Homotopy-theoretic approach to SPT phases in action: $\mathbb{Z}_{16}$ classification of three-dimensional superconductors

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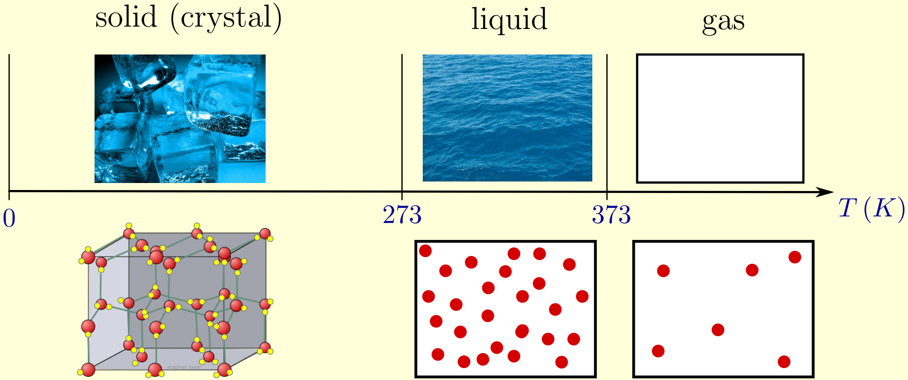
Phases of matter



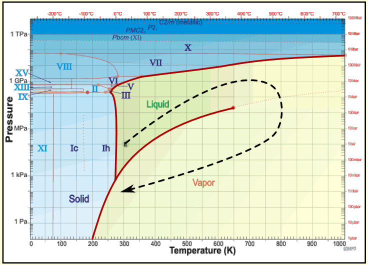
crystal $\not\sim$ liquid

crystal $\not\sim$ gas

liquid $\sim$ gas

Equivalence and non-equivalence

liquid $\sim$ gas because there is a continuous path between the phases



(need to elaborate what "continuous" means)

crystal $\not\sim$ liquid because there is no such path

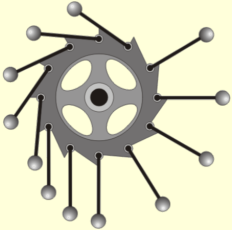
Rather, there is an *invariant* that distinguishes the phases, namely, the *symmetry group*:

– crystal has a discrete translational symmetry

– liquid has a continuous translational symmetry

Invariants in physics and mathematics

Perpetual motion



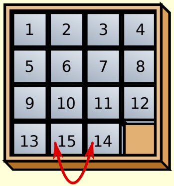
Won't work:

$\boxed{E = \mathrm{const}}$, where

$$E = \sum_j m_j g z_j + \sum_j \frac{m_j \vec{v}_j^{\,2}}{2}$$

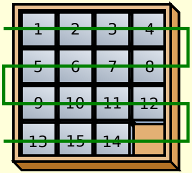
(continuous)

The 15-puzzle



Challenge:
Switch 15 and 14 by moving the pieces inside the box

Impossible:



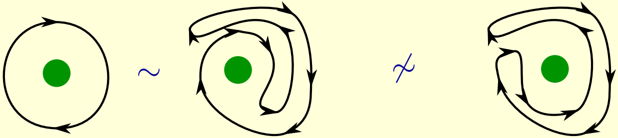
Define a permutation σ by the sequence of numbers along the green line

$$\sigma = (1, 2, 3, 4, 8, 7, 6, 5, 9, 10, 11, 12, 14, 15, 13)$$

$\boxed{\mathrm{sgn}(\sigma) = \mathrm{const}}$ (discrete)

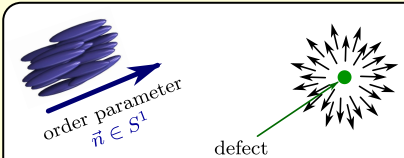
Topology: discrete invariants of continuous objects

- Example: Winding number (the dergee of a map $f:\ S^1 \to S^1$)



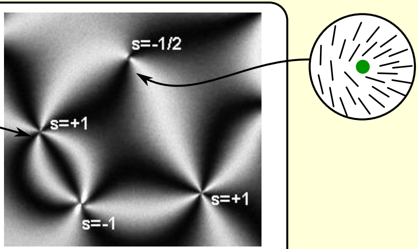
in this cases, $\deg f = -1$ $\qquad\qquad$ $\deg f = 0$

