TOPOLOGICAL FIELD THEORY FOR DEFECTS IN TOPOLOGICAL PHASES

KARLSTADS UNIVERSITET

Jürgen Fuchs

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Motivation

THEME :

☞ *2-d defects in 3-d TFT as models for line defects in topological phases*

some overlap with Z. Wang' s talk — "*approach quite different*"

BARKESHLI-BONDERSON-CHENG-WANG 2014

Motivation

THEMES :

- ☞ *2-d defects in 3-d TFT as models for line defects in topological phases* (TFT as tool)
- ☞ *3-d TFT with defects of any codimension*

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POSSIBLE MOTIVATIONS :

- ☞ Topological line defects in topological phases
- ☞ Gapped interfaces between topological phases

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- ☞ *2-d defects in 3-d TFT as models for line defects in topological phases* (TFT as tool)
- ☞ *3-d TFT with defects of any codimension*

POSSIBLE MOTIVATIONS :

- ☞ Topological line defects in topological phases
- ☞ Gapped interfaces between topological phases
- ☞ TFT with substructures / on stratified spaces
- ☞ Extended TFT / higher categories
- ☞ Defects in general quantum field theory
- ☞ Applications to 2-d rational conformal field theory

Defects in QFT

☞ Codimension-1 defect **QFT**$_1$ | **QFT**$_2$

$=$ interface separating region supporting QFT$_1$ from region supporting QFT$_2$

Defects in QFT

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- ubiquitous in nature
- natural part of the structure of quantum field theory
- *physical boundaries* as special case

QFT$_1$ |

Defects in QFT

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  = interface separating region supporting QFT$_1$ from region supporting QFT$_2$

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- *Topological* defect : correlators do not change when deforming the defect

  without crossing other substructures

- Example : 2-d Ising model

  - ferromagnetic nearest-neighbour interaction
  - change coupling to *anti*-ferromagnetic on all bonds crossed by some line

    $\rightsquigarrow$ topological defect line

Defects in QFT

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- Some general features of topological defects :
  - codimension-2 defects **def_1 | def_2** etc
  - transparent defect
  - invert orientation $\leadsto$ dual defect
  - move two topological defects to coincidence $\leadsto$ fusion product of defects

Defects in QFT

- Codimension-1 defect **QFT$_1$** | **QFT$_2$**

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  - codimension-2 defects **def$_1$** | **def$_2$** etc
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  - invert orientation ↝ dual defect
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- Mathematical formulation : ↝ higher categories

Invertible defects and symmetries

assume : defects form a rigid monoidal category

(proven for 2-d RCFT)

FJELSTAD-ꟻ-RUNKEL-SCHWEIGERT 2008

FRÖHLICH-ꟻ-RUNKEL-SCHWEIGERT 2007

KAPUSTIN-SAULINA 2011

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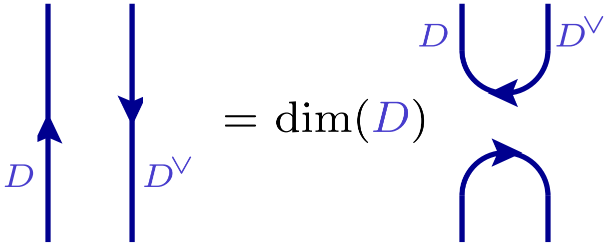

drawn for $d=2$

$\dim(D) = \pm 1$

$\rightsquigarrow$ identity of correlators when applied locally in any configuration of fields & defects

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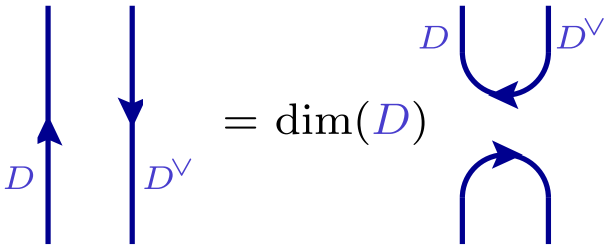

⤳ identity of correlators when applied locally in any configuration of fields & defects

