TOPOLOGICAL FIELD THEORY FOR DEFECTS IN TOPOLOGICAL PHASES
Theme:

2-d defects in 3-d TFT as models for line defects in topological phases

some overlap with Z. Wang’s talk — “approach quite different”
Motivation

**Themes:**

- 2-d defects in 3-d TFT as models for line defects in topological phases (TFT as tool)
- 3-d TFT with defects of any codimension
**Motivation**

**TFT for topological defects**

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**Possible motivations:**
- Topological line defects in topological phases
- Gapped interfaces between topological phases
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TFT for topological defects

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- 2-d defects in 3-d TFT as models for line defects in topological phases (TFT as tool)
- 3-d TFT with defects of any codimension

Possible motivations:
- Topological line defects in topological phases
- Gapped interfaces between topological phases
- TFT with substructures / on stratified spaces
- Extended TFT / higher categories
- Defects in general quantum field theory
- Applications to 2-d rational conformal field theory
Defects in QFT

TFT for topological defects

- Codimension-1 defect $\boxed{\text{QFT}_1 | \text{QFT}_2}$
  - interface separating region supporting $\text{QFT}_1$ from region supporting $\text{QFT}_2$
Defects in QFT

TFT for topological defects

- Codimension-1 defect $\text{QFT}_1 \ \parallel \ \text{QFT}_2$
  - interface separating region supporting $\text{QFT}_1$ from region supporting $\text{QFT}_2$
  - ubiquitous in nature
  - natural part of the structure of quantum field theory
  - *physical boundaries* as special case
Defects in QFT

- Codimension-1 defect $QFT_1 \mid QFT_2$
  = interface separating region supporting $QFT_1$ from region supporting $QFT_2$
  - ubiquitous in nature
  - natural part of the structure of quantum field theory
  - physical boundaries as special case

- Topological defect: correlators do not change when deforming the defect without crossing other substructures

- Example: 2-d Ising model
  - ferromagnetic nearest-neighbour interaction
  - change coupling to anti-ferromagnetic on all bonds crossed by some line
    $\leadsto$ topological defect line
Defects in QFT

- **Codimension-1 defect** $\text{QFT}_1 \mid \text{QFT}_2$
  - interface separating region supporting $\text{QFT}_1$ from region supporting $\text{QFT}_2$
  - ubiquitous in nature
  - natural part of the structure of quantum field theory
  - *physical boundaries* as special case

- **Topological defect**: correlators do not change when deforming the defect without crossing other substructures

- Some general features of topological defects:
  - codimension-2 defects $\text{def}_1 \mid \text{def}_2$ etc
  - transparent defect
  - invert orientation $\sim$ dual defect
  - move two topological defects to coincidence $\sim$ fusion product of defects
Defects in QFT

TFT for topological defects

- Codimension-1 defect $\text{QFT}_1 \parallel \text{QFT}_2$
  
  \[ \text{interface separating region supporting } \text{QFT}_1 \text{ from region supporting } \text{QFT}_2 \]
  
  - ubiquitous in nature
  - natural part of the structure of quantum field theory
  - \textit{physical boundaries} as special case

- \textit{Topological defect}: correlators do not change when deforming the defect without crossing other substructures

- Some general features of topological defects:
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  - transparent defect
  - invert orientation $\rightsquigarrow$ dual defect
  - move two topological defects to coincidence $\rightsquigarrow$ fusion product of defects

- \textit{Mathematical formulation}: $\rightsquigarrow$ higher categories
assume: defects form a rigid monoidal category (proven for 2-d RCFT)

Fjelstad-F-Runkel-Schweigert 2008

Fröhlich-F-Runkel-Schweigert 2007

Kapustin-Saulina 2011
Invertible defects and symmetries

TFT for topological defects

assume: defects form a rigid monoidal category
(proven for 2-d RCFT)

Subclass: invertible topological defects:

\[ D \otimes D^\vee \cong 1 \cong D^\vee \otimes D \]
assume: defects form a rigid monoidal category
( proven for 2-d RCFT )

- Subclass: \textit{invertible} topological defects:
  \[
  D \otimes D^\vee \cong \mathbb{1} \cong D^\vee \otimes D
  \]

- Basic property:
  \[
  D \rightarrow D^\vee = \text{dim}(D) \rightarrow D^\vee
  \]
  drawn for $d = 2$
  \[
  \text{dim}(D) = \pm 1
  \]

\[\leadsto\] identity of correlators when applied locally in any configuration of fields & defects
Invertible defects and symmetries

TFT for topological defects

assume: defects form a rigid monoidal category
(proven for 2-d RCFT)

**Subclass:** invertible topological defects:

\[
D \otimes D^\vee \cong 1 \cong D^\vee \otimes D
\]

**Basic property:**

\[
D \quad D^\vee
\]

\[= \dim(D)\]

\[= \text{identity of correlators when applied locally in any configuration of fields & defects}\]

\[= \text{invertible defects form a group under fusion}\]

\[= \text{act on all data of the theory as a symmetry group}\]

\[= \text{e.g. critical 2-d Ising model: } \mathbb{Z}_2\]

\[= \text{critical three-state Potts model: } \mathbb{S}_3\]
invertible defects and symmetries

TFT for topological defects

assume: defects form a rigid monoidal category
(proven for 2-d RCFT)

Subclass: invertible topological defects:

\[ D \otimes D^\vee \cong 1 \cong D^\vee \otimes D \]

Basic property:

\[ \text{identity of correlators when applied locally in any configuration of fields } \& \text{ defects} \]

invertible defects form a group under fusion

act on all data of the theory as a symmetry group

Example: equalities for bulk field correlators on sphere:

\[ \text{identity of correlators when applied locally in any configuration of fields } \& \text{ defects} \]

invertible defects form a group under fusion

act on all data of the theory as a symmetry group

Example: equalities for bulk field correlators on sphere:
Wrapping of general topological defect around a bulk field:

\[ \phi_D = \sum \text{intermediate defects } D_i \]
Wrapping of general topological defect around a bulk field:

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bulk field turned into disorder field
Wrapping of general topological defect around a bulk field:

\[ \phi_D = \sum_{\text{intermediate defects } D_i} \phi \]

- bulk field turned into disorder field
- wrapping with dual defect turns disorder field back to bulk field if and only if \( D \otimes D^\vee \) is direct sum of invertible defects
- in this case have an order-disorder duality
  - e.g., critical 2-d Ising model: remnant of Kramers-Wannier duality
- again action on all field theoretic quantities
Duality defects

Wrapping of general topological defect around a bulk field:

\[ \phi D = \sum \text{intermediate defects } D_i \]

- bulk field turned into disorder field

- wrapping with dual defect turns disorder field back to bulk field if and only if \( D \otimes D^\vee \) is direct sum of invertible defects

Example: correlator of two Ising spin fields on a torus:

\[
\begin{align*}
\sigma \sigma = & \quad \frac{1}{2} \quad \epsilon_{\mu}^\mu + \frac{1}{2} \\
& \quad + \frac{1}{2} \quad \epsilon_{\mu}^\mu + \frac{1}{2}
\end{align*}
\]
Plan

TFT for topological defects

**Goal**: Similar results for defects in 3-d TFT

**Tasks**:
- Achieve basic understanding of topological defects in 3-d TFT
- Study consequences in relevant classes of models
- Apply insight to topological phases
- Construct 3-d TFT with topological defects mathematically
**Plan**

**TFT for topological defects**

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**Plan**:

- Codimension-1 defects in QFT ✓
Plan

TFT for topological defects

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**Plan:**
- Codimension-1 defects in QFT ✓
- Topological defects in 3-d TFT of Reshetikhin-Turaev type
- Application: Multi-layer systems
- Appendix: Defects in Dijkgraaf-Witten theories
**Plan**

**TFT for topological defects**

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- Codimension-1 defects in QFT ✓
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- Appendix: Defects in Dijkgraaf-Witten theories

**Collaborators:** Jan Priel, Gregor Schaumann, Christoph Schweigert, Alessandro Valentino
**RT-type TFT with defects**

- **RT-type TFT**: symmetric monoidal functor
  \[
  \text{tft}^D_{3,2}: \text{Cobord}_{3,2} \to \text{Vect}
  \]
  resp.
  \[
  \text{tft}^D_{3,2,1}: \text{Cobord}_{3,2,1} \to 2\text{-Vect}
  \]

  - **input**: a modular tensor category \( \mathcal{D} \)
RT-type TFT with defects