- Application: textures in liquid crystals



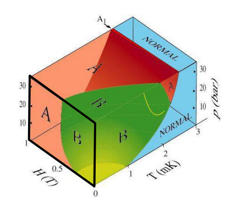
$$f:\ (\text{plane} - \text{defects})\ \to\ S^1$$

winding number is defined by restricting this map to a loop around the defect



nematic liquid crystal between crossed polarizers

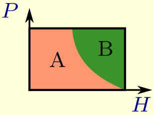
Parameter space



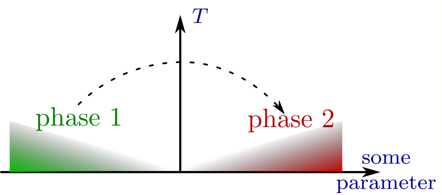
Phases of ^{3}He
in $H - P - T$ coordinates

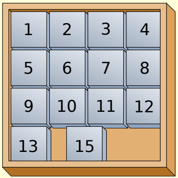
- Potentially infinite number of parameters
(because we may consider arbitrary Hamiltonians)

- We restrict our attention to zero temperature
(because quantum phases are fragile)



- Lifting this restriction may change the definition of a phase





Don't take pieces
out of the box!

Rules of the game

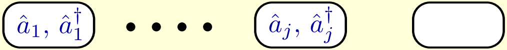
- We consider arbitrary Hamiltonians for fermions (e.g. electrons or ^{3}He atoms) with *local interactions* at $T = 0$.
- The ground state must be *gapped*. (By definition, a gapless state indicates a phase boundary.)
- The *symmetry* is fixed. In this talk:
  - no $\mathrm{U}(1)$ symmetry (the particle number is not conserved due to the presence of condensate),
  - but there is a time-reversal symmetry $\mathcal{T}$.
- We disallow any usual order parameter (i.e. spontaneous symmetry breaking) or topological ordering.
- Two variants of the game: the particles may or may not interact.

Majorana formalism (for discrete systems)

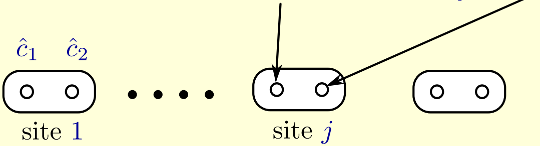
- Hamiltonian in terms of creation and annihilation operators

$$\hat{H} = \sum_{j,k} h_{jk} \hat{a}_j^\dagger \hat{a}_k + \sum_{j,k} \left(\Delta_{jk} \hat{a}_j \hat{a}_k + \Delta_{jk}^* \hat{a}_k^\dagger \hat{a}_j^\dagger \right)$$

spin (if any) is included in the index j



- Majorana operators: $\hat{c}_{2j-1} = \hat{a}_j + \hat{a}_j^\dagger$, $\hat{c}_{2j} = \dfrac{\hat{a}_j - \hat{a}_j^\dagger}{i}$



- Complete problem in terms of the Majorana operators

$$\boxed{\hat{H} = \frac{i}{4} \sum_{j,k} A_{jk} \hat{c}_j \hat{c}_k \,, \qquad \text{where} \quad \hat{c}_k \hat{c}_l + \hat{c}_l \hat{c}_k = 2\delta_{kl}, \quad \hat{c}_k^\dagger = \hat{c}_k}$$

Majorana formalism for continuous systems

- Hamiltonian:

dimensionality
of the physical space

$$\boxed{\hat{H} = \frac{i}{4} \int \eta^T \left(\sum_{j=1}^{n} \Gamma_\mu \partial_\mu + M \right) \eta \, dx}$$

(Any gapped free-fermion phase has a *representative* of this form)

Γ_μ is real symmetric, $\quad \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\delta_{\mu\nu}$