- invertible defects form a group under fusion
- act on *all* data of the theory as a symmetry group
- e.g. critical 2-d Ising model : $\mathbb{Z}_2$
  critical three-state Potts model : $\mathfrak{S}_3$

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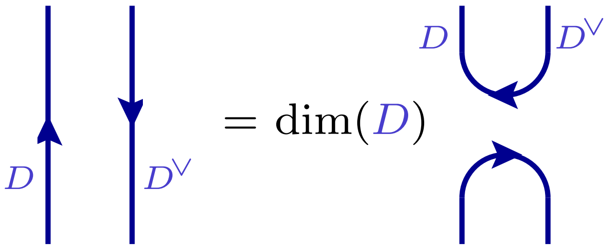
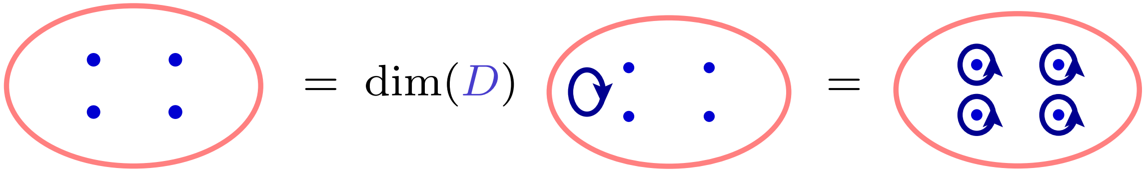


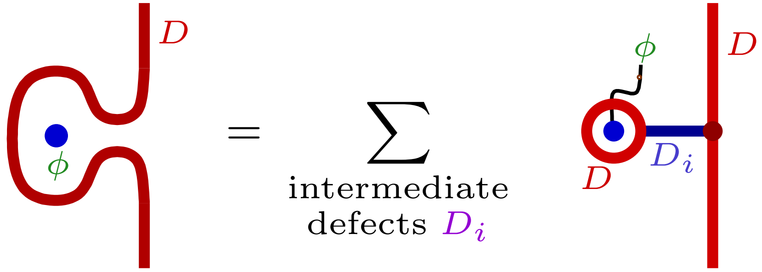
  - ⇝ identity of correlators when applied locally in any configuration of fields & defects
  - ↯ invertible defects form a group under fusion
  - ↯ act on *all* data of the theory as a symmetry group
  - ↯ Example : equalities for bulk field correlators on sphere :



Duality defects

(continuing in $d = 2$)

☞ Wrapping of general topological defect around a bulk field :



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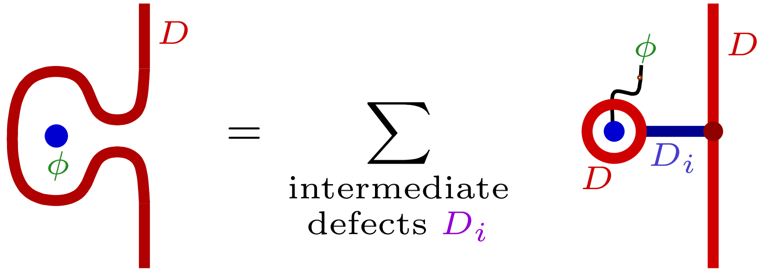

⇝ bulk field turned into disorder field

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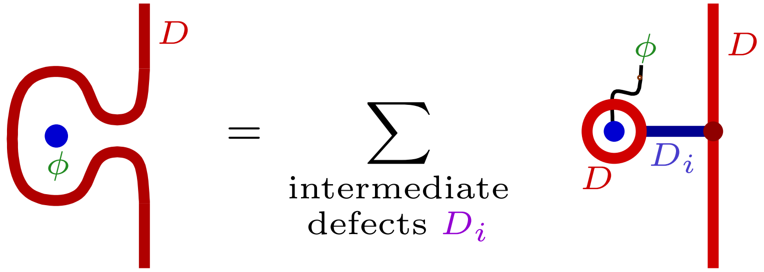