RT-type TFT: symmetric monoidal functor \( tft_{3,2}^D : \text{Cobord}_{3,2} \to \text{Vect} \)

resp. 2-functor \( tft_{3,2,1}^D : \text{Cobord}_{3,2,1} \to 2\text{-Vect} \)

- input: a modular tensor category \( \mathcal{D} \)
- Wilson lines (ribbons) in three-manifolds labeled by objects of \( \mathcal{D} \)
- insertions on Wilson lines / junctions labeled by morphisms of \( \mathcal{D} \)
- 2-d cut-and-paste boundaries on which Wilson lines can end
- state spaces for cut-and-paste boundaries = morphisms spaces \( \text{Hom}_\mathcal{D}(X, 1) \)
RT-type TFT with defects

RT-type TFT: symmetric monoidal functor
\[ \text{tft}_{3,2}^D : \text{Cobord}_{3,2} \rightarrow \text{Vect} \]
resp. 2-functor
\[ \text{tft}_{3,2,1}^D : \text{Cobord}_{3,2,1} \rightarrow \text{2-Vect} \]

- input: a modular tensor category \( D \)
- Wilson lines (ribbons) in three-manifolds labeled by objects of \( D \)
- insertions on Wilson lines / junctions labeled by morphisms of \( D \)
- 2-d cut-and-paste boundaries on which Wilson lines can end
- state spaces for cut-and-paste boundaries = morphisms spaces \( \text{Hom}_D(X,1) \)

RT-type TFT with boundaries and defects:

- include in \( \text{Cobord} \) three-manifolds with physical boundary
- include in \( \text{Cobord} \) three-manifolds with surface defects
RT-type TFT with defects

- **RT-type TFT**: symmetric monoidal functor $\text{tft}^{D}_{3,2} : \text{Cobord}_{3,2} \rightarrow \text{Vect}$
  
  resp. 2-functor $\text{tft}^{D}_{3,2,1} : \text{Cobord}_{3,2,1} \rightarrow 2\text{-Vect}$

  - input: a modular tensor category $D$
  - Wilson lines (ribbons) in three-manifolds labeled by objects of $D$
  - insertions on Wilson lines / junctions labeled by morphisms of $D$
  - 2-d cut-and-paste boundaries on which Wilson lines can end
  - state spaces for cut-and-paste boundaries = morphisms spaces $\text{Hom}_D(X, 1)$

- **RT-type TFT with boundaries and defects**:
  - include three-manifolds with physical boundary and/or surface defects
  - 3-d bulk regions labeled by modular tensor categories $D_1, D_2, \ldots$
    (bulk Wilson lines in such a region labeled by objects of $D_i$)
  - boundary Wilson lines and defect Wilson lines
  - several layers of insertions and of junctions
RT-type TFT with defects

- RT-type TFT: symmetric monoidal functor $\text{tft}^{D}_{3,2} : \text{Cobord}_{3,2} \to \text{Vect}$
  - resp. 2-functor $\text{tft}^{D}_{3,2,1} : \text{Cobord}_{3,2,1} \to 2\text{-Vect}$

  - input: a modular tensor category $D$

  - Wilson lines (ribbons) in three-manifolds labeled by objects of $D$

  - insertions on Wilson lines / junctions labeled by morphisms of $D$

  - 2-d cut-and-paste boundaries on which Wilson lines can end

  - state spaces for cut-and-paste boundaries = morphisms spaces $\text{Hom}_D(X, 1)$

- RT-type TFT with boundaries and defects:

  - Task: construct symmetric monoidal 2-functor $\text{Cobord}^{\partial}_{3,2,1} \to 2\text{-Vect}$
    - for category of cobordisms with corners
RT-type TFT with defects

- RT-type TFT: symmetric monoidal functor \( \text{tft}^{D}_{3,2} : \text{Cobord}_{3,2} \rightarrow \text{Vect} \)

  resp.

  2-functor \( \text{tft}^{D}_{3,2,1} : \text{Cobord}_{3,2,1} \rightarrow 2\text{-Vect} \)

- input: a modular tensor category \( D \)

- Wilson lines (ribbons) in three-manifolds labeled by objects of \( D \)

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- 2-d cut-and-paste boundaries on which Wilson lines can end

- state spaces for cut-and-paste boundaries = morphisms spaces \( \text{Hom}_D(X, 1) \)

- RT-type TFT with boundaries and defects:

  Task: construct symmetric monoidal 2-functor \( \text{Cobord}_{3,2,1}^\partial \rightarrow 2\text{-Vect} \) for category of cobordisms with corners

  In particular:

  - determine labels for physical boundaries / for surface defects
  - determine labels for boundary and defect Wilson lines and for insertions

  Conjecture: Fit together to form bicategories of module categories
Select boundary “a” to some bulk region labeled by a modular tensor category $C$ can contain boundary Wilson lines. Wilson line can contain insertions. Insertions can be composed.

$\rightsquigarrow$ category $\mathcal{W}_a$ of Wilson lines on boundary $a$. 
Select boundary “a” to some bulk region labeled by a modular tensor category $C$ can contain boundary Wilson lines. Wilson line can contain insertions. Insertions can be composed. Boundary Wilson lines can be fused and can be deformed.

$\leadsto$ rigid monoidal category $\mathcal{W}_a$ of Wilson lines on boundary $a$
Select boundary “$a$” to some bulk region labeled by a modular tensor category $C$
- can contain boundary Wilson lines
- Wilson line can contain insertions
- insertions can be composed
- boundary Wilson lines can be fused and can be deformed
- also impose: finitely semisimple etc

$spherical$ fusion category $\mathcal{W}_a$ of Wilson lines on boundary $a$
Select boundary “a” to some bulk region labeled by a modular tensor category $\mathcal{C}$

- fusion category $\mathcal{W}_a$ of Wilson lines on boundary $a$

Postulate process of moving bulk Wilson lines to boundary

- functor $F_a : \mathcal{C} \to \mathcal{W}_a$
Select boundary “a” to some bulk region labeled by a modular tensor category $\mathcal{C}$

$\rightsquigarrow$ fusion category $\mathcal{W}_a$ of Wilson lines on boundary $a$

Postulate process of moving bulk Wilson lines to boundary

$\rightsquigarrow$ functor $F_a : \mathcal{C} \to \mathcal{W}_a$

Impose compatibility of fusion in bulk and boundary

$\rightsquigarrow$ monoidal structure $F_a(U \otimes_C V) \xrightarrow{\sim} F_a(U) \otimes \mathcal{W}_a F_a(V)$
Select boundary “a” to some bulk region labeled by a modular tensor category \( \mathcal{C} \)

- fusion category \( \mathcal{W}_a \) of Wilson lines on boundary \( a \)

Postulate process of moving bulk Wilson lines to boundary

- functor \( F_a : \mathcal{C} \rightarrow \mathcal{W}_a \)

Impose compatibility of fusion in bulk and boundary

- monoidal structure \( F_a(U \otimes \mathcal{C} V) \cong F_a(U) \otimes \mathcal{W}_a F_a(V) \)

Impose independence from details of bulk-to-boundary process

- central structure \( F_a(U) \otimes \mathcal{W}_a X \cong X \otimes \mathcal{W}_a F_a(U) \)
Select boundary \( "a" \) to some bulk region labeled by a modular tensor category \( \mathcal{C} \)

\( \rightsquigarrow \) fusion category \( \mathcal{W}_a \) of Wilson lines on boundary \( a \)

Postulate process of moving bulk Wilson lines to boundary

\( \rightsquigarrow \) functor \( F_a : \mathcal{C} \to \mathcal{W}_a \)

Impose compatibility of fusion in bulk and boundary

\( \rightsquigarrow \) monoidal structure

\[ F_a(U \otimes_C V) \xrightarrow{\simeq} F_a(U) \otimes \mathcal{W}_a F_a(V) \]

Impose independence from details of bulk-to-boundary process

\( \rightsquigarrow \) central structure

\[ F_a(U) \otimes \mathcal{W}_a X \xrightarrow{\simeq} X \otimes \mathcal{W}_a F_a(U) \]

equivalently: choice of lift to Drinfeld center of \( \mathcal{W}_a \)
Select boundary "a" to some bulk region labeled by a modular tensor category $C$

\[ \rightsquigarrow \text{fusion category } \mathcal{W}_a \text{ of Wilson lines on boundary } a \]

Postulate process of moving bulk Wilson lines to boundary

\[ \rightsquigarrow \text{functor } F_a : C \to \mathcal{W}_a \]

Impose compatibility of fusion in bulk and boundary

\[ \rightsquigarrow \text{monoidal structure } F_a(U \otimes_C V) \congrightarrow F_a(U) \otimes_{\mathcal{W}_a} F_a(V) \]

