SPT order and algebraic topology

Xiao-Gang Wen, MIT/PI, IPAM, Jan. 26, 2015
Two kinds of quantum theories: H-type and L-type

- **H-type:**
  - Hilbert space with tensor structure $\mathcal{V} = \bigotimes_i \mathcal{V}_i$
  - A local Hamiltonian operator acting on $\mathcal{V}$
  - Space-time path integral well defined on mapping torus $M_{\text{space}}^{d-1} \rtimes S^1$.

- **L-type:**
  - Tensors for each cell in space-time lattice defined path integral
  - Space-time path integral well defined on any space-time manifold $M_d^\text{space-time}$.

- Lead to different classification of topo. orders and SPT orders
  - H-type models $\rightarrow E_8$ bosonic quantum Hall state
  - L-type models $\rightarrow (E_8)^3$ bosonic quantum Hall state
L-type local bosonic quantum systems

- A L-type local quantum system in $d$-dimensional space-time $M^d$ is described by (use $d = 3$ as an example):
  - a triangulation of $M^3$ with a branching structure,
  - a set of indices $\{v_i\}$ on vertices, $\{e_{ij}\}$ on edges, $\{\phi_{ijk}\}$ on triangles.
  - two real and one complex tensors $T_3 = \{W_{v_0}, A^e_{v_0v_1} (SO_{ij}), C^{\pm e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}_{v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}} (SO_{ij})\}$ for each $0, 1, 3$-cell (vertex, edge, tetrahedron).

- **Unitary condition**
  
  $W_{v_0} > 0, \quad A^e_{v_0v_1} > 0, \quad C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}_{v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}} = (C^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}_{v_0v_1v_2v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}})^*$

[Kong-Wen 14]
Partition function and Correlation function

- **Partition function**:

$$Z = \sum_{v_0, \cdots; e_{01}, \cdots; \phi_{012}, \cdots} \prod_{\text{vertex}} W_{v_0} \prod_{\text{edge}} A_{v_0 v_1}^{e_{01}} \prod_{\text{tetra}} C_{s_{0123} v_0 v_1 v_2 v_3; \phi_{123} \phi_{013} \phi_{023} \phi_{012}}$$

$$\prod T_3$$

where $s_{0123} = \pm$ depends on the orientation of the tetrahedron.

- **Correlation function** on closed space-time manifold – physically measurable quantities:

Modify tensor on a few simplices gives us a new partition function $T_3 \rightarrow \tilde{T}_3$: $Z(M^d) \rightarrow Z[\tilde{T}_3(x), \tilde{T}_3(y), \cdots; M^d]$: 

$$\langle \tilde{T}_3(x) \tilde{T}_3(y) \rangle = \frac{Z[\tilde{T}_3(x), \tilde{T}_3(y); M^d]}{Z(M^d)}$$
Short-range correlated (SRC) system and liquid

- A **infinite-system** is not a single system but a sequence of systems
  - with size of space-time $M^3 \to \infty$ and size of simplices $\sim 1$.
  - each vertex is shared by at most a finite number of simplices.

- A **short-range correlated** (SRC) infinite-system satisfies
  
  $\langle \tilde{T}_3(x) \tilde{T}_3(y) \rangle - \langle \tilde{T}_3(x) \rangle \langle \tilde{T}_3(y) \rangle \sim e^{\frac{|x-y|}{\xi}}$

  for a fixed $\xi$ in the infinite system-size limit;

  (2) systems of different sizes in the sequence can deform into each other and keep the SRC property during the deformation.

  [Zeng-Wen 14]  \[ \rightarrow \text{SRC liquid} \]
Short-range correlated (SRC) liquid phases

- An equivalence relation:
  Two SRC infinite-systems are equivalent if the two sequences can deform into each other while keeping the SRC property during the deformation.

- The resulting equivalent classes are **SRC liquid phases** or **L-type topological orders**
  [Wen 89]
  [Chen-Gu-Wen 10]

How to classify SRC liquid phases (L-type topological orders) in each dimension?
(1) The indices admit a symmetry action $v_i \rightarrow g \cdot v_i$, $e_{ij} \rightarrow g \cdot e_{ij}$, $\phi_{ijk} \rightarrow g \cdot \phi_{ijk}$, where $g \in G$.

(2) $T_{G,3} = \{ W_{v_0}, A^{e_{01}}_{v_0 v_1} (SO_{ij}), C_{v_0 v_1 v_2 v_3; e_{02} e_{13} e_{23}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}} (SO_{ij}) \}$ are invariant under the above symmetry action.

- SRC liquid phases with symmetry:
  
  **Equivalence relation**: Two symmetric SRC infinite-systems can deform into each other while keeping the symmetry property and the SRC property during the deformation.