- bulk field turned into disorder field
- wrapping with dual defect turns disorder field back to bulk field if and only if $D \otimes D^{\vee}$ is direct sum of invertible defects
- in this case have an order-disorder duality

  e.g. critical 2-d Ising model : remnant of Kramers-Wannier duality
- again action on all field theoretic quantities

Duality defects

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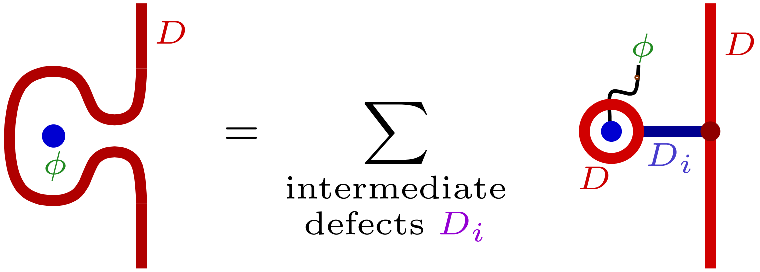

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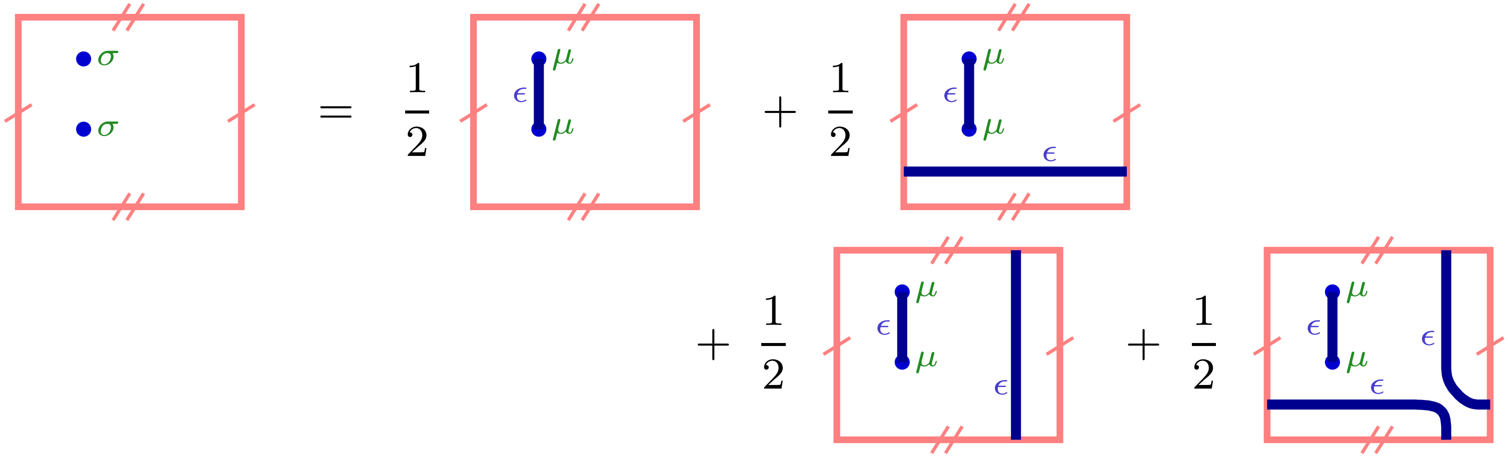
☞ Example : correlator of two Ising spin fields on a torus :



Plan

GOAL : Similar results for defects in 3-d TFT

TASKS :

- ☞ Achieve basic understanding of topological defects in 3-d TFT
- ☞ Study consequences in relevant classes of models
- ☞ Apply insight to topological phases
- ☞ Construct 3-d TFT with topological defects mathematically

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- ☞ Codimension-1 defects in QFT ✓

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- ☞ Application : Multi-layer systems
- ☞ Appendix : Defects in Dijkgraaf-Witten theories

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COLLABORATORS : Jan Priel , Gregor Schaumann ,
Christoph Schweigert , Alessandro Valentino