Impose independence from details of bulk-to-boundary process

\[ \rightsquigarrow \text{central structure } F_a(U) \otimes_{\mathcal{W}_a} X \congrightarrow X \otimes_{\mathcal{W}_a} F_a(U) \]

Postulate naturality:

only reason for being able to consistently move boundary Wilson line $Y \in \mathcal{W}_a$
past any $X \in \mathcal{W}_a$ should be that $Y = F_a(U)$ for some $U \in C$

\[ \rightsquigarrow \text{braided equivalence } C \congrightarrow \mathcal{Z}(\mathcal{W}_a) \]
Categories of boundary Wilson lines

- Select boundary “a” to some bulk region labeled by a modular tensor category $C$
  - fusion category $\mathcal{W}_a$ of Wilson lines on boundary $a$
- Postulate process of moving bulk Wilson lines to boundary
  - functor $F_a : C \to \mathcal{W}_a$
- Impose compatibility of fusion in bulk and boundary
  - monoidal structure $F_a(U \otimes_C V) \cong F_a(U) \otimes \mathcal{W}_a F_a(V)$
- Impose independence from details of bulk-to-boundary process
  - central structure $F_a(U) \otimes \mathcal{W}_a X \cong X \otimes \mathcal{W}_a F_a(U)$
- Postulate naturality:
  - only reason for being able to consistently move boundary Wilson line $Y \in \mathcal{W}_a$
    past any $X \in \mathcal{W}_a$ should be that $Y = F_a(U)$ for some $U \in C$
  - braided equivalence $C \cong \mathcal{Z}(\mathcal{W}_a)$

In short: Compatible boundary condition for bulk region $C$

$= Witt$ trivialization $\tilde{F}_a : C \cong \mathcal{Z}(\mathcal{W}_a)$ for some fusion category $\mathcal{W}_a$
Thus for single boundary condition $a$:

$\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$

In particular, obstruction: no compatible boundary condition unless $[\mathcal{C}] = 0$ in Witt group of modular tensor categories.
Thus for single boundary condition $a$:

\[ C \xrightarrow{\sim} \mathbb{Z}(W_a) \]

- in particular, obstruction: no compatible boundary condition unless \([C] = 0\) in Witt group of modular tensor categories

Other boundary condition $b$:

- other fusion category $\mathcal{W}_b$ of Wilson lines in region $b$
Thus for single boundary condition $a$:

\[ C \xrightarrow{\sim} \mathbb{Z}(\mathcal{W}_a) \]

in particular obstruction: no compatible boundary condition unless $[C] = 0$

in Witt group of modular tensor categories.

Other boundary condition $b$:

- category $\mathcal{W}_{a,b}$
  - of Wilson lines separating boundary region labeled $a$ from region labeled $b$
  - fusion of Wilson lines in region $a$ \( \sim \) functor $\mathcal{W}_a \times \mathcal{W}_{a,b} \rightarrow \mathcal{W}_{a,b}$
  - gives action of $\mathcal{W}_a$ on $\mathcal{W}_{a,b}$: $\mathcal{W}_{a,b}$ is left module category over $\mathcal{W}_a$
  - likewise: $\mathcal{W}_{a,b}$ is right module category over $\mathcal{W}_b$
Thus for single boundary condition $a$:

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- likewise: $\mathcal{W}_{a,b}$ is right module category over $\mathcal{W}_b$
- but also: $\mathcal{W}_{a,b}$ is right module category over $\text{End}_{\mathcal{W}_a}(\mathcal{W}_{a,b})$
Thus for single boundary condition $a$:

\[ C \xrightarrow{\sim} \mathbb{Z}(\mathcal{W}_a) \]

in particular obstruction: no compatible boundary condition unless $[C] = 0$

in Witt group of modular tensor categories

Other boundary condition $b$:

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  - likewise: $\mathcal{W}_{a,b}$ is right module category over $\mathcal{W}_b$
  - but also: $\mathcal{W}_{a,b}$ is right module category over $\mathcal{E}_{\text{nd}} \mathcal{W}_a(\mathcal{W}_{a,b})$

Impose naturality: $\mathcal{E}_{\text{nd}} \mathcal{W}_a(\mathcal{W}_{a,b}) \simeq \mathcal{W}_b$

Consistency check: $\mathbb{Z}(\mathcal{E}_{\text{nd}} \mathcal{W}_a(\mathcal{W}_{a,b})) \simeq \mathbb{Z}(\mathcal{W}_a)$ canonically

Schauenburg 2001
Bicategories of boundary conditions

Thus for single boundary condition $a$: $\mathcal{C} \xrightarrow{\sim} \mathbb{Z}(\mathcal{W}_a)$

- in particular obstruction: no compatible boundary condition unless $[\mathcal{C}] = 0$ in Witt group of modular tensor categories

Other boundary condition $b$:
- category $\mathcal{W}_{a,b}$ of Wilson lines separating boundary region labeled $a$ from region labeled $b$
- fusion of Wilson lines in region $a \rightsquigarrow$ functor $\mathcal{W}_a \times \mathcal{W}_{a,b} \rightarrow \mathcal{W}_{a,b}$
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- likewise: $\mathcal{W}_{a,b}$ is right module category over $\mathcal{W}_b$
- but also: $\mathcal{W}_{a,b}$ is right module category over $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b})$

Impose naturality: $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b}) \simeq \mathcal{W}_b$

$\implies$ can work with a single reference boundary condition $a$

Conjecture: Boundary conditions for $\mathcal{C}$ form the bicategory $\mathcal{W}_a$-Mod of module categories over a fusion category $\mathcal{W}_a$ satisfying $\mathbb{Z}(\mathcal{W}_a) \simeq \mathcal{C}$
Will assume: Boundary conditions given by $\mathcal{W}_a$-Mod.

Then $\mathcal{W}_{b,c} \simeq \mathcal{F}un_{\mathcal{W}_a}(\mathcal{W}_b, \mathcal{W}_c)$ for any pair of boundary conditions $b$, $c$. 
Will assume: Boundary conditions given by $\mathcal{W}_a$-$\text{Mod}$

Then $\mathcal{W}_{b,c} \simeq \text{Fun}_{\mathcal{W}_a}(\mathcal{W}_b, \mathcal{W}_c)$ for any pair of boundary conditions $b$, $c$

Warning:
via $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}($$\mathcal{W}_a$$)$ forget $\mathcal{W}_a$

any $\mathcal{M} \in \mathcal{W}_a$-$\text{Mod}$ has natural structure of $\mathcal{C}$-module category

But not every $\mathcal{C}$-module category of a Witt-trivial $\mathcal{C}$ gives a boundary condition
Will assume: Boundary conditions given by $\mathcal{W}_a$-$\text{Mod}$

Then $\mathcal{W}_{b,c} \simeq \mathcal{F}_{\mathcal{W}_a}(\mathcal{W}_b, \mathcal{W}_c)$ for any pair of boundary conditions $b, c$

Warning:

via $\mathcal{C} \xrightarrow{\simeq} \mathbb{Z}(\mathcal{W}_a) \xrightarrow{\text{forget}} \mathcal{W}_a$

any $\mathcal{M} \in \mathcal{W}_a$-$\text{Mod}$ has natural structure of $\mathcal{C}$-module category

But not every $\mathcal{C}$-module category of a Witt-trivial $\mathcal{C}$ gives a boundary condition

Illustration: Toric code

$\sim$ 2 elementary boundary conditions

BRAVYI-KITAEV 2001
Will assume: Boundary conditions given by \( \mathcal{W}_a\text{-}\text{Mod} \)

Then \( \mathcal{W}_{b,c} \simeq \mathcal{F}un_{\mathcal{W}_a}(\mathcal{W}_b, \mathcal{W}_c) \) for any pair of boundary conditions \( b, c \)

Warning:

via \( C \xrightarrow{\simeq} \mathbb{Z}(\mathcal{W}_a) \xrightarrow{\text{forget}} \mathcal{W}_a \)

any \( \mathcal{M} \in \mathcal{W}_a\text{-}\text{Mod} \) has natural structure of \( C\)-module category

But not every \( C\)-module category of a Witt-trivial \( C \) gives a boundary condition

Illustration: Toric code

- 2 elementary boundary conditions

- \( C = \mathbb{Z}(\text{Vect}(\mathbb{Z}_2)) \)

- 6 inequivalent indecomposable module categories over \( C \)