M is real skew-symmetric

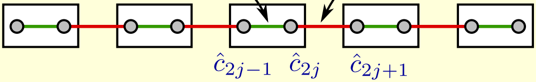
$$\left.\begin{aligned} &M^2 = -1 \\ &M\Gamma_\mu = -\Gamma_\mu M \end{aligned}\right\} \text{no loss of generality for the topological classification}$$

$\Gamma_1, \ldots, \Gamma_n$ are fixed

Different phase are characterized by different M

Example: Majorana wire ($n = 1$)

- Discrete version: $$\hat{H} = \frac{i}{2}\left(u \sum_{j=1}^{m} \hat{c}_{2j-1}\hat{c}_{2j} + v \sum_{j=1}^{m-1} \hat{c}_{2j}\hat{c}_{2j+1} \right)$$



- Continuum limit: $u = 1 - w\,, \quad v = 1 + w$ where $w \ll 1$

$$\hat{H} = \frac{i}{4} \int \eta^T \left(\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\Gamma_1} \frac{\partial}{\partial x} + \underbrace{\begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}}_{M} \right) \eta \, dx$$

In this equation, we may set $w = \pm 1$ (because there is no lattice)

- Different phases

Trivial: $w < 0\,;$ Nontrivial: $w > 0\,.$

Full classification is zero dimensions

(without symmetry)

- General form of matrix M (recall that $M^2 = -1$ *w.l.o.g.*)

$$M = S \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} S^{-1}$$

$S \in \mathrm{O}(2m)$

$M \in \mathrm{O}(2m)/\mathrm{U}(m)$

- The set of M's is a *classifying space for* 0*-dimensional states*:

$\mathcal{F}_0^{(free)} = \mathrm{O}/\mathrm{U}$ (A set that is *homotopy equivalent* to the set of ground states of all gapped free-fermion Hamiltonians)

- Topological invariant: $\mathrm{Pf}\, M \in \mathbb{Z}_2$

(describes the *connected components* of $\mathcal{F}^{(free)}$)

$$\left.\begin{array}{lll} \mathrm{Pf}\, M = +1 & - & \text{even number of particles} \\ \mathrm{Pf}\, M = -1 & - & \text{odd number of particles} \end{array}\right\} \text{counting particles in the ground state}$$

Full classification in all dimensions

(of gapped free-fermion systems without any symmetry)

n	$\mathcal{F}_n^{(free)}$	$\pi_0\left(\mathcal{F}_n^{(free)}\right)$	examples
0	O/U	$\mathbb{Z}_2$	even/odd number of particles
1	O	$\mathbb{Z}_2$	Majorana wire
2	BO $\times\ \mathbb{Z}$	$\mathbb{Z}$	$p_x + ip_y$ superconductor
3	U/O	0	
4	Sp/U	0	
5	Sp	0	
6	BO $\times\ \mathbb{Z}$	$\mathbb{Z}$	
7	U/Sp	0	

Time reversal symmetry

- General form: $$\boxed{\begin{aligned}&\mathcal{T}(\hat{c}_j) = \sum_k T_{jk}\hat{c}_k \\ &\mathcal{T}(i) = -i\end{aligned}}$$ for continuous systems, replace $\hat{c}_j$ with $\hat{\eta}_j(x)$

- Conventional TR symmetry: $T^2 = -1$

$$\mathcal{T}(\hat{a}_{j\uparrow}) = \hat{a}_{j\downarrow}$$
$$\mathcal{T}(\hat{a}_{j\downarrow}) = -\hat{a}_{j\uparrow}$$

$$\mathcal{T}\begin{pmatrix}\hat{c}_{2j-1,\uparrow}\\ \hat{c}_{2j-1,\downarrow}\\ \hat{c}_{2j,\uparrow}\\ \hat{c}_{2j,\downarrow}\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}}_{T}\begin{pmatrix}\hat{c}_{2j-1,\uparrow}\\ \hat{c}_{2j-1,\downarrow}\\ \hat{c}_{2j,\uparrow}\\ \hat{c}_{2j,\downarrow}\end{pmatrix}$$

- Unconventional TR symmetry: $T^2 = 1$

$\mathcal{T}(\hat{c}_l) = \hat{c}_l$ or $\mathcal{T}(\hat{c}_l) = -\hat{c}_l$

(for spinless systems)

Application to 3D superconductors (finally!)