**How to classify SRC liquid phases with symmetry** → **L-type symmetry enriched topological (SET) order**
Examples of SRC liquids

- **Example 1:** Tensors: non-zero only when all the indices are 1
  \[ W_1 = w_0, \ A_{11}^1 = w_1, \ C(\{ v_i = 1, e_{ij} = 1, \phi_{ijk} = 1 \}) = w_3, \]
  All degrees of freedom are frozen to 1.
  \[ \rightarrow \quad Z(M^3) = \prod_{n=0}^N (w_n)^{N_n} \]
  - The above systems with different \( w_n \)'s all belong to the same SRC phase [have a trivial topological order (triTO)].

- **Example 2:** Tensors: non-zero only when all the edge-indices, face indices, etc are 1; the vertex-indices \( v_i \in G \)
  \[ W_{v_i} = w_0, \ A_{v_i v_j}^1 = w_1 \ C(\{ v_i, e_{ij} = 1, \phi_{ijk} = 1 \}) = w_3, \]
  \[ \rightarrow \quad Z(M^3) = |G|^{N_0} \prod_{n=0}^N (w_n)^{N_n} \]
  - The above systems have a symmetry \( G \)

- **A conjecture:** A system with \( Z(M^d) = 1 \) for all closed orientable space-time \( M^d \) has a trivial topological order.
  *Both of the above examples have a trivial topological order.*

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SPT order and algebraic topology
A general picture for SRC phases

- **Non trivial TO w/o symm.** → many phases \([\text{Wen 89}]\)
- **Trivial TO w/o symm.** → one phase (no symm. breaking)
- **Non trivial TO with symm.** → many phases \([\text{Wen 02}]\)
- **Trivial TO with symm.** → many different phases \([\text{Gu-Wen 09}]\)
  
  may be called **symmetry protected trivial** (SPT) phase
  or **symmetry protected topological** (SPT) phase

- **SPT phases** = equivalent class of **symm.** smooth deformation
- **Examples:** 1D Haldane phase [Haldane 83] 2D/3D TI [Kane-Mele 05; Bernevig-Zhang 06] [Moore-Balents 07; Fu-Kane-Mele 07]
SPT phases are SRC (gapped) quantum phases with a certain symmetry, which can be smoothly connected to the same trivial phase if we remove the symmetry $\rightarrow$ trivial topo order.

- **A group cohomology theory of SPT phases:** Using each element in $\mathcal{H}^d[G, U(1)]$ (the $d$ group cohomology class of the group $G$ with $U(1)$ as coefficient), we can construct a exactly soluble path integral in $d$-dimensional space-time, which realize a SPT state with a symmetry $G$.

[Chen-Gu-Liu-Wen 11]
SPT phases – the trivial TO with symmetry

SPT phases are SRC (gapped) quantum phases with a certain symmetry, which can be smoothly connected to the same trivial phase if we remove the symmetry → trivial topo order.

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  [Chen-Gu-Liu-Wen 11]

- How to get the above result? Construct a path integral with symmetry $G$ and $Z_{\text{top}}(M^d) = 1$ on any closed orientable $M^d$. 
L-type SPT state on 1+1D space-time lattice

- Path integral on 1+1D space-time lattice described by tensors

\[ T_2 = \{ W_{v_0}, B_{\pm v_0 v_1 v_2}^{e_01 e_12 e_02} (SO_{ij}) \}_{e_{ij}=1} = \{ W_{v_0} = |G|^{-1}, [\nu_2(v_0, v_1, v_2)]^{1,*} \} \]

\[ e^{-S} = \prod \nu_2^{s_{ijk}}(g_i, g_j, g_k), \quad Z = |G|^{-N_0} \sum e^{-S}, \quad v_i \to g_i, \; g_i \in G \]

where \( \nu^{s_{ijk}}(g_i, g_j, g_k) = e^{-\int_L} \) and \( s_{ijk} = 1,* \)

- The above defines a LN\( \sigma \)M with target space \( G \) on 1+1D space-time lattice.

- The NL\( \sigma \)M will have a symmetry \( G \) if \( g_i \in G \) and

\[ \nu_2(g_i, g_j, g_k) = \nu_2(h g_i, h g_j, h g_k), \; h \in G \]

- We will get a SPT state if we choose \( \nu_2(g_i, g_j, g_h) = 1 \to Z_{\text{top}}(M^2) = 1 \) (which is a trivial SPT state).