RT-type TFT with defects

- ☞ RT-type TFT : symmetric monoidal functor $\mathbf{tft}^{\mathcal{D}}_{3,2} : \mathcal{C}obord_{3,2} \longrightarrow \mathcal{V}ect$

  resp. 2-functor $\mathbf{tft}^{\mathcal{D}}_{3,2,1} : \mathcal{C}obord_{3,2,1} \longrightarrow 2\text{-}\mathcal{V}ect$

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- insertions on Wilson lines / junctions labeled by morphisms of $\mathcal{D}$
- 2-d cut-and-paste boundaries on which Wilson lines can end
- state spaces for cut-and-paste boundaries $=$ morphisms spaces $\mathrm{Hom}_{\mathcal{D}}(X, \mathbf{1})$

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- ☞ RT-type TFT with boundaries and defects :

  - ⇝ include in $\mathcal{C}obord$ three-manifolds with physical boundary
  - ⇝ include in $\mathcal{C}obord$ three-manifolds with surface defects

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- ☞ RT-type TFT with boundaries and defects :

  - ⇝ include three-manifolds with physical boundary and / or surface defects
  - ⇝ 3-d bulk regions labeled by modular tensor categories $\mathcal{D}_1, \mathcal{D}_2, \ldots$

    (bulk Wilson lines in such a region labeled by objects of $\mathcal{D}_i$)
  - ⇝ boundary Wilson lines and defect Wilson lines
  - ⇝ several layers of insertions and of junctions

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  Task : construct symmetric monoidal 2-functor $\mathcal{C}obord^{\partial}_{3,2,1} \longrightarrow 2\text{-}\mathcal{V}ect$
  for category of cobordisms with corners

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In particular :

- ⇝ determine labels for physical boundaries / for surface defects
- ⇝ determine labels for boundary and defect Wilson lines and for insertions

Conjecture : *Fit together to form bicategories of module categories*

Categories of boundary Wilson lines

- ☞ Select boundary "a" to some bulk region labeled by a modular tensor cateory $\mathcal{C}$
  - can contain boundary Wilson lines
  - Wilson line can contain insertions
  - insertions can be composed

$\rightsquigarrow$ category $\mathcal{W}_a$ of Wilson lines on boundary a

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- boundary Wilson lines can be fused and can be deformed

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- insertions can be composed
- boundary Wilson lines can be fused and can be deformed
- also impose : finitely semisimple etc

$\rightsquigarrow$ spherical fusion category $\mathcal{W}_a$ of Wilson lines on boundary a

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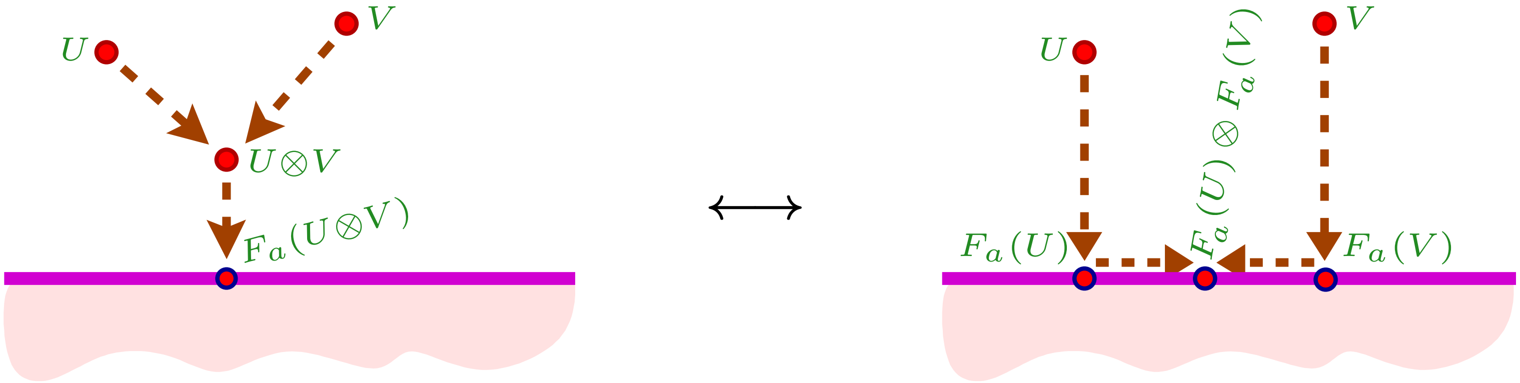