- General form of the Hamiltonian

$$\hat{H} = \frac{i}{4} \int \eta^T \left(\sum_{j=1}^{3} \Gamma_\mu \partial_\mu \, + \, M \right) \eta \, dx$$

$$\boxed{\begin{gathered} \Gamma_1 = \sigma^x \otimes I \otimes I, \qquad \Gamma_2 = \sigma^z \otimes I \otimes I, \qquad \Gamma_3 = (i\sigma^y) \otimes (i\sigma^y) \otimes I \\ M = (i\sigma^y) \otimes \big(\sigma^x \otimes m + \sigma^z \otimes m' \big) \end{gathered}}$$

- TR symmetry: $\quad T = (i\sigma^y) \otimes \sigma^z \otimes I \quad$ (anticommutes with Γ_μ)

$\boxed{\mathcal{T}(M) = -TMT^{-1}} \qquad \mathcal{T}(M) = M \quad \Rightarrow \quad \boxed{M = (i\sigma^y) \otimes \sigma^x \otimes m}$

m is a real symmetric matrix with eigenvalues ± 1

- Topological invariant

$$\boxed{\nu = \text{(number of } +1 \text{ eigenvalues)} - \text{(number of } -1 \text{ eigenvalues)}}$$

Main question

Is ν well-defined in the presence of interactions? In other words,

Can a $\nu = 0$ state be continuously changed into a $\nu \neq 0$ state?

(The intermediate states may include interparticle interactions, but the energy gap must never close.)

Claim: In the presence of interaction, ν is defined modulo 16.

- The $\nu = 16$ phase is connected to the trivial phase by a continuous path. (Shown by explicit construction.)
- If $\nu \not\equiv 0 \pmod{16}$, then there is no continuous path.
  - One has to consider *all* possible quantum states with suitable restrictions: an exact definition is needed.
  - The question can be reduced to the classification of cross sections of a certain fibration up to homotopy (a typical homotopy theory problem).

Acknowledgements

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for teaching me some relevant mathematics.

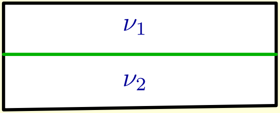
Related (somewhat simpler) question

The boundary between different phases supports gapless modes.

Can those modes be suppressed by suitable interactions without breaking the TR symmetry or creating a topological order?

- Effective boundary theory

$$\hat{H} = \frac{i}{4} \int \eta^T \left(\sum_{j=1}^{2} \Gamma_\mu \partial_\mu \right) \eta \, dx$$



$\nu = \nu_1 - \nu_2$

$$\Gamma_1 = \sigma^x \otimes I, \qquad \Gamma_2 = \sigma^z \otimes I$$

- Possible mass term

$\boxed{M = (i\sigma^y) \otimes m \,, \quad \text{where } m \text{ is a real symmetric } \nu \times \nu \text{ matrix}}$

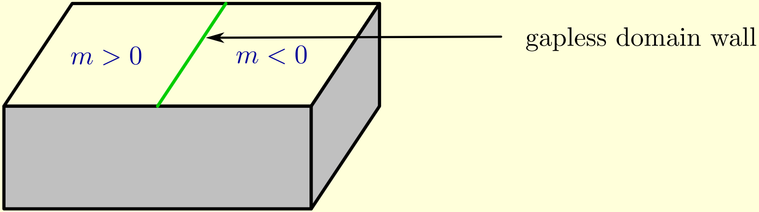
(more generally, a matrix of any size such that
(# of positive e.v.) − (# of negative e.v.) = ν)

- However, $\boxed{\mathcal{T}(m) = -m}$

If $m \neq 0$, then
the symmetry is broken

Dynamic surface mass terms

- Key idea: Let $m = m(x,t)$ fluctuate (as a quantum Bose field)
- Claim: Ergodic, symmetry non-breaking dynamics can be arranged (using a suitable σ-model) *if there are no topological obstructions*
- Example of a topological obstruction (for $\nu = 1$)



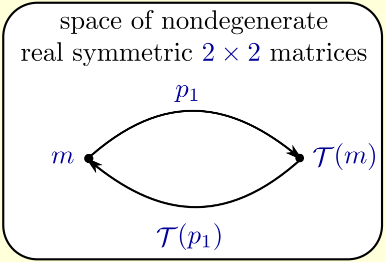
If $m > 0$ then $\mathcal{T}(m) < 0$. There is no continuous path between m and $\mathcal{T}(m)$ in the space of nondegenerate mass terms.

