Symmetry, Defects, and Gauging of Topological Phases of Matter

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Ad: Low Dimensional Higher Categories and Applications

• Math:
  Classification of (2+1)- and (3+1)-TQFTs, not fully extended---
  Invariants of low dimensional manifolds (especially smooth 4D)

• Physics:
  Classification of 2D and 3D symmetry enriched topological order (SET) and symmetry protected topological order (SPT)
Topological phases of matter are TQFTs in Nature and hardware for hypothetical topological quantum computers.
Symmetry and 2D Topological Phases of Matter

We develop a general framework to classify 2D topological order in topological phases of matter with symmetry by using $G$-crossed braided tensor category.

Given a 2D topological order $\mathcal{C}$ and a global symmetry $G$ of $\mathcal{C}$, three intertwined themes on the interplay of symmetry group $G$ and intrinsic topological order $\mathcal{C}$

- **Symmetry Fractionalization**---topological quasi-particles carry fractional quantum numbers of the underlying constituents
- **Defects**---extrinsic point-like defects. Many are non-abelian objects
- **Gauging**---deconfine defects by promoting the global symmetry $G$ to a local $G$ gauge theory
Examples of Topological Phases with Symmetry

$\mathbb{Z}_2$ Toric Code (Kitaev):

$$H_{\mathbb{Z}_2} = \sum_s A_s + \sum_p B_p$$

$$B_p = \prod_{i \in \text{boundary}(p)} \sigma_i^z \quad A_s = \prod_{i \in \text{star}(s)} \sigma_i^x$$

Topological order: $\mathcal{C} = D(\mathbb{Z}_2) = \{1, e, m, \psi\}$

Electric-magnetic duality: $e \leftrightarrow m$
Examples of 2D Topological Phases with Symmetry

$1/m$-Laughlin state

$$\Psi(\{z_i\}) = \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4l_B^2}$$

Topological order is encoded by $U(1)_m \times \{1,e\}$

Topological particle-hole symmetry: $a \leftrightarrow -a$
$Z_2$-Layer Exchange Symmetry: Bilayer FQH States

E.g. Halperin (mml) state

$$\Psi(\{z_i\}, \{w_i\}) = \prod_{i<j} (z_i - z_j)^m (w_i - w_j)^m \prod_{i,j} (z_i - w_j)^l$$
Topological Phases of Matter

Finite-energy topological quasiparticle excitations = anyons

Anyons $a$, $b$, $c$

Anyons are of the same type if they differ only by local operators

Anyons in 2+1 dimensions described mathematically by a Unitary Modular Tensor Category $\mathcal{C}$ = Anyon Model
Symmetry of Quantum Systems ($\mathcal{L}, H$)

Microscopic Symmetry G:

$$R_g: \mathcal{L} \rightarrow \mathcal{L} \quad R_gH = HR_g$$

Should preserve locality of $\mathcal{L}$

Symmetry is on-site if:

$$R_g = \prod_{i=1}^{N} R_{g}^{(i)}$$
Assumptions and Work In Progress

1) The global symmetry G is a finite group
2) Bulk 2D topological order in boson/spin systems=UMTC=anyon model
3) Global symmetry G can be realized as on-site unitary symmetries of the microscopic Hamiltonian, at least at low energies

Partial results in our paper:
• Continuous symmetries such as U(1) charge conservation and SO(3) spin rotation (2/3)
• Fermion systems (1 or 0)
• Time-reversal (1/3)
• Spatial (1/6+1/6)
• Fermion parity (0)
Classification of 2D SETs Topologically

• Given a 2D topological order=UMTC=anyon model $\mathcal{C}$, and a finite group $G$, then $G$-SETs=$G$-crossed braided extensions of $\mathcal{C}$