$\rightsquigarrow$ monoidal structure $F_a(U \otimes_{\mathcal{C}} V) \xrightarrow{\cong} F_a(U) \otimes_{\mathcal{W}_a} F_a(V)$

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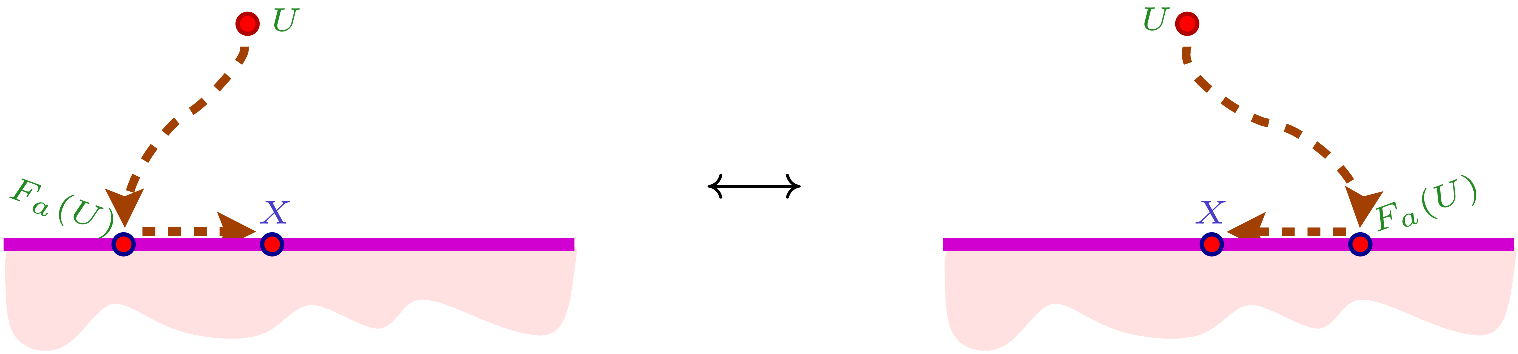

$\rightsquigarrow$ central structure $F_a(U) \otimes_{\mathcal{W}_a} X \xrightarrow{\cong} X \otimes_{\mathcal{W}_a} F_a(U)$

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equivalently : choice of lift

$$\begin{tikzcd} & \mathcal{Z}(\mathcal{W}_a) \arrow[d, "\mathrm{forget}"] \\ \mathcal{C} \arrow[ur, dashed, "\widetilde{F}_a"] \arrow[r, "F_a"'] & \mathcal{W}_a \end{tikzcd}$$

to Drinfeld center of $\mathcal{W}_a$

DAVYDOV-MÜGER-NIKSHYCH-OSTRIK 2013

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- ☞ Postulate naturality :
  only reason for being able to consistently move boundary Wilson line $Y \in \mathcal{W}_a$ past any $X \in \mathcal{W}_a$ should be that $Y = F_a(U)$ for some $U \in \mathcal{C}$

  $\rightsquigarrow$ braided equivalence $\boxed{\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)}$

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In short : Compatible boundary condition for bulk region $\mathcal{C}$

= *Witt trivialization* $\widetilde{F}_a\colon\ \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)$ for some fusion category $\mathcal{W}_a$

Bicategories of boundary conditions

☞ Thus for single boundary condition a : $\boxed{\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)}$

- in particular obstruction : no compatible boundary condition unless $[\mathcal{C}] = 0$ in *Witt group* of modular tensor categories

Bicategories of boundary conditions

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- ☞ Other boundary condition b :

  other fusion category $\mathcal{W}_b$ of Wilson lines in region b

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- ☞ Other boundary condition b :
  - ↝ category $\mathcal{W}_{a,b}$ of Wilson lines separating boundary region labeled a from region labeled b
  - ↝ fusion of Wilson lines in region a $\rightsquigarrow$ functor $\mathcal{W}_a \times \mathcal{W}_{a,b} \longrightarrow \mathcal{W}_{a,b}$
  - ↝ gives action of $\mathcal{W}_a$ on $\mathcal{W}_{a,b}$: $\mathcal{W}_{a,b}$ is left module category over $\mathcal{W}_a$
  - ↝ likewise : $\mathcal{W}_{a,b}$ is right module category over $\mathcal{W}_b$