The second obstruction (for $\nu = 2$)

- Resolving the previous obstruction (now m is a 2×2 matrix)

  Let $m = \sigma^z$ be an admissible value of the dynamic mass term.

  Then $\mathcal{T}(m) = -\sigma^z$



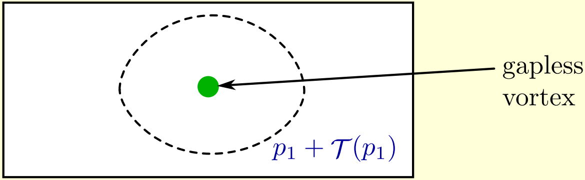
Path from m to $\mathcal{T}(m)$:

$p_1(\theta) = (\cos\theta)\,\sigma^z + (\sin\theta)\,\sigma^x\,,$

where $\theta \in [0, \pi]$.

- New obstruction: No way to interpolate between p_1 and $(\mathcal{T}(p_1))^{-1}$

In the real space:



Further steps

- For $\nu = 4$, the obstruction corresponds to a soft-core soliton

- For $\nu = 8$, one can define a σ-model with the target space S^3

$$m(x,t) = \sum_{k=1}^{4} u_k(x,t)\, m_k, \quad \text{where} \quad u \in S^3, \quad \text{i.e.} \sum_{k=1}^{4} u_k^2 = 1$$

$$m_1 = \sigma^z \otimes I \otimes I, \qquad m_2 = \sigma^x \otimes I \otimes I,$$
$$m_3 = (i\sigma^y) \otimes (i\sigma^y) \otimes I, \qquad m_4 = (i\sigma^y) \otimes \sigma^x \otimes (i\sigma^y)$$

  - The model is TR invariant if we assume that $\mathcal{T}(u) = -u$
  - $m(x,t)$ is nondegenerate because $m_j m_k + m_k m_j = 2\delta_{jk}$

  However, the system is gapless due to a nontrivial WZW term

- For $\nu = 16$, the WZW term vanishes, which can be shown by extending the target space to S^5. The system is fully gapped.

Nontriviality of the $\nu = 8$ phase

- There could be a different, gapped σ-model for $\nu = 8$.

  It would correspond to a map f from S^5 to the space of nondegenerate real symmetric matrices of size 8×8 such that $f(-u) = -f(u)$.

- Reduction to a homotopy theory problem

  Each value of ν corresponds to a cross section of a certain fibration over the classifying space of the symmetry group: $\mathrm{B}\mathbb{Z}_2 = \mathbb{R}P^\infty$.

- Algebraic tools

  Atiah-Hirzebruch spectral sequence

- Result

  The $\nu = 0$ and $\nu = 8$ sections are *not* fiber-wise homotopic.

More general claim: SRE $\Rightarrow$ SPT

- Short-range entangled states in dimension n form some topological space $\mathcal{B}_n$ (for bosons) or $\mathcal{F}_n$ (for fermions).

$$\mathcal{B}_0 = \mathbb{C}P^\infty = K(Z,2), \qquad \mathcal{F}_0 = \mathbb{C}P^\infty \times \mathbb{Z}_2$$

  $\mathcal{B}_n$ and $\mathcal{F}_n$ are also known in dimensions $n = 1, 2$.

- $\mathcal{B}$ and $\mathcal{F}$ are *homotopy spectra*, i.e. $\mathcal{B}_n = \Omega(\mathcal{B}_{n+1}),\ \mathcal{F}_n = \Omega(\mathcal{F}_{n+1})$.

- SPT states with symmetry group G are given by the generalized cohomology $H^n(BG, \mathcal{B})$ or$H^n(BG, \mathcal{F})$. (For the TRS, we need a twisted version.)

