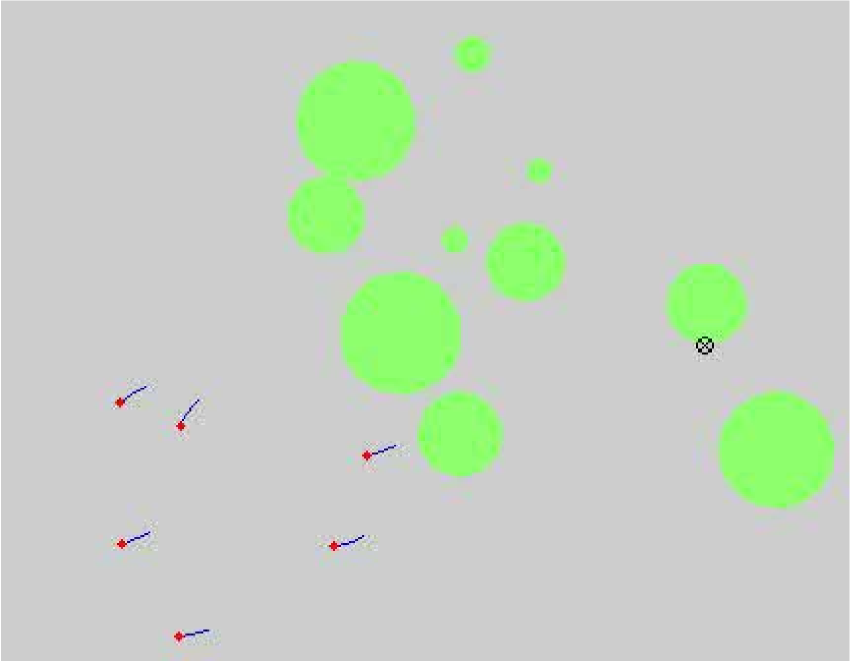
F. Zhang, E.W. Justh, and P.S. Krishnaprasad, CDC 2004.