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  - ↝ likewise : $\mathcal{W}_{a,b}$ is right module category over $\mathcal{W}_b$
  - ↝ but also : $\mathcal{W}_{a,b}$ is right module category over $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b})$

module endofunctors

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- likewise : $\mathcal{W}_{a,b}$ is right module category over $\mathcal{W}_b$
- but also : $\mathcal{W}_{a,b}$ is right module category over $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b})$

☞ Impose naturality : $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b}) \simeq \mathcal{W}_b$

Consistency check : $\mathcal{Z}(\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b})) \simeq \mathcal{Z}(\mathcal{W}_a)$ canonically SCHAUENBURG 2001

Bicategories of boundary conditions

☞ Thus for single boundary condition a : $\boxed{\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)}$

- ↝ in particular obstruction : no compatible boundary condition unless $[\mathcal{C}] = 0$ in *Witt group* of modular tensor categories

☞ Other boundary condition b :

- ↝ category $\mathcal{W}_{a,b}$
  of Wilson lines separating boundary region labeled a from region labeled b
- ↝ fusion of Wilson lines in region a $\rightsquigarrow$ functor $\mathcal{W}_a \times \mathcal{W}_{a,b} \longrightarrow \mathcal{W}_{a,b}$
- ↝ gives action of $\mathcal{W}_a$ on $\mathcal{W}_{a,b}$: $\mathcal{W}_{a,b}$ is left module category over $\mathcal{W}_a$
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☞ Impose naturality : $\mathcal{E}nd_{\mathcal{W}_a}(\mathcal{W}_{a,b}) \simeq \mathcal{W}_b$

$\Longrightarrow$ can work with a single *reference boundary condition* a

☞ Conjecture : *Boundary conditions for $\mathcal{C}$ form the bicategory $\mathcal{W}_a$-$\mathcal{M}od$*
of module categories over a fusion category $\mathcal{W}_a$ satisfying $\mathcal{Z}(\mathcal{W}_a) \simeq \mathcal{C}$

Bicategories of boundary conditions

- ☞ Will assume : Boundary conditions given by $\mathcal{W}_a$-$\mathcal{M}od$
- ☞ Then $\mathcal{W}_{b,c} \simeq \mathcal{F}un_{\mathcal{W}_a}(\mathcal{W}_b, \mathcal{W}_c)$ for any pair of boundary conditions $b\,,\ c$

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- ☞ Warning :

  via $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a) \xrightarrow{\text{forget}} \mathcal{W}_a$

  any $\mathcal{M} \in \mathcal{W}_a$-$\mathcal{M}od$ has natural structure of $\mathcal{C}$-module category

  But not every $\mathcal{C}$-module category of a Witt-trivial $\mathcal{C}$ gives a boundary condition

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Illustration : Toric code

↯ 2 elementary boundary conditions BRAVYI-KITAEV 2001

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Illustration : Toric code

- 2 elementary boundary conditions
- $\mathcal{C} = \mathcal{Z}(\mathcal{V}ect(\mathbb{Z}_2))$
- 6 inequivalent indecomposable module categories over $\mathcal{C}$
- 2 inequivalent indecomposable module categories over $\mathcal{W} = \mathcal{V}ect(\mathbb{Z}_2)$

Bicategories of surface defects

- ☞ Parallel analysis for surface defects :
  - defect d separating bulk regions labeled by $\mathcal{C}_1$ and $\mathcal{C}_2$
  - two monoidal functors $\mathcal{C}_1 \to \mathcal{W}_d$ and $\mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d$ to fusion category $\mathcal{W}_d$

inverse braiding

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  - ↝ combine to central functor $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d$