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• \(\nu(g_i, g_j, g_k)\) give rise to a topological \(\text{NL}\sigma\text{M}\) if 
\[e^{-S_{\text{fixed}}} = \prod \nu^{s_{ijk}}(g_i, g_j, g_k) = 1\]
on any sphere, including a tetrahedron (simplest sphere).
• \(\nu(g_i, g_j, g_k) \in U(1)\)
• On a tetrahedron \(\rightarrow 2\)-cocycle condition

\[\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_2) = 1\]

The solutions of the above equation are called \textbf{group cocycle}.
• The 2-cocycle condition has many solutions:
\(\nu_2(g_0, g_1, g_2)\) and 
\(\tilde{\nu}_2(g_0, g_1, g_2) = \nu_2(g_0, g_1, g_2)\frac{\beta_1(g_1, g_2)\beta_1(g_0, g_1)}{\beta_1(g_0, g_2)}\) are both cocycles. We say \(\nu_2 \sim \tilde{\nu}_2\) (equivalent).
• The set of the equivalent classes of \(\nu_2\) is denoted as
\[H^2[G, U(1)] = \pi_0(\text{space of the solutions})\]
• \(H^2[G, U(1)]\) describes 1+1D SPT phases protected by \(G\).
Topological invariance in topological $\sigma$M

As we change the space-time lattice, the action amplitude $e^{-S}$ does not change:

$$\nu_2(g_0, g_1, g_2)\nu_2^{-1}(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)$$

$$\nu_2(g_0, g_1, g_2)\nu_2^{-1}(g_1, g_2, g_3)\nu_2(g_0, g_2, g_3) = \nu_2(g_0, g_1, g_3)$$

as implied by the cocycle condition:

$$\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_2) = 1$$

The topological $\sigma$MS is a RG fixed-point.
Topological invariance in topological NL$\sigma$Ms

As we change the space-time lattice, the action amplitude $e^{-S}$ does not change:

$$
\nu_2(g_0, g_1, g_2)\nu_2^{-1}(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)
$$

$$
\nu_2(g_0, g_1, g_2)\nu_2^{-1}(g_1, g_2, g_3)\nu_2(g_0, g_2, g_3) = \nu_2(g_0, g_1, g_3)
$$

as implied by the cocycle condition:

$$
\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_2) = 1
$$

The topological NL$\sigma$M is a RG fixed-point.

- The $Z(M^d) = 1$ if $M^d$ can be obtained by gluing spheres. The NL$\sigma$M describes a SPT state with trivial topo. order.
### Bosonic SPT phases from $\mathcal{H}^d[G, U(1)]$

<table>
<thead>
<tr>
<th>Symmetry $G/ d =$</th>
<th>0 + 1</th>
<th>1 + 1</th>
<th>2 + 1</th>
<th>3 + 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(1) \times Z_2^T$ (top. ins.)</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$ ($0$)</td>
<td>$\mathbb{Z}_2$ ($\mathbb{Z}_2$)</td>
<td>$\mathbb{Z}_2^2$ ($\mathbb{Z}_2$)</td>
</tr>
<tr>
<td>$U(1)$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$Z_2^T$ (top. SC)</td>
<td>0</td>
<td>$\mathbb{Z}_2^2$</td>
<td>0</td>
<td>$\mathbb{Z}_2^3$</td>
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<td>$Z_2^T$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T \times \text{trans}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_2$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_3^2$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_8^2$</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T$ (spin sys.)</td>
<td>0</td>
<td>$\mathbb{Z}_2^2$</td>
<td>0</td>
<td>$\mathbb{Z}_2^3$</td>
</tr>
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<td>$U(1) \times Z_2^T \times \text{trans}$</td>
<td>0</td>
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<tr>
<td>$Z_n$</td>
<td>$\mathbb{Z}_n$</td>
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</tr>
</tbody>
</table>

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"$Z_2^T$": time reversal, "trans": translation, $0 \to$ only trivial phase. $(\mathbb{Z}_2) \to$ free fermion result.

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**Note:**
- \text{LRE 1} and \text{LRE 2} indicate topological orders in the tensor categories.
- \text{SRE} indicates symmetry breaking.
- \text{SB-LRE 1} and \text{SB-LRE 2} indicate symmetry breaking in the tensor categories.
- \text{SY-LRE 1} and \text{SY-LRE 2} indicate symmetry breaking in the group cohomology.
- \text{SB-SRE 1} and \text{SB-SRE 2} indicate symmetry breaking in the group cohomology.
Universal probe for SPT orders

- How do you know the constructed NL$\sigma$M ground states carry non-trivial SPT order? How do you probe/measure SPT order?

**Universal probe** = one probe to detect all possible orders.
Universal probe for SPT orders

- How do you know the constructed NL$\sigma$M ground states carry non-trivial SPT order? How do you probe/measure SPT order?
  
  **Universal probe** = one probe to detect all possible orders.

- Universal probe for crystal order = X-ray diffraction:
Universal probe for SPT orders

- How do you know the constructed NL$\sigma$M ground states carry non-trivial SPT order? How do you probe/measure SPT order?
  
  **Universal probe** = one probe to detect all possible orders.

- Universal probe for crystal order
  
  = X-ray diffraction:

- Partitional function as an universal probe, but $Z_{top}^{SPT}(M^d) = 1 \rightarrow$ does not work.

- Twist the symmetry by “gauging” the symmetry on $M^d$
  
  $\rightarrow A - G$ gauge field.
  
  $\rightarrow Z_{top}^{SPT}(A, M^d) \neq 1$.