Boundary-Following Simulation

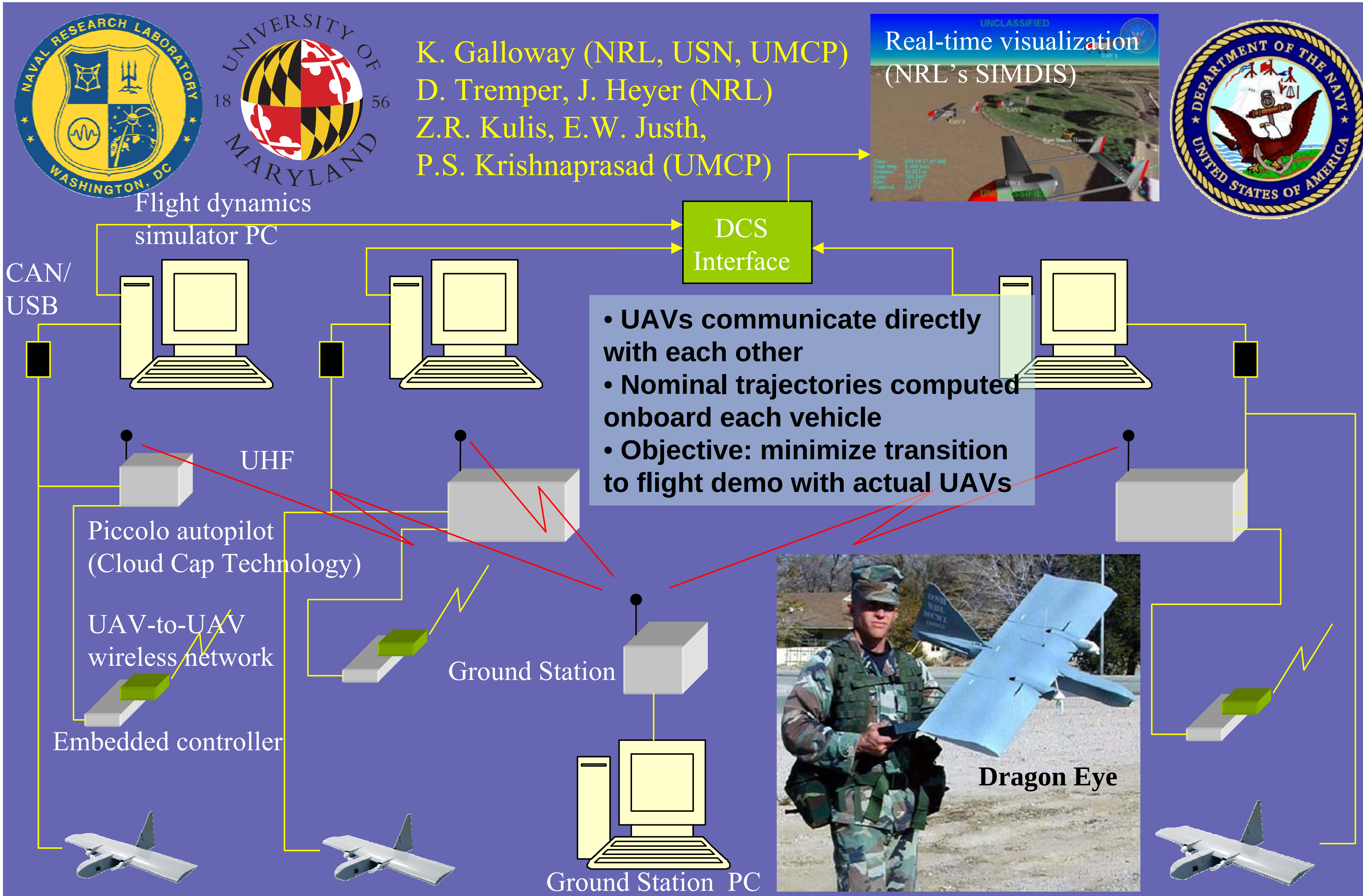




Obstacle-Avoidance Simulation

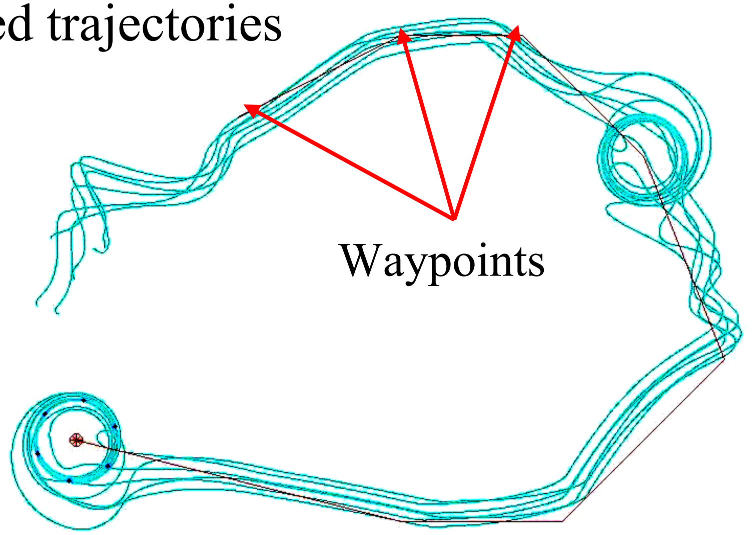


UAV Hardware-in-the-Loop Testbed

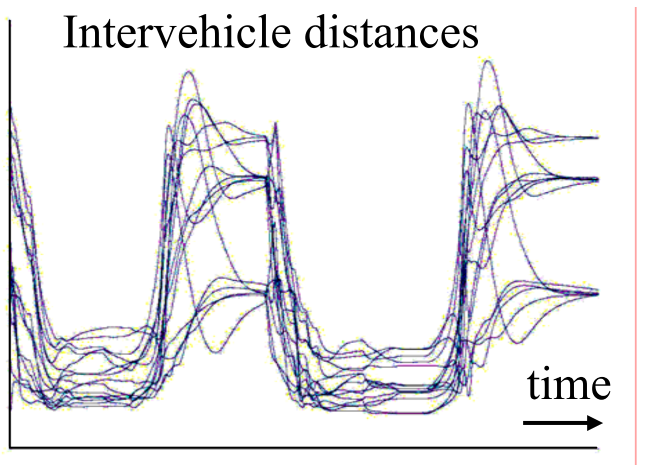


Performance Criteria

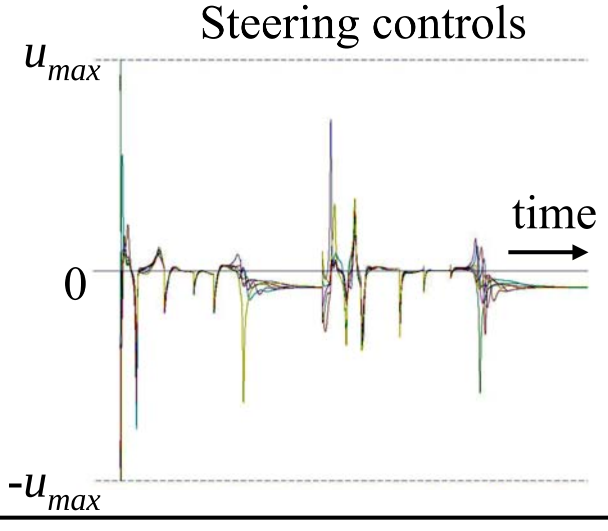
• Faithful following of waypoint-specified trajectories

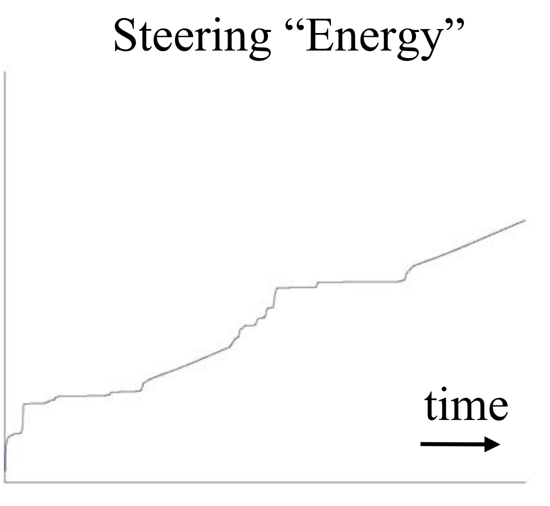


• Sufficient separation between vehicles (to avoid collisions)



• Minimize steering: for UAVs, turning requires considerably more energy than straight, level flight. Maneuverability is also limited.





Time Discretization (Planar System)

$$\dot{\mathbf{r}}_j = \mathbf{x}_j$$
$$\dot{\mathbf{x}}_j = \mathbf{y}_j u_j$$
$$\dot{\mathbf{y}}_j = -\mathbf{x}_j u_j$$

- Compute $u_1(t_m)$, $u_2(t_m)$, ..., $u_n(t_m)$, where t_m=mT for m=1, 2, ..., and let

$$u_j(t) = u_j(t_m), \quad \forall t \in [t_m, t_{m+1}).$$

- Piecewise constant controls allow the vehicle positions to be computed using simple formulas:

$$\begin{bmatrix} \mathbf{x}_j(t_{m+1}) & \mathbf{y}_j(t_{m+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_j(t_m) & \mathbf{y}_j(t_m) \end{bmatrix} \begin{bmatrix} \cos\left(u_j(t_m)T\right) & -\sin\left(u_j(t_m)T\right) \\ \sin\left(u_j(t_m)T\right) & \cos\left(u_j(t_m)T\right) \end{bmatrix}$$

$$\mathbf{r}_j(t_{m+1}) = \frac{1}{u_j(t_m)} \begin{bmatrix} \mathbf{x}_j(t_m) & \mathbf{y}_j(t_m) \end{bmatrix} \begin{bmatrix} \sin\left(u_j(t_m)T\right) \\ 1-\cos\left(u_j(t_m)T\right) \end{bmatrix} + \mathbf{r}_j(t_m)$$