- Approximations:

$$K(\mathbb{Z}, n+2) \to \mathcal{B}_n \qquad D_{n+2} \to \mathcal{F}_n$$

$$H^{n+2}(BG, \mathbb{Z}) \to H^n(BG, \mathcal{B}) \qquad H^{n+2}(BG, D) \to H^n(BG, \mathcal{F})$$

What do we know about $\mathcal{F}_3$?

- Begin with $\mathcal{F}_3^{(free)} = \mathrm{Sp/U} = \{M : \ M\Gamma_\mu = -\Gamma_\mu M, \ M^2 = 1\}$ for $\mu = 1, 2, 3$

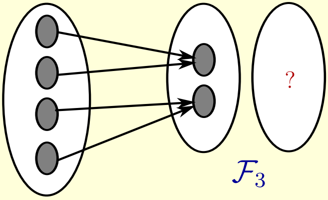
- Consider the map $\mathcal{F}_3^{(free)} \to \mathcal{F}_3$

k	$\pi_k(\mathcal{F}_3^{(free)})$	$\pi_k(\mathcal{F}_3)$
0	0	?
1	$\mathbb{Z}$	$\mathbb{Z}$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$	$\mathbb{Z}_2$
4	0	0
5	$\mathbb{Z}$	$\mathbb{Z}$
6	0	0
7	0	0
8	0	0
9	$\mathbb{Z}$	0
$\vdots$	$\vdots$	$\vdots$

Connected components of $\mathcal{F}_3$ $\sim$ $(\mathrm{Sp/U})_5$ (Postnikov truncation).

Fixed points of the symmetry group (discussion)

SPT phases of the time-reversal symmetry are *connected components of the fixed point space* of a certain action of $G = \mathbb{Z}_2$ on $\mathcal{F}_3^{(free)}$ or $\mathcal{F}_3$.

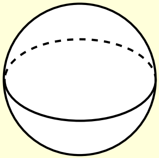


$\mathcal{F}_3$

$\mathcal{F}_3^{(free)}$

- Idea: Reduce the fixed point problem to pure homotopy theory
- *A priory*, it might not work. Example: G acts of EG as well as on a point: (no fixed points vs 1 fixed point). But $EG \sim \mathrm{pt}$.

$$\begin{array}{ccc} EG & \qquad & E\mathbb{Z}_2 = S^\infty \\ \downarrow p & & \downarrow p \\ BG & & B\mathbb{Z}_2 = RP^\infty \end{array}$$



Reductons

$$\begin{tikzcd} \text{Fixed points} \arrow[r, "\text{automatic}"] & \text{Homotopy fixed points} \arrow[l, bend left, "\text{works for } \mathcal{F} \text{ (but not } \mathcal{F}^{(free)}\text{)}"] \end{tikzcd}$$

$$\boxed{(\text{Homotopy fixed point in } X) \overset{def}{=} (G\text{-equivariant map } f : EG \to X)}$$

- Alternative description of homotopy fixed points using the Borel construction: X is replaced with $\widetilde{X} = (X \times EG)/G$.

$$\begin{tikzcd} \widetilde{X} = (X \times EG)/G \arrow[d, "p \text{ (defined by the group action on } X\text{)}"] \\ BG = EG/G \arrow[u, bend right, dashed, "\widetilde{f}"'] \end{tikzcd}$$

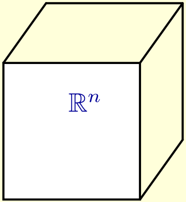
$$\widetilde{f} : EG \to (X \times EG)/G$$

$$\widetilde{f}(v) = (f(v), v) \bmod G$$

This description is purely homotopic (does not use the group action)

From homotopy f.p. to actual f.p.

- Let $f : EG \to \mathcal{F}_n$ be a G-equivariant map. We try to define a G-invariant quantum state as a uniform superposition over all maps $m : \mathbb{R}^n \to EG$ (fluctuating order parameter).



$$|\Psi\rangle = \int |m\rangle \otimes \underbrace{\big|\psi(f \circ m)\big\rangle}_{\substack{\text{state of fermions that looks like} \\ f(m(x)) \in \mathcal{F}_n \text{ near each point } x \in \mathbb{R}^n}} Dm$$

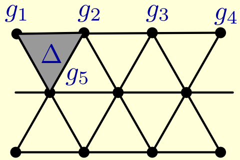
- Issues:
  - Too big a space to integrate over. (Need some cutoff.)
  - Relative phase factors of $|\psi(f \circ m)\rangle$.