Deligne product

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  - ⇝ combine to central functor $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \to \mathcal{W}_d$
  - ⇝ naturality $\rightsquigarrow$ braided equivalence $\boxed{\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a)}$
  - ⇝ obstruction : no defects between $\mathcal{C}_1$ and $\mathcal{C}_2$ unless $[\mathcal{C}_1] = [\mathcal{C}_2]$ in Witt group

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- ☞ *Defects separating $\mathcal{C}_1$ from $\mathcal{C}_2$ form the bicategory $\mathcal{W}_d$-$\mathcal{M}od$*

  of module categories over a fusion category $\mathcal{W}_d$ satisfying $\mathcal{Z}(\mathcal{W}_d) \simeq \mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}}$

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- Example : Canonical Witt trivialization $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$ ($\mathcal{C}$ modular)
  - defects separating $\mathcal{C}$ from itself $=$ $\mathcal{C}$-module catgeories

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  - defects separating $\mathcal{C}$ from itself $=$ $\mathcal{C}$-module catgeories
  - *regular* $\mathcal{C}$-module category $(\mathcal{C}, \otimes)$ $\rightsquigarrow$ transparent defect $\mathcal{T}$
  - serves as monoidal unit for fusion of surface defects
  - Wilson lines separating transparent defect from itself $=$ ordinary Wilson lines

Bicategories of surface defects

- Parallel analysis for surface defects :
  - defect d separating bulk regions labeled by $\mathcal{C}_1$ and $\mathcal{C}_2$
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  - *regular* $\mathcal{C}$-module category $(\mathcal{C}, \otimes)$ $\rightsquigarrow$ transparent defect $\mathcal{T}$
- Example : *Turaev-Viro* TFT : $\mathcal{C}_1 \simeq \mathcal{Z}(\mathcal{A}_1)$ and $\mathcal{C}_2 \simeq \mathcal{Z}(\mathcal{A}_2)$

  $\rightsquigarrow$ $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{A}_1) \boxtimes \mathcal{Z}(\mathcal{A}_2^{\mathrm{op}}) \simeq \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2^{\mathrm{op}})$

  $\rightsquigarrow$ defects separating $\mathcal{C}_1$ from $\mathcal{C}_2$ form bicategory $\mathcal{A}_1\text{-}\mathcal{A}_2\text{-}\mathcal{B}imod$

KITAEV-KONG 2012

Defects for multi-layer systems

- ☞ Classification of module categories over a general modular tensor category $\mathcal{D}$ out of reach

  (even finding *any* indecomposable $\mathcal{D}$-module besides $(\mathcal{D},\otimes)$ can be hard)

- ☞ Side remark :

  bijection between indecomposable $\mathcal{D}$-module categories and

  modular invariant torus partition functions for the rational conformal field theory based on $\mathcal{D}$

Defects for multi-layer systems

- ☞ Classification of module categories over a general modular tensor category $\mathcal{D}$ out of reach
- ☞ TFT for N-layer system : modular tensor category $\mathcal{D} = \mathcal{C}^{\boxtimes N}$ with $\mathcal{C}$ modular tensor category for each single layer

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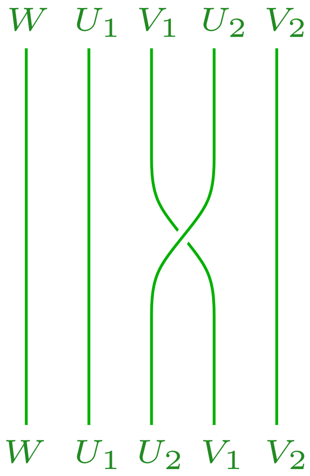
- ☞ Classification of module categories over a general modular tensor category $\mathcal{D}$ out of reach
- ☞ TFT for N-layer system : modular tensor category $\mathcal{D} = \mathcal{C}^{\boxtimes N}$
- ☞ Generic non-trivial right $\mathcal{D}$-module category : $\mathcal{P} \equiv \mathcal{P}_{\mathcal{D}} := (\mathcal{C}, \triangleleft, \alpha)$

  with $W \triangleleft (U_1 \boxtimes \cdots \boxtimes U_N) = W \otimes U_1 \otimes \cdots \otimes U_N$