$$\theta_j(t_{m+1}) = \theta_j(t_m) + u_j(t_m)T$$

$$\begin{bmatrix} \mathbf{x}_j(t_{m+1}) & \mathbf{y}_j(t_{m+1}) \end{bmatrix} = \begin{bmatrix} \cos\theta_j(t_{m+1}) & -\sin\theta_j(t_{m+1}) \\ \sin\theta_j(t_{m+1}) & \cos\theta_j(t_{m+1}) \end{bmatrix}$$

$$\mathbf{r}_j(t_{m+1}) = T \begin{bmatrix} \mathbf{x}_j(t_m) & \mathbf{y}_j(t_m) \end{bmatrix} \begin{bmatrix} \operatorname{sinc}\left(u_j(t_m)T\right) \\ \mathrm{q}\left(u_j(t_m)T\right) \end{bmatrix} + \mathbf{r}_j(t_m)$$

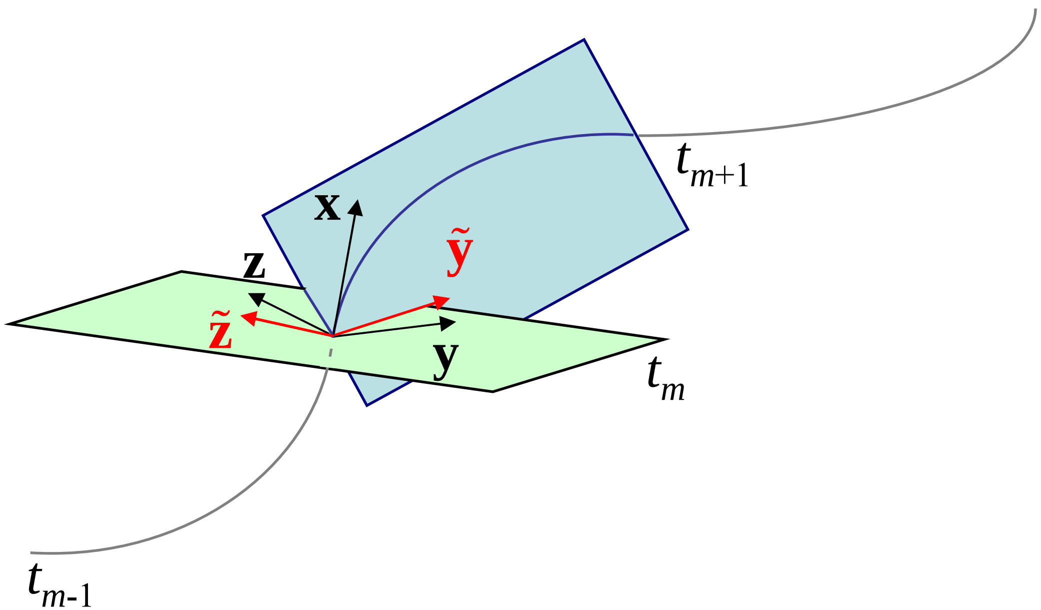
$$\mathrm{q}(\tau) = \frac{1\text{-}\cos(\tau)}{\tau} \approx \frac{\tau}{2!} - \frac{\tau^3}{4!} + \cdots$$ (use, e.g., a sinusoidal approximation)

Time Discretization (3D System)

• Piecewise constant controls: $t_m = mT$ for $m=1, 2, ...,$ and

$$u_j(t) = u_j(t_m), \quad v_j(t) = v_j(t_m), \quad \forall t \in [t_m, t_{m+1}).$$

• Corresponding to $u_j(t_m)$ and $v_j(t_m)$, $\exists(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$ such that $(\mathbf{x}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}})$ is orthonormal, and the trajectory for $t \in [t_m, t_{m+1})$ lies strictly in the $\mathbf{x}$-$\tilde{\mathbf{y}}$ plane.