• SETs are in 1-1 correspondence with set $[BG, \text{BPic}(\mathcal{C})]$ of homotopy classes of maps between classifying spaces $BG$ and $\text{BPic}(\mathcal{C})$, where $\text{BPic}(\mathcal{C})$ is the classifying space of the categorical 2-group $\text{Pic}(\mathcal{C})$ with $\pi_1 = \text{Aut}(\mathcal{C})$, $\pi_2 = \mathcal{A}$, $\pi_3 = \mathbb{C}\setminus\{0\}$, and $\pi_i = 0$ for $i>3$. (ENO 2010)

• Note that $[BG, \text{BPic}(\mathcal{C})] = \pi_0(X^Y)$, where $X=\text{BPic}(\mathcal{C})$, $Y=BG$. Do higher homotopy groups $\pi_i(X^Y)$, $i>0$, of the mapping space $X^Y$ have physical meaning and significance?
Classification of G-crossed Extensions of a UMTC $\mathcal{C}$ Algebraically

Etingof, Nikshych, Ostrik (2010)

$\mathcal{C}_G^X$ classified by $([\rho], [t], [\alpha])$

$[\rho] : G \to \text{Aut}(\mathcal{C})$

If a primary obstruction in $H^3_\rho(G, \mathcal{A})$ vanishes, then choose $[t] \in H^2_\rho(G, \mathcal{A})$

If a secondary obstruction in $H^4(G, U(1))$ vanishes, then choose $[\alpha] \in H^3(G, U(1))$
Fine print: Symmetry, Defects, and Gauging

1. Skeletonizing G-Crossed Braided Tensor Category to Obtain Numerical Version of G-Crossed Braided Tensor Category

2. Applying G-Crossed Braided Tensor Categories to Physics:
   General Classification and Characterization of Symmetry-Enriched 2D Topological Order
2D Topological Order = UMTC = Anyon Model $\mathcal{C}$

A modular tensor category = a non-degenerate braided spherical fusion category:
a collection of numbers \( \{ L, N_{ab}^c, F_{d;nm}^{abc}, R_c^{ab} \} \) that satisfy some polynomial constraint equations.

6j symbols for recoupling

\[
\begin{align*}
\sum_{f,\mu,\nu} [F_d^{abc}]_{(e,\alpha,\beta)(f,\mu,\nu)} &= \sum_{\nu} [R_c^{ab}]_{\mu,\nu}.
\end{align*}
\]

Pentagons for 6j symbols

R-symbol for braiding

\[
\begin{align*}
\sum_{\nu} [R_c^{ab}]_{\mu,\nu} &= \sum_{\nu} [R_c^{ab}]_{\mu,\nu}.
\end{align*}
\]

Hexagons for R-symbols
Examples

• Pointed: $\mathcal{C}(A, q)$, $A$ finite abelian group, $q$ non-deg. quadratic form on $A$.

• $\text{Rep}(D^\omega G)$, $\omega$ a 3-cocycle on $G$ a finite group.

• Quantum groups/Kac-Moody algebras: subquotients of $\text{Rep}(U_q\mathfrak{g})$ at $q = e^{\pi i/l}$ or level $k$ integrable $\mathfrak{g}$-modules, e.g.
  - $\text{SU}(N)_k = \mathcal{C}(\mathfrak{sl}_N, N + k)$,
  - $\text{SO}(N)_k$,
  - $\text{Sp}(N)_k$,
  - for $\gcd(N, k) = 1$, $\text{PSU}(N)_k \subset \text{SU}(N)_k$ “even half”

• Drinfeld center: $Z(\mathcal{D})$ for spherical fusion category $\mathcal{D}$.

• Rank-finiteness (see E. Rowell’s poster).
Topological and Global Symmetry

The **categorical symmetry group** $\text{Aut}(\mathcal{C})$ of an anyon model $\mathcal{C}$ consists of all permutations of anyon types and transformations of fusion states $\{|a,b,c,\mu\rangle\}$ that preserve all defining data up to gauge freedom. In math jargon, all braided tensor auto-equivalences of $\mathcal{C}$.