Lattice regularization

- Degrees of freedom: Spins $g_1, g_2, \ldots \in G$ on lattice sites and continuous fermions in between.

- We use Milnor's model: $EG = G * G * G * \cdots$ (infinite join).
  For each k, define $X_k = \underbrace{G * \cdots * G}_{k+1 \text{ copies of } G} \subseteq EG$.
  Note that $\pi_0(X_k) = \pi_{k-1}(X_k) = 0$.

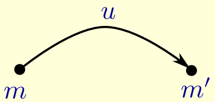
- For each n-simpex Δ with vertex spins $g_{s_0}, \ldots, g_{s_n}$ there is a standard map $m_{g_{s_0},\ldots,g_{s_n}} : \Delta \to X_n$ (by the definition of the join). We construct a map $m_{g_1,\ldots,g_N} : \mathbb{R}^n \to X_n$ from such local patches.



- Finally: $$|\Psi\rangle = \sum_{g_1,\ldots,g_N} |g_1, \ldots, g_N\rangle \otimes \big|\psi(f \circ m_{g_1,\ldots,g_N})\big\rangle$$

Phase factors

Step1: $$\boxed{\left|\psi(f\circ m')\right\rangle = U(u)\left|\psi(f\circ m)\right\rangle}$$



adiabatic evolution over path
$u : \mathbb{R}^n \times [0,1] \to X_{n+1} \subseteq EG$

Berry phase

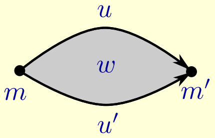
- New problem: Consider two paths, u and u'.

  $U(u)^{-1}U(u) = e^{2\pi i\varphi(u,u')}$. Step 1 only works if $\varphi(u,u') = 0$.

  To achieve that, we will modify $U(u)$ with *local counter-terms*.

Step 2: Consider u, $u' : \mathbb{R}^n \times [0,1] \to EG$.

Since $\pi_{n+1}(EG) = 0$, there exists some $w : \mathbb{R}^n \times [0,1]^2 \to EG$



$$\boxed{\varphi(u,u') = \int_{\mathbb{R}^n\times[0,1]^2} w^*(f^*\Omega)}$$ — WZW action

(Ω is a G-invariant $(n+2)$-form on $\mathcal{F}_n$.)

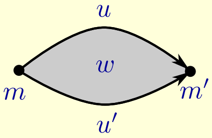
Cancelling the phase factors

Step 3: The WZW action is topologically trivial in this case because we are working in the contractible space EG.

$$f^*\Omega \text{ is an } (n+2)\text{-form on } EG, \qquad \boxed{f^*\Omega = d\omega}$$

Recall that Ω is invariant under the group action. By averaging ω over G, we can make it invariant too.

Step 4: Define $$\boxed{V(u) = U(u)\cdot\exp\left(-2\pi i\int_{\mathbb{R}^n\times[0,1]} u^*(\omega)\right)}$$



$$V(u)^{-1}V(u) = e^{2\pi i\varphi(u,u')}\cdot\exp\left(-2\pi i\underbrace{\left(\int (u')^*(\omega) - \int u^*(\omega)\right)}_{\int w^*(d\omega)=\varphi(u,u')}\right) = 1.$$

Conclusion

The classification of SPT phases splits into two problems:

- Find the spaces $\mathcal{B}_n$ and $\mathcal{F}_n$.
  - Known for $n = 0, 1$.
  - Sort of known for $n = 2$.
  - Solving the problem in higher dimensions (and proving the answer for $n = 2$) requires some new methods.
- Twisted generalized cohomoly: Classify cross sections of a certain fiber bundle with fiber $\mathcal{B}_n$ (or $\mathcal{F}_n$) over BG.
  - Can be solved using homotopy theory tools (Atiyah-Hirzebruch spectral sequence).
  - Requires advanced technical skills.