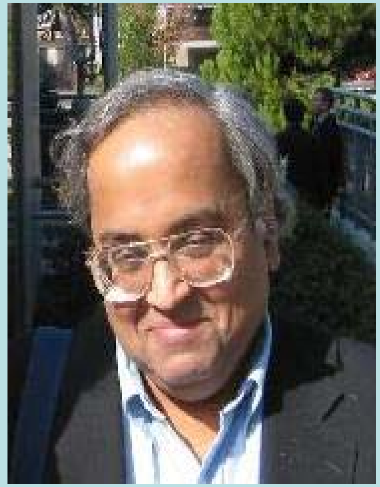


• Use the planar constant-steering-control formulas (in the $\mathbf{x}$-$\tilde{\mathbf{y}}$ plane).

Motion Camouflage: Bat Data

Echolocating FM bat,
Eptesicus fuscus

Kaushik Ghose



P.S. Krishnaprasad

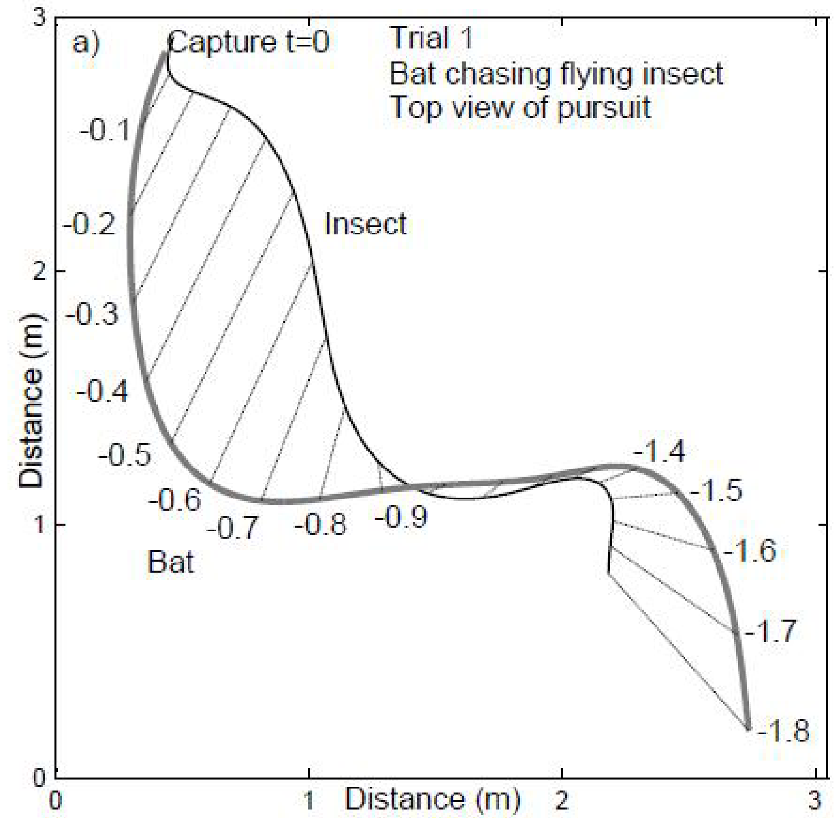
Cynthia Moss

Timothy K. Horiuchi

Neuroscience and Cognitive
Science Program (NACS)