Given an anyon model $\mathcal{C}$, its $\text{Aut}(\mathcal{C})$ is classified by a triple $(\Pi_1, \Pi_2, \kappa)$, where $\Pi_1$ is the classes of braided tensor auto-equivalences of $\mathcal{C}$, $\Pi_2=\mathcal{A}$ the abelian anyons of $\mathcal{C}$, and $\kappa \in H^3(\Pi_1,\Pi_2)$ a cohomology class.

$\Pi_1=\text{Aut}(\mathcal{C})$ will be called the **topological symmetry group** of $\mathcal{C}$.

Given a group $G$, a **global $G$-symmetry** of $\mathcal{C}$ is $\rho: G \rightarrow \text{Aut}(\mathcal{C})$ --- a group homomorphism.
Symmetries of Abelian Anyon Models

• An **abelian anyon model** is given by a pair \( \mathcal{C}=(A,q) \),
where \( A \) is a commutative finite group and
\( q(x) \) is the topological twist of anyon type \( x \in A, \ q: A \to U(1) \).

• The topological symmetry group \( \Pi_1=\text{Aut}(\mathcal{C}) \)
is the group \( O(A,q)=\{s \in \text{Aut}(A): q(s(x))=q(x) \text{ for all } x \in A \} \)
and \( \Pi_2=A \).

• \( U(1)_3: \ A=\mathbb{Z}_3, \ q(x)=\{1,e^{2\pi i/3}, e^{2\pi i}\}, \ \Pi_1=\mathbb{Z}_2, \ \Pi_2=\mathbb{Z}_3. \)

• Toric code and 3fermion: both \( A=\mathbb{Z}_2 \oplus \mathbb{Z}_2 =\{1,\ e, m, \psi\} \) and
\( q(x)=\{1,1,1,-1\} \) or \( q(x)=\{1,-1,-1,-1\} \), so \( \Pi_1 = \mathbb{Z}_2 \) or \( S_3 \).
Origin of Symmetry Fractionalization: Topological Symmetry Is Categorical

Given a global symmetry \((G, \rho)\) realized as symmetries \(R_g\) of a Hamiltonian with a local Hilbert space \(L(Y;l)\), then \(L(Y;l)=\bigoplus L_{\lambda_i}\) according to energy levels \(\lambda_i\). The ground state manifold \(L_{\lambda_0}\) further decomposes as \(V(Y;t)\bigotimes L^{loc}_{\lambda_0}(Y;l)\), where \(V(Y;t)\) is the topological part and \(L^{loc}_{\lambda_0}(Y;l)\) the local part. On-site symmetries \(R_g\) act on \(L_{\lambda_0}= V(Y;t)\bigotimes L^{loc}_{\lambda_0}(Y;l)\) split as \(\rho_g \bigotimes \prod_l R^l_g\).

Anyon states in \(V(Y;t)\) are universality classes up to local actions, so global symmetry actions are not exact. Hence, projective local actions on \(L^{loc}_{\lambda_0}(Y;l)\) are allowed to compensate for the overall phases from the global actions. Since projective representations of \(G\) are classified by \(H^2(G, U(1))\), can symmetry fractionalizations be classified by \(H^2(G, U(1))\)?

The separation of global symmetry into topological and local parts requires subtle consistency:

1. A potential obstruction;
2. The coefficient for \(H^2\) is not \(U(1)\), but \(\Pi_2=\text{abelian anyons}\).
Global Symmetry $G$  

$$\rho: G \rightarrow Aut(\mathcal{C})$$

![Diagram showing the action of $G$ on three points $a$, $b$, and $c$, and their images under $g$.]