[Levin-Gu 12; Hung-Wen 13]
Universal topo. inv.: “gauged” partition function

\[
\frac{Z(A, M^d)}{Z(0, M^d)} = \frac{\int Dg e^{-\int \mathcal{L}(g^{-1}(d-iA)g)}}{\int Dg e^{-\int \mathcal{L}(g^{-1}dg)}} = e^{-i2\pi \int W_{\text{topinv}}(A)}
\]

- \(W_{\text{topinv}}(A)\) and \(W'_{\text{topinv}}(A)\) are equivalent if

\[
W'_{\text{topinv}}(A) - W_{\text{topinv}}(A) = \frac{1}{\lambda_g} \text{Tr}(F^2) + \cdots
\]

- The equivalent class of the gauge-topological term \(W_{\text{topinv}}(A)\) is the topological invariant that probe different SPT state.

- The topological invariant \(W_{\text{topinv}}(A)\) are Chern-Simons terms or Chern-Simons-like terms.

- Such Chern-Simons-like terms are classified by

\[
H^{d+1}(BG, \mathbb{Z}) = \mathcal{H}^d[G, U(1)]
\]

[Diekgraaf-Witten 92]

\(The\ \text{topological\ invariant} \ W_{\text{topinv}}(A)\ \text{can probe all the SPT states}\) (constructed so far)
Topo. terms for $U(1)$ SPT state: $a \equiv \frac{A}{2\pi}$, $c_1 \equiv \frac{dA}{2\pi}$, $ac_1 \equiv a \wedge c_1$

<table>
<thead>
<tr>
<th>$d = \frac{d}{2}$</th>
<th>$\mathcal{H}^d[U(1)]$</th>
<th>$W_{\text{top inv}}^d$</th>
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<tbody>
<tr>
<td>$0 + 1$</td>
<td>$\mathbb{Z}$</td>
<td>$a$</td>
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<td>$3 + 1$</td>
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U(1) SPT phases and their physical properties

- Topo. terms for $U(1)$ SPT state: $a \equiv \frac{A}{2\pi}$, $c_1 \equiv \frac{dA}{2\pi}$, $ac_1 \equiv a \wedge c_1$

- In $0 + 1D$, $W_{\text{top inv}}^1 = k\frac{A}{2\pi} = ka$.

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- Topo. terms for U(1) SPT state: \( a \equiv \frac{A}{2\pi}, \quad c_1 \equiv \frac{dA}{2\pi}, \quad ac_1 \equiv a \wedge c_1 \)

- In 0 + 1D, \( W_{\text{topinv}}^1 = k \frac{A}{2\pi} = ka \).

\[
\text{Tr}(U_{\theta}^{\text{twist}} e^{-H}) = e^{i k \oint_{S^1} A^{\text{twist}}} = e^{i k \theta}
\]

\( \rightarrow \) ground state carries charge \( k \)

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**$U(1)$ SPT phases and their physical properties**

- Topo. terms for $U(1)$ SPT state: $a \equiv \frac{A}{2\pi}, c_1 \equiv \frac{dA}{2\pi}, ac_1 \equiv a \wedge c_1$
- In $0 + 1D$, $W^1_{\text{topinv}} = k\frac{A}{2\pi} = ka$.
  \[
  \text{Tr}(U^\text{twist}_\theta e^{-H}) = e^{ik\oint_{S^1} A^\text{twist}} = e^{ik\theta}
  \rightarrow \text{ground state carries charge } k
  \]
- In $2 + 1D$, $W^3_{\text{topinv}} = k\frac{AdA}{(2\pi)^2} = kac_1$.

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\( U(1) \) SPT phases and their physical properties

- Topo. terms for \( U(1) \) SPT state: \( a \equiv \frac{A}{2\pi}, \ c_1 \equiv \frac{dA}{2\pi}, \ ac_1 \equiv a \wedge c_1 \)

- In \( 0 + 1D \), \( W^1_{\text{topinv}} = kA = ka \).
  \[
  \text{Tr}(U^\text{twist}_\theta e^{-H}) = e^{ik \oint_{s_1} A^{\text{twist}}} = e^{ik\theta}
  \]
  \( \rightarrow \) ground state carries charge \( k \)

- In \( 2 + 1D \), \( W^3_{\text{topinv}} = k\frac{AdA}{(2\pi)^2} = kac_1 \).
  \( \rightarrow \) Hall conductance \( \sigma_{xy} = 2k\frac{e^2}{h} \)
  \( \rightarrow \) The edge of \( U(1) \) SPT phase must be gapless: left/right movers + anomalous \( U(1) \) symm.