http://www.bsos.umd.edu/psyc/batlab/index.html

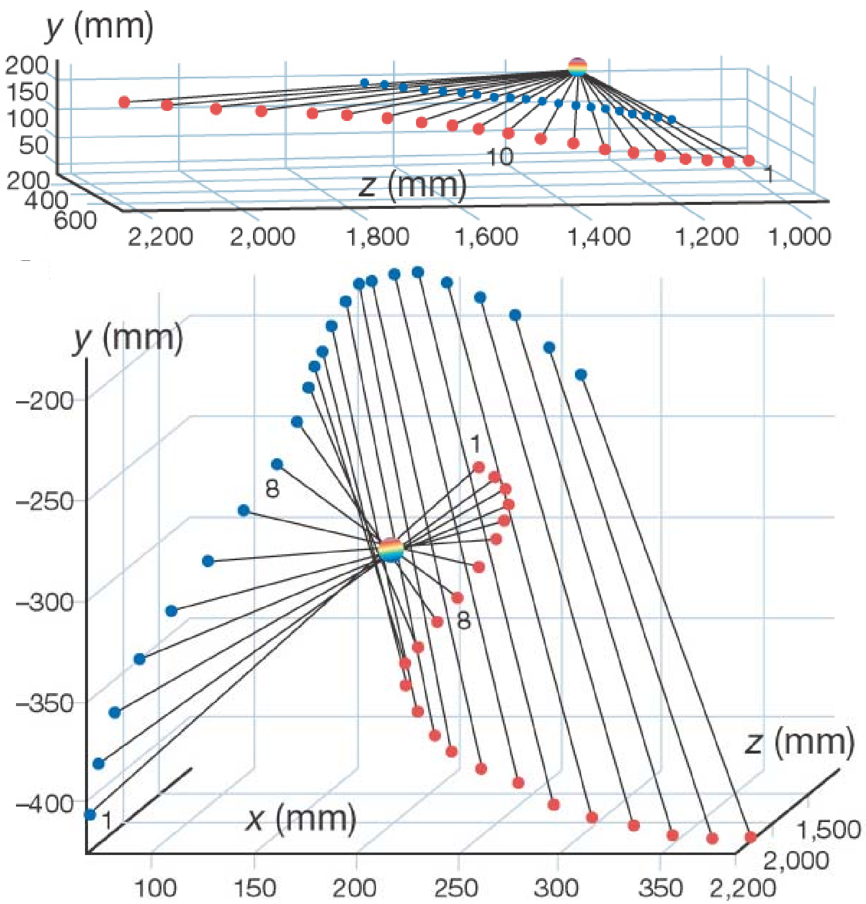
Target is preying mantis,
Parasphendale agrionina.
Hearing organ blocked by vaseline.



K. Ghose, T.K. Horiuchi, P.S. Krishnaprasad, and C. Moss, Preprint, 2005

Motion Camouflage: Dragonfly Data

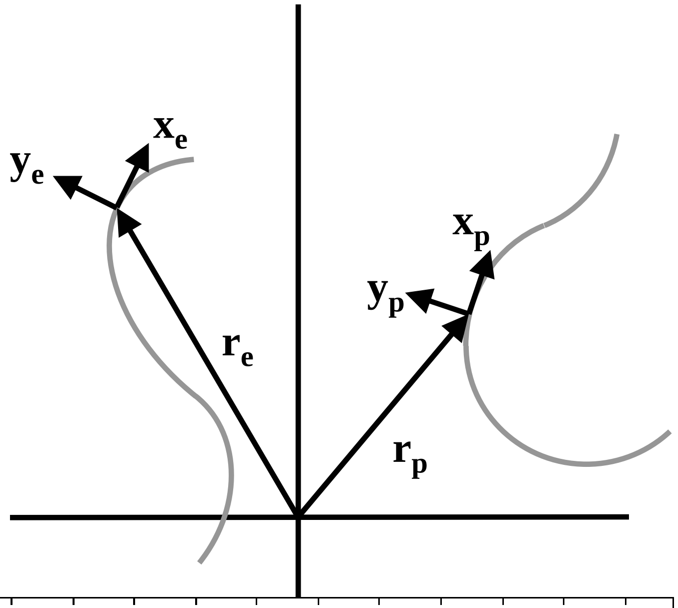
3-D reconstruction of territorial interaction of two male dragonflies *Hemianax papuensis*.
Shadower – blue; Shadowee – red



Thanks to Mandyam V. Srinivasan (Australian National University) for inspiration and discussions

From A.K. Mizutani, J.S.Chahl, and M.V. Srinivasan, “Motion camouflage in dragonflies,”*Nature*, vol. 423, p. 604, 2003. (with permission)

Steering Law for Motion Camouflage



Frame equations

$$
\begin{aligned}
\dot{\mathbf{r}}_p &= \mathbf{x}_p \\
\dot{\mathbf{x}}_p &= \mathbf{y}_p u_p \\
\dot{\mathbf{y}}_p &= -\mathbf{x}_p u_p \\
\dot{\mathbf{r}}_e &= \nu \mathbf{x}_e \\
\dot{\mathbf{x}}_e &= \nu \mathbf{y}_e u_e \\
\dot{\mathbf{y}}_e &= -\nu \mathbf{x}_e u_e
\end{aligned}
$$

$$0 < \nu < 1$$

Motion camouflage condition:

$$\mathbf{r} \triangleq \mathbf{r}_p - \mathbf{r}_e = \lambda \mathbf{r}_\infty$$