**Natural Isomorphism**

$$g_a \equiv \rho_g(a)$$

$$\rho_{gh} = \kappa_{g,h} \rho_g \rho_h$$

$$\rho_g(|a, b; c\rangle) = U_g(ga, gb; gc)|ga, gb; gc\rangle$$

$\rho$ leads to an obstruction $o_3(\rho) \in H^3_\rho(G, \mathcal{A})$

$\mathcal{A} \subseteq \mathcal{C}$  

Abelian anyons
Symmetry Localization

Ground state is symmetric: \[ R_g |\Psi_0\rangle = |\Psi_0\rangle \]

Consider state with two anyons:

\[ R_g |\Psi_{a,\bar{a};0}\rangle = U^{(1)}_{g} U^{(2)}_{g} \rho_c |\Psi_{a,\bar{a};0}\rangle = U^{(1)}_{g} U^{(2)}_{g} U_g( g_a, g_{\bar{a}}; 0) |\Psi_{g_a, g_{\bar{a}};0}\rangle \]
Symmetry Fractionalization

Anyons can form a projective representation

$$U^{(j)}_g U^{(j)}_h \neq U^{(j)}_{gh}$$

Even if $$R_g R_h = R_{gh}$$

General Result: Symmetry Fractionalization

1. Requires $$o_3(\rho) = 0$$ ($$H^3_\rho(G, \mathcal{A})$$ obstruction must vanish)

2. Classified by $$H^2_\rho(G, \mathcal{A})$$

$$\mathcal{A} \subseteq \mathcal{C}$$

Abelian anyons
Symmetry Fractionalization Mathematically

The obstruction $o_3(\rho) = \rho^*(\kappa) \in H^3(G, \Pi_2)$:

the pull back of the class $\kappa$ in $(\Pi_1, \Pi_2, \kappa)$ to $H^3(G, \Pi_2)$ by the global symmetry $\rho : G \to \Pi_1$.

If $o_3(\rho) = 0$, then possible symmetry fractionalizations form a torsor over $H^2(G, \Pi_2)$.

A set $X$ is a torsor over a group $G$ if $X$ has a transitive free action of $G$. 
Vanishing of Symmetry Fractionalization Obstruction

**Theorem:**
The obstruction to symmetry fractionalization vanishes if either
1) the global symmetry $\rho$ does not permute anyon types or
2) the anyon model is abelian with all 6j symbols trivial, i.e. the associativity 3-cocycle $\omega$ is trivial.

It follows that the obstructions to symmetry fractionalizations for toric code and 3fermion all vanish.
Given a topological phase with symmetry $G$, extrinsic point-like defects can be introduced by modifying the original Hamiltonian $H_0$.
Defects Confined

Defects are NOT finite-energy quasiparticle excitations/anyons

Cannot be described by original UMTC

Mathematics: G-Crossed Braided Tensor Category

We would like to have methods to systematically compute all properties of defects (fusion rule, braiding ,etc)
G-Graded Fusion

Topologically distinct types of g-defects

\[ C_G = \bigoplus_{g \in G} C_g \]

\( C_g \) contains collection of g-defects. Module category

\[ a_g \times b_h = \sum_{c \in C_{gh}} N_{ab}^c \]

\[ D_g = D_0 \]
Obstructions to Defectification

• Obstruction $o_3(\rho)$ to symmetry fractionalization is also the obstruction to a consistent fusion rule for $C_g$. If $o_3(\rho)=0$, then consistent fusion rules are in 1-1 correspondence with symmetry fractionalization classes $(\rho,t)$.

• Pentagons lead to a secondary obstruction $o_4(\rho,t)\in H^4(G, U(1))$ to consistently defectify.

• If $o_4(\rho,t)=0$, possible defectifications form a torsor over $H^3(G,U(1))$.