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**U(1) SPT phases and their physical properties**

- **Topo. terms for U(1) SPT state:** \( a \equiv \frac{A}{2\pi}, \ c_1 \equiv \frac{dA}{2\pi}, \ ac_1 \equiv a \wedge c_1 \)

- In 0 + 1D, \( W_{\text{topinv}}^1 = kA \)

  \[
  \text{Tr}(U^\text{twist}_\theta e^{-H}) = e^{i k \oint S^1 A^{\text{twist}}} = e^{i k \theta}
  \]

  \( \rightarrow \) ground state carries charge \( k \)

- In 2 + 1D, \( W_{\text{topinv}}^3 = k\frac{A dA}{(2\pi)^2} = kac_1 \)

  \( \rightarrow \) Hall conductance \( \sigma_{xy} = 2k \frac{e^2}{h} \)

  \( \rightarrow \) The edge of U(1) SPT phase must be gapless: left/right movers + anomalous U(1) symm.

- **Probe:** \( 2\pi m \) flux in space \( M^2 \) induces \( km \) unit of charge

  \( \rightarrow \) Hall conductance \( \sigma_{xy} = 2ke^2/h \)

<table>
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<th>( d )</th>
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<th>( W_{\text{topinv}}^d )</th>
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<td>( ac_1 )</td>
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<tr>
<td>3 + 1</td>
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</table>
$U(1)$ SPT phases and their physical properties

- Topo. terms for $U(1)$ SPT state: $a \equiv \frac{A}{2\pi}$, $c_1 \equiv \frac{dA}{2\pi}$, $ac_1 \equiv a \wedge c_1$

- In $0 + 1$D, $W^1_{\text{topinv}} = k \frac{A}{2\pi} = ka$.
\[
\text{Tr}(U^\text{twist}_\theta e^{-H}) = e^{i k \oint_{S^1} A^\text{twist}} = e^{i k \theta}
\]
→ ground state carries charge $k$

- In $2 + 1$D, $W^3_{\text{topinv}} = k \frac{AdA}{(2\pi)^2} = kac_1$.
→ Hall conductance $\sigma_{xy} = 2k \frac{e^2}{h}$
→ The edge of $U(1)$ SPT phase must be gapless: left/right movers + anomalous $U(1)$ symm.

- **Probe:** $2\pi m$ flux in space $M^2$ induces $km$ unit of charge → Hall conductance $\sigma_{xy} = 2ke^2/h$.

- **Mechanism:** Start with $2+1$D bosonic superfluid: proliferate vortices → trivial Mott insulator. Prohibit vortex + $2k$-charge → $U(1)$ SPT state labeled by $k$. 

<table>
<thead>
<tr>
<th>$d$</th>
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<th>$W^d_{\text{topinv}}$</th>
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A mechanism for 2+1D $U(1) \rtimes Z_2^T$ SPT state

- 2+1D boson superfluid + gas of vortex → boson Mott insulator.
- 2+1D boson superfluid + gas of $S^z$-vortex → boson topological insulator ($U(1) \rtimes Z_2^T$ SPT state)

- The boson superfluid + spin-1 system

$$S^z\text{-vortex} = \text{vortex} + (S_z = +1)\text{-spin}$$

$$\text{anti } S^z\text{-vortex} = \text{anti-vortex} + (S_z = -1)\text{-spin}$$

Probing 2+1D $U(1) \rtimes Z_2^T$ SPT state [Liu-Gu-Wen 14]

Let $\Phi_{\text{vortex}}$ be the creation operator of the vortex. Then

$$T^{-1}\Phi_{\text{vortex}} T = \Phi_{\text{vortex}}^\dagger,$$

or

$$T^{-1}\Phi_{S^z\text{-vortex}} T = -\Phi_{S^z\text{-vortex}}^\dagger.$$
$\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ does not produce all the SPT phases with symm. $G$: Topological states and anomalies

SPT order from $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$

- SPT state with on-site symmetry
- Theory with gauge (symm.) anomaly

Topologically ordered gravitational anomaly state

Topological order (PSG, ???)

Xiao-Gang Wen, MIT/PI, IPAM, Jan. 26, 2015
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Topological order

- Topologically ordered state
- Effective theory with gravitational anomaly

SET phases (PSG, ???)

SPT phases (???)

---

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SPT order from $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$

- SPT state with on-site symmetry
- theory with gauge (symm.) anomaly

- SPT state with on-site symmetry
- theory with mixed gauge–grav. anomaly

Topologically ordered state

effective theory with gravitational anomaly

SPT order beyond $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$

- TO 1
- TO 2
- triTO

SET phases (PSG, ???)

- SY–TO 1
- SY–TO 2
- SY–TO 3
- SY–TO 4
- SY–triTO 1
- SY–triTO 2
- SY–triTO 3
- SY–triTO 4

SPT phases (???)
CS-like topological terms $\leftrightarrow$ SPT and topo. orders

Pure SPT order within $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W^d_{\text{topinv}} = ac_1$

$W^d_{\text{topinv}}$ only depend on $A$ – the gauge $G$-connection
Pure SPT order within $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W^d_{\text{topinv}} = ac_1$

$W^d_{\text{topinv}}$ only depend on $A$ – the gauge $G$-connection.