- 2 inequivalent indecomposable module categories over \( \mathcal{W} = \text{Vect}(\mathbb{Z}_2) \)
Parallel analysis for **surface defects**:

- defect $d$ separating bulk regions labeled by $C_1$ and $C_2$
- two monoidal functors $C_1 \to \mathcal{W}_d$ and $C_2^{rev} \to \mathcal{W}_d$ to fusion category $\mathcal{W}_d$
Parallel analysis for surface defects:

- defect $d$ separating bulk regions labeled by $C_1$ and $C_2$
- two monoidal functors $C_1 \rightarrow \mathcal{W}_d$ and $C_2^{\text{rev}} \rightarrow \mathcal{W}_d$ to fusion category $\mathcal{W}_d$
- combine to central functor $C_1 \boxtimes C_2^{\text{rev}} \rightarrow \mathcal{W}_d$

Deligne product
Parallel analysis for surface defects:

- defect $d$ separating bulk regions labeled by $C_1$ and $C_2$.
- two monoidal functors $C_1 \to \mathcal{W}_d$ and $C_2^{\text{rev}} \to \mathcal{W}_d$ to fusion category $\mathcal{W}_d$.
- combine to central functor $C_1 \boxtimes C_2^{\text{rev}} \to \mathcal{W}_d$.
- naturality $\sim$ braided equivalence $C_1 \boxtimes C_2^{\text{rev}} \sim \mathbb{Z}(\mathcal{W}_a)$.
- obstruction: no defects between $C_1$ and $C_2$ unless $[C_1] = [C_2]$ in Witt group.
Parallel analysis for surface defects:

- defect $d$ separating bulk regions labeled by $C_1$ and $C_2$
- two monoidal functors $C_1 \to \mathcal{W}_d$ and $C_2^{\text{rev}} \to \mathcal{W}_d$ to fusion category $\mathcal{W}_d$
- combine to central functor $C_1 \boxtimes C_2^{\text{rev}} \to \mathcal{W}_d$
- naturality $\sim$ braided equivalence $C_1 \boxtimes C_2^{\text{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$

Defects separating $C_1$ from $C_2$ form the bicategory $\mathcal{W}_d$-Mod of module categories over a fusion category $\mathcal{W}_d$ satisfying $\mathcal{Z}(\mathcal{W}_d) \simeq C_1 \boxtimes C_2^{\text{rev}}$
Parallel analysis for surface defects:

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Defects separating $C_1$ from $C_2$ form the bicategory $\mathcal{W}_d$-$\text{Mod}$ of module categories over a fusion category $\mathcal{W}_d$ satisfying $\mathbb{Z}(\mathcal{W}_d) \simeq C_1 \boxtimes C_2^{\text{rev}}$

Example: Canonical Witt trivialization $C \boxtimes C^{\text{rev}} \sim \mathbb{Z}(C)$ (C modular)

- defects separating $C$ from itself $=$ $C$-module categories
Parallel analysis for surface defects:
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\[
C_1 \boxtimes C_{2}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{W}_d)
\]

Defects separating $C_1$ from $C_2$ form the bicategory $\mathcal{W}_d\text{-Mod}$ of module categories over a fusion category $\mathcal{W}_d$ satisfying $\mathcal{Z}(\mathcal{W}_d) \simeq C_1 \boxtimes C_{2}^{\text{rev}}$

Canonical Witt trivialization $C \boxtimes C^{\text{rev}} \simeq \mathcal{Z}(C)$
- defects separating $C$ from itself = $C$-module categories
- regular $C$-module category $(C, \otimes) \sim$ transparent defect $\mathcal{T}$
- serves as monoidal unit for fusion of surface defects
- Wilson lines separating transparent defect from itself = ordinary Wilson lines
Bicategories of surface defects

- Parallel analysis for surface defects:
  - defect \( d \) separating bulk regions labeled by \( C_1 \) and \( C_2 \)
  - two monoidal functors \( C_1 \to \mathcal{W}_d \) and \( C_2^{\text{rev}} \to \mathcal{W}_d \) to fusion category \( \mathcal{W}_d \)
  - combine to central functor \( C_1 \boxtimes C_2^{\text{rev}} \to \mathcal{W}_d \)
  - naturality \( \sim \) braided equivalence

- Defects separating \( C_1 \) from \( C_2 \) form the bicategory \( \mathcal{W}_d\text{-Mod} \)
  - of module categories over a fusion category \( \mathcal{W}_d \) satisfying \( \mathcal{Z}(\mathcal{W}_d) \simeq C_1 \boxtimes C_2^{\text{rev}} \)

- Canonical Witt trivialization \( C \boxtimes C^{\text{rev}} \sim \mathcal{Z}(C) \)
  - defects separating \( C \) from itself \( = C\)-module categories
  - regular \( C\)-module category \( (C, \otimes) \sim \) transparent defect \( T \)

- Example: Turaev-Viro TFT: \( C_1 \simeq \mathcal{Z}(A_1) \) and \( C_2 \simeq \mathcal{Z}(A_2) \)
  \( \sim \) \( C_1 \boxtimes C_2^{\text{rev}} \simeq \mathcal{Z}(A_1) \otimes \mathcal{Z}(A_2^{\text{op}}) \simeq \mathcal{Z}(A_1 \boxtimes A_2^{\text{op}}) \)
  \( \sim \) defects separating \( C_1 \) from \( C_2 \) form bicategory \( A_1\text{-}A_2\text{-Bimod} \)

- Defects separating \( C_1 \) from \( C_2 \) form the bicategory \( \mathcal{W}_d\text{-Mod} \)
  - of module categories over a fusion category \( \mathcal{W}_d \) satisfying \( \mathcal{Z}(\mathcal{W}_d) \simeq C_1 \boxtimes C_2^{\text{rev}} \)

- Canonical Witt trivialization \( C \boxtimes C^{\text{rev}} \sim \mathcal{Z}(C) \)
  - defects separating \( C \) from itself \( = C\)-module categories
  - regular \( C\)-module category \( (C, \otimes) \sim \) transparent defect \( T \)

- Example: Turaev-Viro TFT: \( C_1 \simeq \mathcal{Z}(A_1) \) and \( C_2 \simeq \mathcal{Z}(A_2) \)
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  \( \sim \) defects separating \( C_1 \) from \( C_2 \) form bicategory \( A_1\text{-}A_2\text{-Bimod} \)
Defects for multi-layer systems

TFT for topological defects

- Classification of module categories over a general modular tensor category \( \mathcal{D} \) out of reach
  
  (even finding any indecomposable \( \mathcal{D} \)-module besides \( (\mathcal{D}, \otimes) \) can be hard)

- Side remark:
  
  bijection between indecomposable \( \mathcal{D} \)-module categories and modular invariant torus partition functions for the rational conformal field theory based on \( \mathcal{D} \)
Defects for multi-layer systems

- Classification of module categories over a general modular tensor category $\mathcal{D}$ out of reach
- TFT for $N$-layer system: modular tensor category $\mathcal{D} = \mathcal{C} \boxtimes N$
  with $\mathcal{C}$ modular tensor category for each single layer
Defects for multi-layer systems

Classification of module categories over a general modular tensor category \( \mathcal{D} \) out of reach

TFT for \( N \)-layer system: modular tensor category \( \mathcal{D} = \mathcal{C} \boxtimes N \)

Generic non-trivial right \( \mathcal{D} \)-module category: \( \mathcal{P} \equiv \mathcal{P}_D := (\mathcal{C}, \triangleleft, \alpha) \)

with \( W \triangleleft (U_1 \boxtimes \cdots \boxtimes U_N) = W \otimes U_1 \otimes \cdots \otimes U_N \)

and mixed associativity constraint for \( N = 2 \)

\[ \begin{array}{cccc}
  W & U_1 & V_1 & U_2 & V_2 \\
  W & U_1 & U_2 & V_1 & V_2 \\
\end{array} \]
Defects for multi-layer systems

- Classification of module categories over a general modular tensor category $\mathcal{D}$ out of reach
- TFT for $N$-layer system: modular tensor category $\mathcal{D} = \mathcal{C} \boxtimes N$
- Generic non-trivial right $\mathcal{D}$-module category: $\mathcal{P} \equiv \mathcal{P}_\mathcal{D} := (\mathcal{C}, \triangleleft, \alpha)$
  
  with $W \triangleleft (U_1 \boxtimes \cdots \boxtimes U_N) = W \otimes U_1 \otimes \cdots \otimes U_N$

  and mixed associativity constraint for $N = 2$

  $W \otimes U_1 \otimes V_1 \otimes U_2 \otimes V_2$

  $= (W \triangleleft (U_1 \boxtimes V_1)) \triangleleft (U_2 \boxtimes V_2)$

  $= W \triangleleft ((U_1 \boxtimes V_1) \otimes_\mathcal{D} (U_2 \boxtimes V_2))$