• If both obstructions=0, a defect theory is determined by $(G, \rho, t, \alpha)$, where $\alpha\in H^3(G,U(1))$. 
G-Crossed Braiding

\[ R^{a_g b_h} = b_h \times \bar{a}_g b_h \]

\[ [U_0(a, b; c)]_{\mu\nu} = \delta_{\mu\nu} \]

\[ U_k(0, 0; 0) = U_k(a, 0; a) = U_k(0, b; b) = 1 \]

\[ \eta_0(g, h) = 1 \]

\[ \eta_x(0, 0) = \eta_x(g, 0) = \eta_x(0, h) = 1 \]
G-Crossed Heptagon

G-Crossed version of hexagon equation
G-Crossed Data: Skeletonization

G-Crossed UBTC $\mathcal{C}_G^X$ characterized by data

$$\{L, N^c_{ab}, F^{abc}_d, R^b_c, \eta_a(g, h), U_k(a, b; c)\}$$

Subject to consistency equations

Inequivalent solutions $\leftrightarrow$ Distinct SET phases

Gauge-Invariant quantities = Topological invariants of SET
Gauge Transformations

(1) Vertex basis gauge transformations (Old type)

\[ |a, b; c, \mu\rangle = \sum_{\mu'} \left[ \Gamma_c^{ab} \right]_{\mu\mu'} |a, b; c, \mu'\rangle \]

\[ \left[ \tilde{R}^{a}_{cgh} b_{h} \right]_{\mu\nu} = \sum_{\mu', \nu'} \left[ \Gamma_c^b \tilde{F}_a \right]_{\mu\mu'} \left[ R^{a}_{cgh} b_{h} \right]_{\mu'\nu'} \left( \left[ \Gamma_c^{ab} \right]^{-1} \right)_{\nu'\nu} \]

\[ \left[ \tilde{U}_k (a, b; c) \right]_{\mu\nu} = \sum_{\mu', \nu'} \left[ \Gamma_c^{k \mu \nu} \right]_{\mu\mu'} \left[ U_k (a, b; c) \right]_{\mu'\nu'} \left( \left[ \Gamma_c^{ab} \right]^{-1} \right)_{\nu'\nu} \]

\[ \left[ \tilde{F}^{abc}_{d} \right]_{(e, \alpha, \beta)(f, \mu, \nu)} = \sum_{\alpha', \beta', \mu', \nu'} \left[ \Gamma_c^{ab} \right]_{\alpha\alpha'} \left[ \Gamma_c^{ec} \right]_{\beta\beta'} \left[ \tilde{F}^{abc}_{d} \right]_{(e, \alpha', \beta')(f, \mu', \nu')} \left( \left[ \Gamma_c^{bc} \right]^{-1} \right)_{\mu'\mu} \left( \left[ \Gamma_c^{af} \right]^{-1} \right)_{\nu'\nu} \]
(2) Symmetry Action Gauge Transformations (New Type)

Associated with natural isomorphism \( \tilde{\rho}_g = \Upsilon_g \rho_g \)

\[
\begin{align*}
\tilde{U}_k (a, b; c)_{\mu\nu} &= \frac{\gamma_a(k)\gamma_b(k)}{\gamma_c(k)} [U_k (a, b; c)]_{\mu\nu} \\
\tilde{R}_{cgh}^{agbh} \big|_{\mu\nu} &= \gamma_a(h) \big[ R_{cgh}^{agbh}\big]_{\mu\nu} \\
\tilde{\eta}_x (g, h) &= \frac{\gamma_x(gh)}{\gamma_{x}(h)^{\gamma_x}(g)} \eta_x (g, h) \\
\gamma_0 (h) &= \gamma_a(0) = 1
\end{align*}
\]
Invariants of Modular Tensor Category

\[
\text{MTC } \mathcal{C} \overset{\sim}{\leftrightarrow} \text{RT (2+1)-TQFT } (V, Z)
\]

• Pairing \( \langle Y^2, \mathcal{C} \rangle = V(Y^2; \mathcal{C}) \in \text{Rep}(\mathcal{M}(Y^2)) \) for a surface \( Y^2 \), \( \mathcal{M}(Y^2) = \) mapping class group