Topological order: $W^d_{\text{topinv}} = \omega_3, \frac{1}{2}w_2w_3$

$W^d_{\text{topinv}}$ only depend on $\Gamma$ – the gravitational $SO$-connection. $p_1$ is the first Pontryagin class, $d\omega_3 = p_1$, and $w_i$ is the Stiefel-Whitney classes.

[Kapustin 14]
CS-like topological terms $\leftrightarrow$ SPT and topo. orders

Pure SPT order within $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W_{\text{topinv}}^d = ac_1$

$W_{\text{topinv}}^d$ only depend on $A$ – the gauge $G$-connection

Invertible topological order (iTO): $W_{\text{topinv}}^d = \omega_3, \frac{1}{2}w_2w_3$

Topo. order w/ no topo. excitations

$W_{\text{topinv}}^d$ only depend on $\Gamma$ – the gravitational $SO$-connection

$p_1$ is the first Pontryagin class, $d\omega_3 = p_1$, and $w_i$ is the Stiefel-Whitney classes.

- The $\mathbb{Z}$-class of 2+1D iTO’s are generated by $\omega_3$, a $(E_8)^3$ state.
  $E_8$ state is anomalous as a L-type theory, but not as a H-type
- $\mathbb{Z}_2$-class of 4+1D iTO’s are generated by $\frac{1}{2}w_2w_3$. [Kapustin 14]
CS-like topological terms ↔ SPT and topo. orders

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Mixed SPT order beyond $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W^d_{\text{topinv}} = c_1\omega_3$
$W^d_{\text{topinv}}$ depend on both $A$ and $\Gamma$
A model to realize all (?) bosonic pure STP orders, mixed SPT orders, and invertible topological orders

• NL$σ$M (group cohomology) approach to pure SPT phases:
  1. NL$σ$M+topo. term: $\frac{1}{2\lambda} |\partial g|^2 + 2\pi i W(g^{-1}\partial g)$, $g \in G$
  2. Add symm. twist: $\frac{1}{2\lambda} |(\partial - iA)g|^2 + 2\pi i W[(\partial - iA)g]$
  3. Integrate out matter field: $Z_{\text{fixed}} = e^{2\pi i \int W_{\text{topinv}}(A)}$

But not one-to-one. Need to quotient out something $\Gamma$. 

Xiao-Gang Wen, MIT/PI, IPAM, Jan. 26, 2015
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- **G × SO∞** NLσM (group cohomology) approach:
  1. **NLσM:** \( \frac{1}{2\lambda}|\partial g|^2 + 2\pi i W(g^{-1}\partial g), \ g \in G \times SO \)
  2. Add twist: \( \frac{1}{2\lambda}|(\partial - iA - i\Gamma)g|^2 + 2\pi i W[(\partial - iA - i\Gamma)g] \)
  3. Integrate out matter field: \( Z_{\text{fixed}} = e^{2\pi i \int W_{\text{topinv}}(A,\Gamma)} \)

All possible topo. terms are classified by \( H^d(G \times SO, \mathbb{R}/\mathbb{Z}) \) and pure/mixed SPT orders, and iTOs are classified by \( H^d(G \times SO, \mathbb{R}/\mathbb{Z}) = H^d(G, \mathbb{R}/\mathbb{Z}) \oplus \bigoplus_{k=1}^{d-1} H^k(G, \mathbb{R}/\mathbb{Z}) \oplus H^d(SO, \mathbb{R}/\mathbb{Z}) \) but not one-to-one. Need to quotient out something \( \Gamma \) \( d(G) \).
A model to realize all (?) bosonic pure STP orders, mixed SPT orders, and invertible topological orders

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  3. Integrate out matter field: \( Z_{\text{fixed}} = e^{2\pi i \int W_{\text{topinv}}(A)} \)

- **\( G \times SO_\infty \) NLσM (group cohomology) approach:**
  1. NLσM: \( \frac{1}{2\lambda}|\partial g|^2 + 2\pi i W(g^{-1}\partial g), \ g \in G \times SO \)
  2. Add twist: \( \frac{1}{2\lambda}|(\partial - iA - i\Gamma)g|^2 + 2\pi i W[(\partial - iA - i\Gamma)g] \)
  3. Integrate out matter field: \( Z_{\text{fixed}} = e^{2\pi i \int W_{\text{topinv}}(A,\Gamma)} \)

- All possible topo. terms are classified by \( \mathcal{H}^d(G \times SO, \mathbb{R}/\mathbb{Z}) \)

**Pure/mixed SPT orders, and iTOs are classified by**

\[
\mathcal{H}^d(G \times SO, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \oplus \bigoplus_{k=1}^{d-1} \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})
\]

But not one-to-one. Need to quotient out something \( \Gamma^d(G) \).
Trying to classify bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- **Pure STP**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ one-to-one
- **Mixed SPT**: $\bigoplus_{k=1}^{d-1} \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]$ many-to-one
- **iTO’s**: $\mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$ many-to-one