  $= W \triangleleft ((U_1 \otimes_\mathcal{D} U_2) \boxtimes (V_1 \otimes_\mathcal{D} V_2))$

  braiding in $\mathcal{C}$

  categorification of fact that commutative ring $R$ is $R \otimes_\mathcal{Z} R$-module
Defects for multi-layer systems

- Classification of module categories over a general modular tensor category $\mathcal{D}$ out of reach
- TFT for $N$-layer system: modular tensor category $\mathcal{D} = \mathcal{C} \boxtimes N$
- Generic non-trivial right $\mathcal{D}$-module category: $\mathcal{P} \equiv \mathcal{P}_\mathcal{D} := (\mathcal{C}, \triangleleft, \alpha)$ with $W \triangleleft (U_1 \boxtimes \cdots \boxtimes U_N) = W \otimes U_1 \otimes \cdots \otimes U_N$
- Generalization: a $\mathcal{D}$-module category for every permutation of the $N$ factors $\mathcal{C}$
  - Side remark: corresponding to permutation modular invariants in RCFT
Defects for multi-layer systems

TFT for topological defects

- Classification of module categories over a general modular tensor category $\mathcal{D}$ out of reach
- TFT for $N$-layer system: modular tensor category $\mathcal{D} = C^\otimes N$
- Generic non-trivial right $\mathcal{D}$-module category: $\mathcal{P} \equiv \mathcal{P}_\mathcal{D} := (C, \prec, \alpha)$ with $W \prec (U_1 \otimes \cdots \otimes U_N) = W \otimes U_1 \otimes \cdots \otimes U_N$
- Generalization: a $\mathcal{D}$-module category for every permutation of the $N$ factors $C$
- Side remark: corresponding to permutation modular invariants in RCFT
- From now on restrict to two-layer system $\mathcal{D} = C \otimes C$
  - two generic $\mathcal{D}$-module categories $\mathcal{D} \equiv \mathcal{T}$ and $\mathcal{P}$
  - right action $\mathcal{P} \times \mathcal{D} \rightarrow \mathcal{P}$
Defects for multi-layer systems

- Classification of module categories over a general modular tensor category $\mathcal{D}$ out of reach

- TFT for $N$-layer system: modular tensor category $\mathcal{D} = \mathcal{C} \boxtimes N$

- Generic non-trivial right $\mathcal{D}$-module category: $\mathcal{P} \equiv \mathcal{P}_\mathcal{D} := (\mathcal{C}, \triangleleft, \alpha)$
  with $W \triangleleft (U_1 \boxtimes \cdots \boxtimes U_N) = W \otimes U_1 \otimes \cdots \otimes U_N$

- Generalization: a $\mathcal{D}$-module category for every permutation of the $N$ factors $\mathcal{C}$
  
  Side remark: corresponding to permutation modular invariants in RCFT

- From now on restrict to two-layer system $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$

  - two generic $\mathcal{D}$-module categories $\mathcal{D} \equiv \mathcal{T}$ and $\mathcal{P}$
  - right action $\mathcal{P} \times \mathcal{D} \rightarrow \mathcal{P}$
  - form part of a $\mathbb{Z}_2$-equivariant modular category
  - thus further fusion functors $\mathcal{D} \times \mathcal{P} \rightarrow \mathcal{P}$ and $\mathcal{P} \times \mathcal{P} \rightarrow \mathcal{D}$
  - derivable from a $\mathbb{Z}_2$-equivariant topological field theory
The $C\boxtimes C$-module $\mathcal{P}$

- $\mathcal{D}$-module category $\mathcal{P}$ realizable as category $A_{\mathcal{P}}\text{-mod}$ of left $A_{\mathcal{P}}$-modules in $\mathcal{D}$
  - $A_{\mathcal{P}} = \bigoplus_{i \in I_C} S_i^\vee \boxtimes S_i$ as object
  - algebra structure determined by fusion of simple objects in $\mathcal{C}$:

\[
m = \bigoplus_{i,j,k \in I_C} \sum_{\alpha=1}^{N_{i,j}^k} \overline{e_i^\alpha \boxtimes e_j^\alpha \boxtimes e_k^\alpha} \]

\[
\eta = \overline{e_1^\alpha \boxtimes 1_{\mathcal{CP}}} \]

References:

BARMEIER-J-F-RUNKEL-SCHWEIGERT 2010
BARMEIER-SCHWEIGERT 2011
The $\mathcal{C} \boxtimes \mathcal{C}$-module $\mathcal{P}$

$\mathcal{D}$-module category $\mathcal{P}$ realizable as category $A_{\mathcal{P}}$-$\text{mod}$ of left $A_{\mathcal{P}}$-modules in $\mathcal{D}$

$A_{\mathcal{P}} = \bigoplus_{i \in I_C} S_i^\vee \boxtimes S_i$

symmetric special Frobenius algebra:

$$\Delta = \bigoplus_{i,j,k \in I_C} \frac{\dim(S_i) \dim(S_j)}{\text{Dim}(\mathcal{C}) \dim(S_k)} \sum_{\alpha} f_\alpha$$

$$\eta = r_{A_{\mathcal{P}}} \triangleright 1 \otimes 1$$
The $C \otimes C$-module $\mathcal{P}$

- $\mathcal{P}$ realizable as category $A_{\mathcal{P}}$-$\text{mod}$ of left $A_{\mathcal{P}}$-modules in $\mathcal{D}$
  - $A_{\mathcal{P}} = \bigoplus_{i \in I_C} S_i^{\vee} \otimes S_i$
  - symmetric special Frobenius algebra
  - Azumaya algebra:
    - braided induction functors $\alpha_{A_{\mathcal{P}}}^\pm : \mathcal{C} \to A_{\mathcal{P}}$-$\text{bimod}$ are monoidal equivalences
      $$U \mapsto (A_{\mathcal{P}} \otimes U, m \otimes \text{id}_U, (m \otimes \text{id}_U) \circ (\text{id}_{A_{\mathcal{P}}} \otimes c_{U,A_{\mathcal{P}}}), c_{A_{\mathcal{P}},U}^{-1})$$
The $C \boxtimes C$-module $\mathcal{P}$

- $D$-module category $\mathcal{P}$ realizable as category $A_\mathcal{P}$-mod of left $A_\mathcal{P}$-modules in $D$
  - $A_\mathcal{P} = \bigoplus_{i \in I_C} S_i^\vee \boxtimes S_i$

- symmetric special Frobenius Azumaya algebra

- Analogously
  \[
  \bigoplus_{i_1, i_2, \ldots, i_N \in I_C} (S_{i_1} \boxtimes S_{i_2} \boxtimes \cdots \boxtimes S_{i_N})^\oplus \bigoplus_{i_1, i_2, \ldots, i_N} \text{for } N > 2
  \]
  \[
  N_{i_1, i_2, \ldots, i_N} = \dim \text{Hom}_C(S_{i_1} \otimes S_{i_2} \otimes \cdots \otimes S_{i_N}, 1)
  \]
The $\mathcal{C} \boxtimes \mathcal{C}$-module $\mathcal{P}$

**TFT for topological defects**

- $\mathcal{D}$-module category $\mathcal{P}$ realizable as category $A_\mathcal{P}$-$\text{mod}$ of left $A_\mathcal{P}$-modules in $\mathcal{D}$

  - $A_\mathcal{P} = \bigoplus_{i \in I_C} S_i^V \boxtimes S_i$

  - symmetric special Frobenius Azumaya algebra

- For $A$ Azumaya $\Psi_A := (\alpha_A^+)^{-1} \circ \alpha_A^-$
  
    describes transmission of bulk Wilson lines through surface defect $A$-$\text{mod}$
The $\mathcal{C} \otimes \mathcal{C}$-module $\mathcal{P}$

**TFT for topological defects**

- $\mathcal{D}$-module category $\mathcal{P}$ realizable as category $A_{\mathcal{P}}$-$\text{mod}$ of left $A_{\mathcal{P}}$-modules in $\mathcal{D}$
  - $A_{\mathcal{P}} = \bigoplus_{i \in I_C} S_i^\vee \otimes S_i$
  - symmetric special Frobenius Azumaya algebra