• Pairing \( Z_{X,L,\mathcal{C}} = \langle (X^3, L_C), \mathcal{C} \rangle \in \mathbb{C} \) for colored framed oriented links \( L_c \) in 3-mfd \( X^3 \)

\[ \text{fix } \mathcal{C}, Z_{X,L,\mathcal{C}} \text{ invariant of } (X^3, L_C) \]

\[ \text{fix } (X^3, L_C), Z_{X,L,\mathcal{C}} \text{ invariant of } \mathcal{C} \]

\[ \text{fix } Y^2, V(Y^2; \mathcal{C}) \text{ invariant of } \mathcal{C} \]
Quantum Dimensions, Twists, and S-matrix: Unknot and Hopf Link

Quantum Dimension

$$d_a = \ a$$

Twist

$$\theta_a = \frac{1}{d_a}$$

Total Quantum Dimension

$$D = \sqrt{\sum_{a \in C} d_a^2}$$

$$= \sum_{c,\mu} \frac{d_c}{d_a} [R_c^{aa}]_{\mu\mu}$$

$$S_{ab} = D^{-1} \sum_c N_{\bar{a}b}^{c} \frac{\theta_c}{\theta_a \theta_b} d_c = \frac{1}{D} \ a \ b$$
For a Unitary Modular Tensor Category,

\[(ST)^3 = e^{i\pi c/4} C\quad S^2 = C\quad C^2 = I\]

\[T_{ab} = \theta_a \delta_{ab}\]

Dimension of ground state Hilbert space on torus = \(|C|\)

\[|a\rangle_l = \sum_{b \in C} S_{ab} |b\rangle_m\]
Topological Twists

Type (2) Gauge transformations:

\[ \dot{\theta}_{a_g} = \gamma_{a_g} (g) \theta_{a_g} \]

Twist of defects is not gauge-invariant, as expected
Topological S-Matrix

\[ S_{ag\ bh} = \frac{1}{D_0} \sum_{c, \mu} d_c \frac{\theta_c}{\theta_{\bar{a}} \theta_b} \frac{[U_{gh}(\bar{a}, b; c)]_{\mu\mu}}{\eta_{\bar{a}}(\bar{g}, h) \eta_b(h, \bar{g})} \]

Type II Gauge transformations:

\[ \tilde{S}_{ag\ bh} = \gamma_{\bar{a}}(h) \gamma_b(\bar{g}) S_{ag\ bh} \]

G-Crossed Verlinde Formula:

\[ N_{ag\ bh}^c = \sum_{x_0 \in C_{0, h}^g} \frac{S_{ag\ x_0} S_{bh\ x_0} S_{cgh\ x_0}^*}{S_{0\ x_0}} \eta_x(\bar{h}, \bar{g}) \]
Extended Verlinde Algebra

\[ \mathcal{V}^{\text{ext}} = \bigoplus_{(g,h), gh = hg} \mathcal{V}_{(g,h)} \]

\[ S^{(g,h)} : \mathcal{V}_{(g,h)} \to \mathcal{V}_{(h, \bar{g})} \]

\[ T^{(g,h)} : \mathcal{V}_{(g,h)} \to \mathcal{V}_{(g, gh)} \]

\[ \dim \mathcal{V}_{(g,h)} = |C^h_g| \quad \quad C^h_g = \{ a \in C_g \mid h a = a \} \]

\[ |C^g_h| = |C^h_g| \quad \quad |C_g| = |C^g_0| \]
For G-Crossed UBTC, define modular matrices:

\[ S_{a g b h}^{(g, h)} = \frac{S_{a g b h}}{U_h(a, \bar{a}; 0)} \]

\[ T_{a g b g}^{(g, h)} = \eta_a(g, h) \theta_{a g} \delta_{a g b g} \]

\[ C_{a g b g}^{(g, h)} = \frac{1}{U_h(b, b; 0) \eta_b(h, \bar{h}) \delta_{a g b g}} \]

\[ (ST)^3 = e^{i \pi c/4} C \quad S = S^\dagger C \quad C^2 = 1 \]

Unitarity of \( S \) \( \rightarrow \) Representation of \( \text{SL}(2, \mathbb{Z}) \): Homotopy TQFT
Gauging Global Symmetry G