- Pure STP orders are classified by $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$  
  CS-like topo. inv. $W^d_{\text{topinv}}(A)$ are classified by $H^{d+1}(BG, \mathbb{Z})$  
  Pure STP orders are also classified by $W^d_{\text{topinv}}(A)$

- But iTOs are not one-to-one classified by $W^d_{\text{topinv}}(\Gamma_{SO})$ in $H^{d+1}(BSO, \mathbb{Z})$, because different $W^d_{\text{topinv}}(\Gamma_{SO})$ and $\tilde{W}^d_{\text{topinv}}(\Gamma_{SO})$ may satisfy $W^d_{\text{topinv}}(\Gamma_{SO}) = \tilde{W}^d_{\text{topinv}}(\Gamma_{SO})$ when $\Gamma_{SO}$ is the $SO$-connection of the tangent bundle of $M^d$.

- $W^d_{\text{topinv}}, \tilde{W}^d_{\text{topinv}} \rightarrow$ the same iTO $\rightarrow$ iTO$^d = \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})/\Gamma^d$
Trying to classify bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- **Pure STP orders**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ (the black entries below)
- **iTO’s**: $\text{iTO}^d = \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})/\Gamma^d$ (using Wu class and $Sq^n$)
- **Mixed SPT order** $\bigoplus_{k=1}^{d-1} \mathcal{H}^k(G, \text{iTO}^{d-k}) \subset \frac{\bigoplus \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]}{\Gamma^d(G)}$

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<th>2+1</th>
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</tbody>
</table>

Xiao-Gang Wen, MIT/PI, IPAM, Jan. 26, 2015
Trying to classify bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- **Pure STP orders**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ (the black entries below)
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</tr>
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- **Probe mixed SPT order described by** $\mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]$: put the state on $M^d = M^k \times M^{d-k}$ and add a $G$-symmetry twist on $M^k$. Induce a state on $M^{d-k}$ described by $\mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})$ → a iTO state in iTO$^{d-k}$
A mechanism for $Z_2^T$ mixed SPT state in 3+1D

[Vishwanash-Senthil 12, Kapustin 14, Wen 14]

The $Z_2^T$ mixed SPT states are classified by
$\mathcal{H}^1(Z_2^T, iTO^3) = \mathbb{Z}_2$

The topological invariant for a $Z_2^T$ mixed SPT state (bosonic topological super fluid with time reversal symmetry) is
$W_{\text{topinv}}^4 = \frac{1}{2} p_1$ [W 14] ($W_{\text{topinv}}^4 = \frac{1}{6} p_1$ [VS 12, K 14])

- Start with a T-symmetry breaking state. Proliferate the symmetry breaking domain walls to restore the T-symmetry.
  $\rightarrow$ a trivial SPT state.

- Bind the domain walls to $(E_8)^3$ [W 14] ($E_8$ [VS 12, K 14]) quantum Hall state, and then proliferate the symmetry breaking domain walls to restore the T-symmetry.
  $\rightarrow$ a non-trivial $Z_2^T$ SPT state.
$\mathbb{Z}_2$ SPT phases and their physical properties

- Topological terms:

$$\oint A_{\mathbb{Z}_2} = 0, \pi; \quad a_1 \equiv \frac{A_{\mathbb{Z}_2}}{\pi};$$

<table>
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<tr>
<th>$d =$</th>
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<tr>
<td>3 + 1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Z₂ SPT phases and their physical properties

- **Topological terms:**
  \[ \oint A Z₂ = 0, \pi; \ a₁ ≡ \frac{A Z₂}{\pi} ; \]

- In 0 + 1D, \( W_{\text{topinv}}^1 = k \frac{A Z₂}{2\pi} = ka₁. \)
  \[ \text{Tr}(U^\text{twist}_π e^{-H}) = e^{2\pi i \oint S₁ W_{\text{topinv}}} = e^{i k \pi} = ±1 \]
  \[ \rightarrow \text{ground state } Z₂\text{-charge: } k = 0, 1 \]

<table>
<thead>
<tr>
<th>( d = )</th>
<th>( \mathcal{H}^d[Z₂] )</th>
<th>( W_{\text{topinv}}^d )</th>
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<tbody>
<tr>
<td>0 + 1</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \frac{1}{2}a₁ )</td>
</tr>
<tr>
<td>1 + 1</td>
<td>0</td>
<td>( \frac{1}{2}a₁^3 )</td>
</tr>
<tr>
<td>2 + 1</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \frac{1}{2}a₁^3 )</td>
</tr>
<tr>
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Topological terms:

\[ \oint A_{\mathbb{Z}_2} = 0, \pi; \quad a_1 \equiv \frac{A_{\mathbb{Z}_2}}{\pi}; \]

<table>
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<tr>
<th>(d)</th>
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<th>(W^{d}_{\text{topinv}})</th>
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\[ \text{Tr}(U_{\pi}^{\text{twist}} e^{-H}) = e^{2\pi i \oint_{S^1} W_{\text{topinv}}} = e^{i k \pi} = \pm 1 \]

\(\rightarrow\) ground state \(\mathbb{Z}_2\)-charge: \(k = 0, 1\)

In 2 + 1D, \(\int_{M^3} W_{\text{topinv}}^{3} = \int_{M^3} \frac{1}{2} a_1^3\).