- For Azumaya $\Psi_A := (\alpha_A^+)^{-1} \circ \alpha_A^-$
  describes transmission of bulk Wilson lines through surface defect $A$-$\text{mod}$
  - $\alpha_{A_{\mathcal{P}}}^+(U \otimes V) \cong \alpha_{A_{\mathcal{P}}}^-(V \otimes U)$ by direct calculation
    - transmission of bulk Wilson lines through $\mathcal{P}$ permutes the layers
The $C \boxtimes C$-module $\mathcal{P}$

- **$D$-module category** $\mathcal{P}$ realizable as category $A_{\mathcal{P}}$-$\text{mod}$ of left $A_{\mathcal{P}}$-modules in $D$
  - $A_{\mathcal{P}} = \bigoplus_{i \in I_C} S_i^\vee \boxtimes S_i$
  - symmetric special Frobenius Azumaya algebra

- For $A$ Azumaya $\Psi_A := (\alpha_A^+)^{-1} \circ \alpha_A^-$ describes transmission of bulk Wilson lines through surface defect $A$-$\text{mod}$
  - $\alpha^+_A(U \boxtimes V) \cong \alpha^-_A(V \boxtimes U)$

- Braided induction for tensor products:

$$\begin{align*}
\alpha^-_{A_1 \otimes A_2} & \Rightarrow \nu \Rightarrow \beta^- \\
\beta^- & \Rightarrow \bar{\nu} \Rightarrow \beta^+ \\
\beta^+ & \Rightarrow \text{Id} \Rightarrow \alpha^+_A_{A_1 \otimes A_2}
\end{align*}$$
The $C \otimes C$-module $\mathcal{P}$

- $\mathcal{D}$-module category $\mathcal{P}$ realizable as category $A_{\mathcal{P}}\text{-mod}$ of left $A_{\mathcal{P}}$-modules in $\mathcal{D}$
  - $A_{\mathcal{P}} = \bigoplus_{i \in I_C} S_i^v \boxtimes S_i$
  - symmetric special Frobenius Azumaya algebra
- For Azumaya $A$, $\Psi_A := (\alpha_A^+)^{-1} \circ \alpha_A^-$
describes transmission of bulk Wilson lines through surface defect $A\text{-mod}$
  - $\alpha_{A_{\mathcal{P}}}^+ (U \boxtimes V) \cong \alpha_{A_{\mathcal{P}}}^- (V \boxtimes U)$
- Braided induction for tensor products
  - $\Psi_{A_1 \otimes A_2} = \Psi_{A_1} \circ \Psi_{A_2}$ as monoidal functors if $A_{1,2}$ Azumaya
  - $A_{\mathcal{P}} \otimes A_{\mathcal{P}}$ Morita equivalent to $1_{\mathcal{D}}$
The $C \Box C$-module $\mathcal{P}$

**TFT for topological defects**

- $\mathcal{D}$-module category $\mathcal{P}$ realizable as category $A_{\mathcal{P}}$-$\text{mod}$ of left $A_{\mathcal{P}}$-modules in $\mathcal{D}$
  - $A_{\mathcal{P}} = \bigoplus_{i \in I_C} S_i^V \boxtimes S_i$
  - symmetric special Frobenius Azumaya algebra
- For $A$ Azumaya $\Psi_A := (\alpha_A^+)^{-1} \circ \alpha_A^-$ describes transmission of bulk Wilson lines through surface defect $A$-$\text{mod}$
  - $\alpha_A^+ (U \boxtimes V) \cong \alpha_A^- (V \boxtimes U)$
- Braided induction for tensor products
  - $\Psi_{A_1} \otimes A_2 = \Psi_{A_1} \circ \Psi_{A_2}$ as monoidal functors if $A_{1,2}$ Azumaya
  - $A_{\mathcal{P}} \otimes A_{\mathcal{P}}$ Morita equivalent to $1_{\mathcal{D}}$
- Fusion rules:
  - $\mathcal{T} \boxtimes_{\mathcal{D}} \mathcal{P} \cong \mathcal{P}$
  - $\mathcal{P} \boxtimes_{\mathcal{D}} \mathcal{P} \cong \mathcal{T}$
- Categories of defect Wilson lines:
  - $\mathcal{F}un_{\mathcal{D}}(\mathcal{T}, \mathcal{P}) \cong (1_{\mathcal{D}} \otimes A_{\mathcal{P}})$-$\text{mod} \cong A_{\mathcal{P}}$-$\text{mod} \cong \mathcal{C}$
  - $\mathcal{F}un_{\mathcal{D}}(\mathcal{P}, \mathcal{T}) \cong \mathcal{C}$
  - $\mathcal{E}nd_{\mathcal{D}}(\mathcal{T}) \cong \mathcal{D} \cong \mathcal{E}nd_{\mathcal{D}}(\mathcal{P})$
Relation with extended TFT

More general Wilson lines:

\[ \text{Diagram of more general Wilson lines} \]
Relation with extended TFT

More general Wilson lines:

Via extended TFT $\text{tft}^D_{3,2,1}$ assign categories:

$$\text{Cobord}_{3,2,1} \rightarrow 2\text{-Vect}$$

$$M \mapsto \text{tft}^D_{3,2,1}(M)$$

1-manifold category

e.g. circle: $\text{tft}^D_{3,2,1}(\mathbb{S}) = \mathcal{D}$
Relation with extended TFT

More general Wilson lines:

Via extended TFT $tft_{3,2,1}^D$ assign categories: \[ \text{Cobord}_{3,2,1} \rightarrow 2\text{-Vect} \]

Circle with defect points: use cover functor \[ M \mapsto \text{two-sheeted cover } \tilde{M} \]

locally:

$M$ $\tilde{M}$
Relation with extended TFT

More general Wilson lines:

Via extended TFT $\text{tft}^D_{3,2,1}$ assign categories:

\[ \text{Cobord}_{3,2,1} \longrightarrow \text{2-Vect} \]

\[ M \mapsto \text{tft}^D_{3,2,1}(M) \]

Circle with defect points: use cover functor

$M \mapsto$ two-sheeted cover $\tilde{M}

\begin{align*}
\text{tft}^{\mathbb{Z}_2;D}_{3,2,1}(\mathbb{S}_{n_T},n_P) \\
= \text{tft}^C_{3,2,1}(\mathbb{S}_{n_T},n_P) \\
= \begin{cases} 
\text{tft}_C(\mathbb{S} \sqcup \mathbb{S}) \simeq \text{tft}_C(\mathbb{S}) \boxtimes \text{tft}_C(\mathbb{S}) = C \boxtimes C = D & \text{for } n_P \text{ even} \\
\text{tft}_C(\mathbb{S}) = C & \text{for } n_P \text{ odd}
\end{cases}
\end{align*}

reproducing the previous results for categories of defect Wilson lines
Relation with extended TFT

More general Wilson lines:

Via extended TFT $\text{tft}_{3,2,1}^D$ assign functors to 2-manifolds

General surfaces with Wilson lines:
More general Wilson lines:

Via extended TFT $\text{tft}_{3,2,1}^D$ assign functors to 2-manifolds

General surfaces with Wilson lines:
functor $\text{tft}_{3,2,1}^D(\partial_+ \Sigma \xrightarrow{\Sigma} \partial_- \Sigma)$
e.g. pair of pants
$Y \leftrightarrow \boxtimes : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$
Relation with extended TFT

More general Wilson lines:

Via extended TFT $tft^{D}_{3,2,1}$ assign functors to 2-manifolds

General surfaces with Wilson lines:

functor $tft^{D}_{3,2,1}(\partial_\Sigma \xrightarrow{\Sigma} \partial_+ \Sigma)$

e.g. pair of pants

$Y \mapsto \bigstar : D \times D \to D$

General case:

e.g. via cover functor: pair of pants $(n_1, n_2, n_3)$ $\mathcal{P}$-defects on $\partial Y$

$Y_{n_1, n_2, n_3} \mapsto \begin{cases} \bigstar & \text{for } n_1 + n_2 \text{ even} \\ < & \text{for } n_1 + n_2 \text{ odd} \end{cases}$
Spaces of conformal blocks

Surface without defect lines with $\partial_+ \Sigma = \emptyset$ and $g_{\Sigma} = 0$ and $\pi_0(\partial \Sigma) = m$ gives functor

$$\mathcal{D} \boxtimes m \to \text{Vect}$$

$$U_1 \boxtimes \cdots \boxtimes U_m \mapsto \text{Hom}_\mathcal{D}(U_1 \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} U_m, 1_{\mathcal{D}})$$
Surface without defect lines with \( \partial_+ \Sigma = \emptyset \) and \( g_\Sigma = 0 \) and \( \pi_0(\partial \Sigma) = m \) gives functor

\[
\mathcal{D}^\otimes m \longrightarrow \text{Vect}
\]

\[
U_1 \otimes \cdots \otimes U_m \longmapsto \text{Hom}_D(U_1 \otimes_D \cdots \otimes_D U_m, 1_D)
\]