Given a topological order $\mathcal{C}$, then gauging $(G, \rho, t, \alpha)$ of $\mathcal{C}$ is:

**Step I:**
Defectify $\mathcal{C}$, $\mathcal{C}_G^x = \bigoplus_g \mathcal{C}_g$, where $\mathcal{C}_e = \mathcal{C}$.

**Step II:**
Orbifold $\mathcal{C}_G^x$, a new topological order $\mathcal{C}/G = (\mathcal{C}_G^x)^G$.

Gauging deconfines defects and leads to a topological phase transition from $\mathcal{C}$ to $\mathcal{C}/G$. 
Gauged Theory

Objects in $\mathcal{C}/G$

$$[a] = \{g^a, \forall g \in G\} \quad \quad G_a = \{g \in G | g^a = a\}$$

$$\pi_a = \text{irreducible projective representation of} \quad G_a$$

$$\pi_a(g)\pi_a(h) = \eta_a(g, h)\pi_a(gh) \quad g, h \in G_a$$

$$([a], \pi_a) \in \mathcal{C}/G$$

Flux-Charge composite
General Results

• The anyon model $\mathcal{C}/G = (\mathcal{C}_G^x)^G$ contains a sub-category Rep(G).
• $D^2_{\mathcal{C}/G} = D^2_{\mathcal{C}} |G|^2$. Same central charge.
• Gauging done sequentially if $N \subset G$ normal: first $N$ and then $G/N$.
• If $\mathcal{C}$ is a quantum double, then $\mathcal{C}/G$ a double.
• $\mathcal{C}$ and $\mathcal{C}/G$ same up to doubles.

• Inverse process of gauging:
  When $\text{Rep}(G)$ in $\mathcal{C}/G = (\mathcal{C}_G^x)^G$ condensed, $\mathcal{C}$ recovered.
Consider p-h symm. of $\mathbb{Z}_3$---No symm. fractionalization as $H^2(\mathbb{Z}_2, \mathbb{Z}_3)=0$.

**Defectification:**
Only one twist defect $g$ in $C_1$: $g \otimes g = 1 + a + \bar{a}$. This theory is NOT braided---Tambara-Yamagami theory for $\mathbb{Z}_3$. But it has a $G$-crossed braiding. There are two ways to have an defect as $H^3(\mathbb{Z}_2, U(1))=\mathbb{Z}_2$.

**Gauging:**
Taking the equivariant quotient results either $SU(2)_4$ or its cousin Jones-Kauffman theory at $r=6$---two metaplectic theories corresponding to the two classes in $H^3(\mathbb{Z}_2, U(1))=\mathbb{Z}_2$ as above.
Braided G-crossed $Z_3$-Tambara-Yamagami

The 6j symbols for the $Z_3$-Tambara-Yamagami theory is (unlisted admissible 6j symbols and R-symbols=1):

$$F_{g}^{agb} = F_{b}^{gag} = \chi(a,b), \quad F_{g,ab}^{ggg} = \frac{\kappa}{\sqrt{3}} \chi^{-1}(a,b),$$

where $\chi(a,b)$ is a symmetric bi-character of $Z_3$ and $\kappa = \pm 1$, $g=$defect and $a, b \in Z_3$.

It is known that this theory is NOT braided.

But it is G-crossed braided:

$$R_{g}^{ga} = R_{g}^{ag} = \omega^{2a^2} \quad \text{and} \quad R_{a}^{gg} = (-i\kappa)^{1/2} \omega^{a^2}, \quad a=0,1,2.$$
Modular G-crossed Category

• The extended Verlinde algebra has 4 sectors: $V_{0,0}$, $V_{0,1}$, $V_{1,0}$, $V_{1,1}$, and $\tilde{s}$-, $\tilde{t}$-matrices form a rep. of $SL(2,\mathbb{Z})$. Below the $s,t$ are those of the $\mathbb{Z}_3$ theory.