Here we do not view \(a_1\) as 1-form but as 1-cocycle \(a_1 \in H^1(M^3, \mathbb{Z}_2)\), and \(a_1^3 \equiv a_1 \cup a_1 \cup a_1\):

\[ \int_{M^3} a_1 \cup a_1 \cup a_1 = 0 \text{ or } 1 \rightarrow e^{2\pi i \oint_{M^3} W_{\text{topinv}}} = e^{\pi i \oint_{M^3} a_1^3} = \pm 1 \]
**Z\textsubscript{2} SPT phases and their physical properties**

- **Topological terms:**
  \[ \oint A = 0, \pi; a_1 \equiv \frac{A_{\text{Z}_2}}{\pi}; \]

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- **In 0 + 1D,** \(W^1_{\text{topinv}} = k \frac{A_{\text{Z}_2}}{2\pi} = ka_1\).
  \[
  \text{Tr}(U_{\pi}^{\text{twist}} e^{-H}) = e^{2\pi i \oint S^1 W_{\text{topinv}}}
  = e^{i k \pi \oint S^1 a_1} = e^{i k \pi} = \pm 1
  \]

  \(\rightarrow\) ground state \(\text{Z}_2\)-charge: \(k = 0, 1\)

- **In 2 + 1D,** \(\int_{M^3} W^3_{\text{topinv}} = \int_{M^3} \frac{1}{2}a_1^3\).

  Here we do not view \(a_1\) as 1-form but as 1-cocycle \(a_1 \in H^1(M^3, \mathbb{Z}_2)\), and \(a_1^3 \equiv a_1 \cup a_1 \cup a_1\):
  \[
  \int_{M^3} a_1 \cup a_1 \cup a_1 = 0\text{ or }1 \rightarrow e^{2\pi i \oint_{M^3} W_{\text{topinv}}} = e^{\pi i \oint_{M^3} a_1^3} = \pm 1
  \]

- **Poincaré duality:** 1-cocycle \(a_1 \leftrightarrow 2\)-cycle \(N^2\) (2D submanifold)

  \(N^2\) is the surface across which we do the \(\mathbb{Z}_2\) symmetry twist.

  Choose \(M^3 = M^2 \times S^1\)

  As we go around \(S^1\):
  \[
  \int_{M^3} a_1^3 = \# \text{ of loop creation/annihilation} + \# \text{ of line reconnection}
  \]
  \(\xrightarrow{(a)}\) \(\xrightarrow{(b)}\) \(\xrightarrow{(c)}\)

Xiao-Gang Wen, MIT/PI, IPAM, Jan. 26, 2015

SPT order and algebraic topology
Assume the edge of a $\mathbb{Z}_2$ SPT phase is gapped with no symmetry breaking. We use $\mathbb{Z}_2$ twist try to create excitations (called $\mathbb{Z}_2$ domain walls) at the edge. We may naively expect those $\mathbb{Z}_2$-domain walls are trivial, but they are not. They have a non-trivial fusion property: different fusion order can differ by a $-$ sign.

So the $\mathbb{Z}_2$ domain walls on the boundary form a non-trivial fusion category. → the bulk state must carry a non-trivial topological order.
The boundary of the 2+1D $\mathbb{Z}_2$ SPT state has a 1+1D bosonic global $\mathbb{Z}_2$ anomaly [Chen-Wen 12]

**The 1+1D bosonic global $\mathbb{Z}_2$ anomaly** → The edge of $\mathbb{Z}_2$ SPT phase must be gapless or symmetry breaking.

- One realization of the edge is described by 1+1D XY model or $U(1)$ CFT. The primary field (vertex operator) $V_{l,m}$ has dimensions $(h_R, h_L) = \left(\frac{(l+2m)^2}{8}, \frac{(l-2m)^2}{8}\right)$.
- The $\mathbb{Z}_2$ symmetry action $V_{l,m} \rightarrow (-)^{l+m}V_{l,m}$

Such a 1+1D $\mathbb{Z}_2$ symmetry is anomalous:

1. The XY model has no UV completion in 1+1D such that the $\mathbb{Z}_2$ symmetry is realized as an on-site symm. $U = \prod_i \sigma_i^x$.
2. If we gauge the $\mathbb{Z}_2$, the 1+1D $\mathbb{Z}_2$ gauge theory has no UV completion in 1+1D as a bosonic theory.
3. The XY model has a UV completion as boundary of 2+1D lattice theory w/ the $\mathbb{Z}_2$ symmetry realized as an on-site symm.