- $\mathbf{r}_\infty$ is a fixed unit vector
- λ is a time-dependent scalar

Equivalent infinitesimal condition (we assume $|\mathbf{r}| \neq 0$):

$$\dot{\mathbf{r}} - \left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}} \right) \frac{\mathbf{r}}{|\mathbf{r}|} = 0$$

Transverse component of relative velocity = 0.

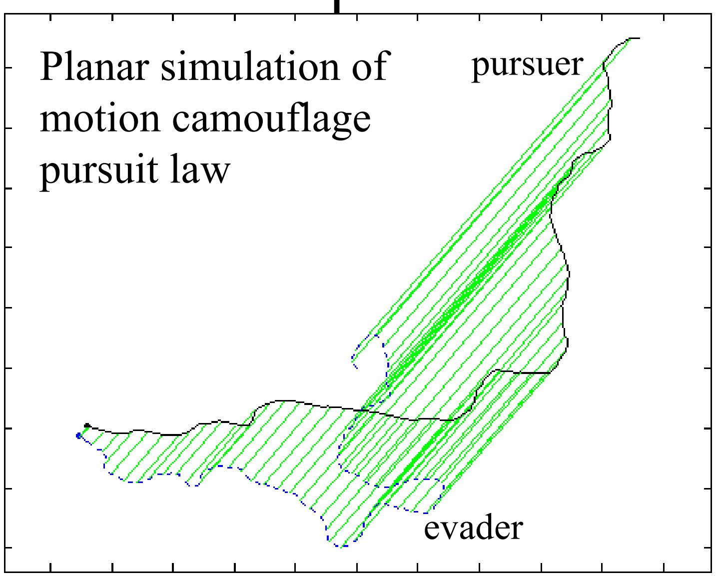
Distance function:

$$\Gamma = \frac{\frac{d}{dt}|\mathbf{r}|}{|\dot{\mathbf{r}}|} = \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$$

Drive Γ to -1 for motion camouflaged pursuit.

Pursuer control law:

$$u_p = -\mu \left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}}^{\perp} \right)$$



E.W. Justh and P.S. Krishnaprasad, “Steering laws for motion camouflage,” arXiv:math.OC/0508023, 2005.

Formations vs. Motion Camouflage

Formation Control	Motion Camouflage
Cooperation	Conflict
Mechanism for collision avoidance	Objective is intercept (i.e., collision)
Study asymptotic convergence to a relative equilibrium (a.k.a, shape equilibrium, formation)	Finite time problem: Study accessibility of the state of motion camouflage: drive cost function Γ to $\Gamma \leq -1 + \varepsilon$, for $\varepsilon > 0$ arbitrarily small.
3-D modeling with curves and moving frames – specifically, natural Frenet frames	

References

1. E.W. Justh and P.S. Krishnaprasad, "Natural frames and interacting particles in three dimensions," *Proc. IEEE Conf. Decision and Control*, pp. 2841-2846, 2005 (arXiv:math.OC/0503390v1).
2. E.W. Justh and P.S. Krishnaprasad, "Steering laws for motion camouflage," preprint, 2005 (arXiv:math.OC/0508023).
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4. F. Zhang, E.W. Justh, and P.S. Krishnaprasad, "Boundary following using gyroscopic control," *Proc. IEEE Conf. Decision and Control*, pp. 5204-5209, 2004.
5. E.W. Justh and P.S. Krishnaprasad, "Steering laws and continuum models for planar formations," *Proc. IEEE Conf. Decision and Control*, pp. 3609-3614, 2003.
6. E.W. Justh and P.S. Krishnaprasad, "A simple control law for UAV formation flying," Institute for Systems Research Technical Report TR 2002-38, 2002 (see http://www.isr.umd.edu).

See also http://www.isr.umd.edu/~justh