= space of conformal blocks

= space of ground states of topological phase

generalizes to higher genus

dimension computed by Verlinde formula
Surface without defect lines with $\partial^+ \Sigma = \emptyset$ and $g_\Sigma = 0$ and $\pi_0(\partial \Sigma) = m$
gives functor $\mathcal{D}^m \longrightarrow \text{Vect}$
$U_1 \boxtimes \cdots \boxtimes U_m \longmapsto \text{Hom}_\mathcal{D}(U_1 \otimes_\mathcal{D} \cdots \otimes_\mathcal{D} U_m, \mathbf{1}_\mathcal{D})$

General surface:
- $m_0$ boundary circles $\bigcirc$ with even number of $\mathcal{P}$-defects
- $m_1$ boundary circles $\bigcirc$ with odd number of $\mathcal{P}$-defects
Spaces of conformal blocks

Surface without defect lines with $\partial_+ \Sigma = \emptyset$ and $g_\Sigma = 0$ and $\pi_0(\partial \Sigma) = m$

gives functor

$$D \boxtimes m \longrightarrow \text{Vect}$$

$$U_1 \boxtimes \cdots \boxtimes U_m \longmapsto \text{Hom}_D(U_1 \otimes_D \cdots \otimes_D U_m, 1_D)$$

General surface:

$m_0$ boundary circles $\bigcirc$ with even number of $\mathcal{P}$-defects

$m_1$ boundary circles $\bigcirc$ with odd number of $\mathcal{P}$-defects

gives functor

$$D \boxtimes m_0 \boxtimes C \boxtimes m_1 \longrightarrow \text{Vect}$$

expressible as a composite of functors in pair-of-pants decomposition of $\Sigma$

glue $\mathbb{Z}_2$-covers of pairs of pants $\leadsto$ branched twofold cover $\tilde{\Sigma}$

compatible with gluing of surfaces with defects

$tft^{\mathbb{Z}_2;D}_{3,2,1}(\Sigma) = tft^C_{3,2,1}(\tilde{\Sigma})$
Spaces of conformal blocks

TFT for topological defects

- Surface without defect lines with \( \partial_+ \Sigma = \emptyset \) and \( g_\Sigma = 0 \) and \( \pi_0(\partial \Sigma) = m \) gives functor \( \mathcal{D} \boxtimes m \longrightarrow \text{Vect} \)

\[
U_1 \boxtimes \cdots \boxtimes U_m \longmapsto \text{Hom}_\mathcal{D}(U_1 \otimes_\mathcal{D} \cdots \otimes_\mathcal{D} U_m, 1_\mathcal{D})
\]

- General surface:
  - \( m_0 \) boundary circles \( \bigcirc \) with even number of \( \mathcal{P} \)-defects
  - \( m_1 \) boundary circles \( \bigcirc \) with odd number of \( \mathcal{P} \)-defects

  gives functor \( \mathcal{D} \boxtimes m_0 \boxtimes \mathcal{C} \boxtimes m_1 \longrightarrow \text{Vect} \)

- Generalized Verlinde formula via ordinary Verlinde formula for \( \text{tft}^C_{3,2,1}(\tilde{\Sigma}) \)
  - boundary circle with even number of \( \mathcal{P} \)-defects labeled by \( U \boxtimes \tilde{U} \in \mathcal{D} = \mathcal{C} \boxtimes \mathcal{C} \)
    (pre-image on \( \tilde{\Sigma} \) consisting of two circles)
  - boundary circle with odd number of \( \mathcal{P} \)-defects labeled by \( V \in \mathcal{C} \)
    (pre-image on \( \tilde{\Sigma} \) consisting of one circle)
Spaces of conformal blocks

TFT for topological defects

- Surface without defect lines with $\partial_+ \Sigma = \emptyset$ and $g_\Sigma = 0$ and $\pi_0(\partial \Sigma) = m$
  gives functor $\mathcal{D} \boxtimes m \longrightarrow \text{Vect}$
  $U_1 \boxtimes \cdots \boxtimes U_m \longmapsto \text{Hom}_\mathcal{D}(U_1 \otimes_\mathcal{D} \cdots \otimes_\mathcal{D} U_m, 1_\mathcal{D})$

- General surface:
  $m_0$ boundary circles $\bigcirc$ with even number of $\mathcal{P}$-defects
  $m_1$ boundary circles $\bigcirc$ with odd number of $\mathcal{P}$-defects
  gives functor $\mathcal{D} \boxtimes m_0 \boxtimes C \boxtimes m_1 \longrightarrow \text{Vect}$

- Generalized Verlinde formula via ordinary Verlinde formula for $\text{tft}_3,2,1^C(\tilde{\Sigma})$
  boundary circle with even number of $\mathcal{P}$-defects labeled by simple $U_i \boxtimes \tilde{U}_i \in \mathcal{D}$
  boundary circle with odd number of $\mathcal{P}$-defects labeled by simple $V_j \in \mathcal{C}$

$$\dim_C(\text{tft}_3^\mathcal{D}(\Sigma; \{U_i \boxtimes \tilde{U}_i\}, \{V_j\})) = \sum_{n \in I_\mathcal{C}} (S_{0,n})^{2\chi-m_1} \prod_{i=1}^{m_0} \frac{S_{U_i,n}}{S_{0,n}} \frac{S_{\tilde{U}_i,n}}{S_{0,n}} \prod_{j=1}^{m_1} \frac{S_{V_j,n}}{S_{0,n}}$$
**Spaces of conformal blocks**

- Surface without defect lines with $\partial_+ \Sigma = \emptyset$ and $g_\Sigma = 0$ and $\pi_0(\partial \Sigma) = m$

  gives functor

  $\mathcal{D} \boxtimes m \rightarrow \text{Vect}$

  $U_1 \boxtimes \cdots \boxtimes U_m \mapsto \text{Hom}_D(U_1 \otimes_D \cdots \otimes_D U_m, 1_D)$

- General surface:

  - $m_0$ boundary circles $\bigcirc$ with even number of $\mathcal{P}$-defects
  - $m_1$ boundary circles $\bigcirc$ with odd number of $\mathcal{P}$-defects

  gives functor

  $\mathcal{D} \boxtimes m_0 \boxtimes \mathcal{C} \boxtimes m_1 \rightarrow \text{Vect}$

- **Generalized Verlinde formula** via ordinary Verlinde formula for $\text{tft}_{3,2,1}^C(\tilde{\Sigma})$

  - boundary circle with even number of $\mathcal{P}$-defects labeled by simple $U_i \boxtimes \tilde{U}_i \in \mathcal{D}$
  - boundary circle with odd number of $\mathcal{P}$-defects labeled by simple $V_j \in \mathcal{C}$

  e.g. $\dim_{\mathbb{C}}(\text{tft}_D^D(S^2; \emptyset, \{V, V, \ldots, V\})) = \sum_{n \in I_C} (S_{0,n})^{4-2m_1} (S_V,n)^{m_1}$
Spaces of conformal blocks

TFT for topological defects

Surface without defect lines with $\partial_+ \Sigma = \emptyset$ and $g_\Sigma = 0$ and $\pi_0(\partial \Sigma) = m$
gives functor

$$D \boxtimes m \longrightarrow \text{Vect}$$

$$U_1 \boxtimes \cdots \boxtimes U_m \longmapsto \text{Hom}_D(U_1 \otimes \cdots \otimes U_m, 1_D)$$

General surface:

$m_0$ boundary circles $\bigcirc$ with even number of $P$-defects
$m_1$ boundary circles $\bigcirc$ with odd number of $P$-defects

gives functor

$$D \boxtimes m_0 \boxtimes C \boxtimes m_1 \longrightarrow \text{Vect}$$

Generalized Verlinde formula via ordinary Verlinde formula for $\text{tft}_3,2,1^{C}(\tilde{\Sigma})$

boundary circle with even number of $P$-defects labeled by simple $U_i \boxtimes \tilde{U}_i \in D$
boundary circle with odd number of $P$-defects labeled by simple $V_j \in C$

e.g.