• The extended $\tilde{s}$-matrix $\tilde{s}=
\begin{pmatrix}
    s & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & -\kappa
\end{pmatrix}$

• The extended $\tilde{t}$ matrix $\tilde{t}=
\begin{pmatrix}
    t & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & (-i\kappa)^{1/2} \\
    0 & 0 & (-i\kappa)^{1/2} & 0
\end{pmatrix}$
Gauging As Construction of New UMTCs

• 3-fermion theory (toric code sister): $SO(8)_1$ with $G=S_3$

• $S$, $T$-matrices:

\[
\begin{array}{cccccccccccc}
1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 & 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} \\
1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 & -3\sqrt{2} & -3\sqrt{2} & -3\sqrt{2} & -3\sqrt{2} \\
2 & 2 & 4 & 6 & 6 & -4 & -4 & -4 & 0 & 0 & 0 & 0 \\
3 & 3 & 6 & -3 & -3 & 0 & 0 & 0 & -3\sqrt{2} & -3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} \\
3 & 3 & 6 & -3 & -3 & 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2} & -3\sqrt{2} & -3\sqrt{2} \\
4 & 4 & -4 & 0 & 0 & b & c & a & 0 & 0 & 0 & 0 \\
4 & 4 & -4 & 0 & 0 & c & a & b & 0 & 0 & 0 & 0 \\
4 & 4 & -4 & 0 & 0 & a & b & c & 0 & 0 & 0 & 0 \\
3\sqrt{2} & -3\sqrt{2} & 0 & -3\sqrt{2} & 3\sqrt{2} & 0 & 0 & 0 & 0 & 6 & -6 & 0 \\
3\sqrt{2} & -3\sqrt{2} & 0 & -3\sqrt{2} & 3\sqrt{2} & 0 & 0 & 0 & 0 & -6 & 6 & 0 \\
3\sqrt{2} & -3\sqrt{2} & 0 & 3\sqrt{2} & -3\sqrt{2} & 0 & 0 & 0 & 6 & -6 & 0 & 0 \\
3\sqrt{2} & -3\sqrt{2} & 0 & 3\sqrt{2} & -3\sqrt{2} & 0 & 0 & 0 & -6 & 6 & 0 & 0 \\
\end{array}
\]

\[
a = -8\cos\frac{2\pi}{9}, \quad b = -8\sin\frac{\pi}{9}, \quad c = 8\cos\frac{\pi}{9}.
\]


\begin{tabular}{|c|c|c|}
\hline
Label & $d$ & $\theta$ \\
\hline
$(I,+)$ & 1 & 1 \\
$(I,-)$ & 1 & 1 \\
{$a,a$} & 2 & 1 \\
$(Y,+)$ & 3 & $-1$ \\
$(Y,-)$ & 3 & $-1$ \\
{$w,w$} & 4 & $\alpha^{-1/3}$ \\
{$wa,\bar{w}a$} & 4 & $\omega\alpha^{-1/3}$ \\
{$w\bar{a},\bar{w}a$} & 4 & $\omega^2\alpha^{-1/3}$ \\
$(\sigma_+,+,)$ & $3\sqrt{2}$ & $e^{i\frac{5\pi}{8}}$ \\
$(\sigma_+,-)$ & $3\sqrt{2}$ & $-e^{-i\frac{4\pi}{8}}$ \\
$(\sigma_+,-)$ & $3\sqrt{2}$ & $-e^{-i\frac{4\pi}{8}}$ \\
$(\sigma_-,+) & $3\sqrt{2}$ & $e^{-i\frac{4\pi}{8}}$\\
\hline
\end{tabular}

$v = 1, \omega = e^{2\pi i/3}, \alpha = e^{4\pi i/3}$
Summary

We skeletonize an existing mathematical theory and formulate it into a physical theory with full computational power for symmetry, defects, and gauging of 2D topological phases. It provides a general framework to classify symmetry enriched 2D topological phases of matter.