$$\dim_C(\text{tft}_D(S^2; \emptyset, \{V, V, \ldots, V\})) = \sum_{n \in I_C} (S_{0,n})^{4-2m_1} (S_{V,n})^{m_1}$$
depends on genus type $V$

modular S-matrix of $C$
APPENDIX
Defects in Dijkgraaf-Witten theories

Dijkgraaf-Witten theories

- input data: finite group $G$ and cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$
- $\mathcal{C} = D^\omega(G)-\text{mod} \cong \mathbb{Z}(\text{Vect}(G)^\omega)$, Turaev-Viro type
- $\omega$ gives holonomy on closed three-manifolds $\sim$ topological bulk Lagrangian
- two-step gauge-theoretic construction:

\[
\text{Cobord}_{3,2,1} \xrightarrow{\text{Bun}} \text{SpanGrp} \xrightarrow{[-,\text{Vect}^\tau]} 2\text{-Vect}
\]

- twisted linearization

\[
\tau \in H^2(G//G, \mathbb{C}^\times) \quad \text{obtained by transgression}
\]

FREED 1995
MORTON 2013

WILLERTON 2008
Dijkgraaf-Witten theories

- input data: finite group $G$ and cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$
- $\mathcal{C} = D\omega(G)\text{-mod} \cong Z(\text{Vect}(G)\omega)$ Turaev-Viro type
- $\omega$ gives holonomy on closed three-manifolds $\rightsquigarrow$ topological bulk Lagrangian
- two-step gauge-theoretic construction:

\[
\begin{array}{ccc}
\text{Cobord}_{3,2,1} & \xrightarrow{\text{Bun}} & \text{SpanGrp} \\
\xrightarrow{[-,\text{Vect}]^\tau} & & \xrightarrow{} \text{2-Vect}
\end{array}
\]
twisted linearization

- extends to TFT with boundaries and defects via (bi)relative manifolds and (bi)relative bundles
Dijkgraaf-Witten theories

- input data: finite group $G$ and cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$
- $\mathcal{C} = D^\omega(G)\text{-mod} \simeq Z(\text{Vect}(G)^\omega)$ Turaev-Viro type
- $\omega$ gives holonomy on closed three-manifolds $\leadsto$ topological bulk Lagrangian
- two-step gauge-theoretic construction:
  \[
  \text{Cobord}_{3,2,1} \overset{\text{Bun}}{\longrightarrow} \text{SpanGrp} \overset{[-,\text{Vect}]^\tau}{\longrightarrow} 2\text{-Vect}
  \]
  twisted linearization
- extends to TFT with boundaries and defects
  - category of relative bundles for smooth map $j: Y \to X$ and group homomorphism $\iota: H \to G$
  - objects: $G$-bundle $P_G \to X$ and $H$-bundle $P_H \to Y$
  - with isomorphism $\alpha: \text{Ind}_H^G(P_H) \cong j^*P_G$
  - morphisms: bundle morphisms

\[
\begin{align*}
  P_G \overset{\varphi_G}{\longrightarrow} P'_G \\
  P_H \overset{\varphi_H}{\longrightarrow} P'_H \\
  \text{Ind}_H^G(P_H) \overset{\alpha}{\longrightarrow} j^*P_G \\
  \text{Ind}_H^G(P_H) \overset{\alpha'}{\longrightarrow} j^*P'_G
\end{align*}
\] s.t.

\[
\begin{align*}
  \text{Ind}_H^G(\varphi_H) \downarrow \\
  \text{Ind}_H^G(P'_H) \overset{\alpha'}{\longrightarrow} j^*P'_G
\end{align*}
\]
Dijkgraaf-Witten theories

- input data: finite group $G$ and cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$
- $\mathcal{C} = D^\omega(G)\text{-mod} \simeq Z(\text{Vect}(G)^\omega)$ Turaev-Viro type
- $\omega$ gives holonomy on closed three-manifolds $\sim$ topological bulk Lagrangian
- two-step gauge-theoretic construction:
  \[
  \text{Cobord}_{3,2,1} \xrightarrow{\text{Bun}} \text{SpanGrp} \xrightarrow{[-, \text{Vect}]^\tau} 2\text{-Vect}
  \] twisted linearization
- extends to TFT with boundaries and defects

Example: category for circle $S$ with one defect point $p$
- to interval $S\setminus\{p\}$ assign group $G$ with cocycle $\omega$
- to $p$ assign homomorphism $\iota: H \to G \times G$ with cochain $\theta \in C^2(H, \mathbb{C}^\times)$
- $\text{Bun}$ gives action groupoid $G\backslash G \times G/\sim H$
- twisted linearization gives $[G \backslash G \times G/\sim H, \text{Vect}]^{\tau, \omega, \theta}$

find $\tau_{\omega, \theta}((\gamma_1, \gamma_2); (g, h), (g', h')) = [\theta(h', h)]^{-1}$

\[
\omega(g', g, \gamma_1) [\omega(g', g\gamma_1\iota_1(h)^{-1}, \iota_1(h))]^{-1}\omega(g'g\gamma_1\iota_1(h)^{-1}\iota_1(h')^{-1}, \iota_1(h'), \iota_1(h))
\]

\[
[\omega(g', g, \gamma_2)]^{-1}\omega(g', g\gamma_2\iota_2(h)^{-1}, \iota_2(h)) [\omega(g'g\gamma_2\iota_2(h)^{-1}\iota_2(h')^{-1}, \iota_2(h'), \iota_2(h))^{-1}
\]
Defects in Dijkgraaf-Witten theories

Dijkgraaf-Witten theories

- input data: finite group $G$ and cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$

- $\mathcal{C} = D^\omega(G)\text{-mod} \simeq Z(\operatorname{Vect}(G)^\omega)$ Turaev-Viro type

- $\omega$ gives holonomy on closed three-manifolds $\sim$ topological bulk Lagrangian

- two-step gauge-theoretic construction:
  \[
  \text{Cobord}_{3,2,1} \xrightarrow{\widetilde{\text{Bun}}} \text{SpanGrp} \xrightarrow{[-,\operatorname{Vect}]^\tau} 2\text{-Vect}
  \]
  twisted linearization

extends to TFT with boundaries and defects

Example: category for circle $\mathbb{S}$ with one defect point $p$

- to interval $\mathbb{S}\setminus\{p\}$ assign group $G$ with cocycle $\omega$

- to $p$ assign homomorphism $\iota: H \to G \times G$ with cochain $\theta \in C^2(H, \mathbb{C}^\times)$

- $\widetilde{\text{Bun}}$ gives action groupoid $G \backslash G \times G \backslash_{H} H$

- twisted linearization gives $[G \backslash G \times G \backslash_{H} H, \operatorname{Vect}]^{\tau_{\omega,\theta}}$

- thus equivalent to category of $G \times G$-graded vector spaces $\bigoplus_{g_1,g_2 \in G} V(g_1,g_2)$

- with $\tau_{\omega,\theta}$-twisted $G \times H$-action $\pi_{g,h}: V(g_1,g_2) \to V(gg_1,gg_2)\iota(h)^{-1}$

- equivalent to category of $A_{G_{\text{diag}}}-A_{H,\theta}$-bimodules in $\operatorname{Vect}(G)^\omega \boxtimes \operatorname{Vect}(G)^{\omega^{-1}}$
Further developments

A few other available results:

- Transmission functors for invertible defects realize bijection invertible $\mathcal{A}$-bimodule categories $\leftrightarrow$ braided auto-equivalences of $\mathcal{Z}(\mathcal{A})$

- Gauge-theoretic description of symmetries of abelian Dijkgraaf-Witten theories $O_q(A \oplus A^\ast)$ generated by

\[
\varphi \oplus (\varphi^\ast)^{-1} \quad \text{with} \quad \varphi \in \text{Aut}(A)
\]

\[
(g, \chi) \mapsto (g, \chi + \beta(g, -)) \quad \text{with} \quad \beta \text{ alternating bicharacter (}B\text{-field)}
\]

Electric-magnetic dualities
Further developments

- A few other available results:
  - transmission functors for invertible defects realize bijection invertible $\mathcal{A}$-bimodule categories $\leftrightarrow$ braided auto-equivalences of $\mathbb{Z}(\mathcal{A})$
  - gauge-theoretic description of symmetries of abelian Dijkgraaf-Witten theories
  - simplicial constructions à la TV/BW
  - deconfining of twist defects
  - interpretation of categories arising as $\text{tft}_{3,2,1}^{\mathbb{Z}(\mathcal{A})}(\mathcal{S})$ as category-valued trace $\otimes$ for 1-morphisms in the tricategory of finite tensor categories

- Among next steps:
  - formulation of Dijkgraaf-Witten results in terms of relative Deligne product and $\otimes$ so as to extend to all Turaev-Viro TFTs