  and mixed associativity constraint
  for $N = 2$



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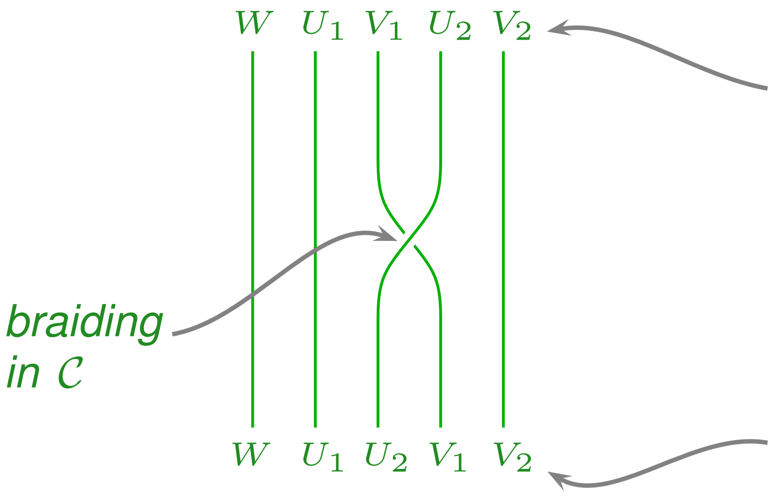

$$W \otimes U_1 \otimes V_1 \otimes U_2 \otimes V_2$$

$$= \big(W \triangleleft (U_1 \boxtimes V_1)\big) \triangleleft (U_2 \boxtimes V_2)$$

$$= W \triangleleft \big((U_1 \boxtimes V_1) \otimes_{\mathcal{D}} (U_2 \boxtimes V_2)\big)$$

$$= W \triangleleft \big((U_1 \otimes_{\mathcal{D}} U_2) \boxtimes (V_1 \otimes_{\mathcal{D}} V_2)\big)$$

$$W \otimes U_1 \otimes V_1 \otimes U_2 \otimes V_2$$

categorification of fact that *commutative* ring R is $R \otimes_{\mathbb{Z}} R$-module

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  Side remark : corresponding to permutation modular invariants in RCFT

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- ☞ From now on restrict to two-layer system $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$
  - ⇝ two generic $\mathcal{D}$-module categories $\mathcal{D} \equiv \mathcal{T}$ and $\mathcal{P}$
  - ⇝ right action $\mathcal{P} \times \mathcal{D} \to \mathcal{P}$

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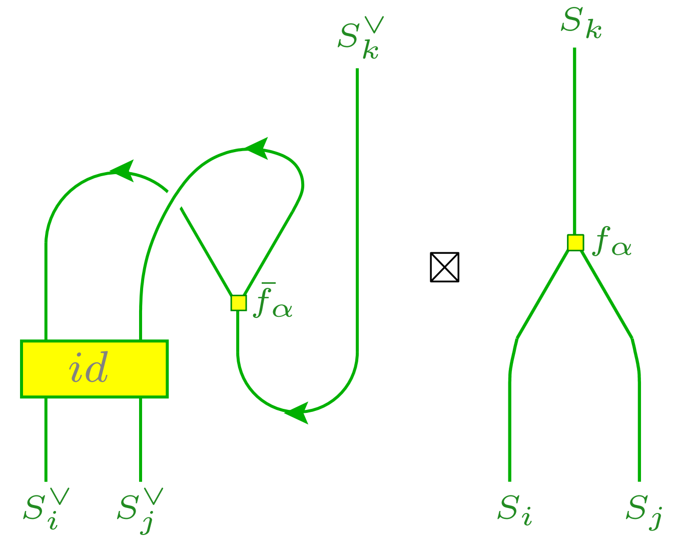
  - ⇝ right action $\mathcal{P} \times \mathcal{D} \to \mathcal{P}$
  - ⇝ form part of a $\mathbb{Z}_2$-*equivariant* modular category
  - ⇝ thus further fusion functors $\mathcal{D} \times \mathcal{P} \to \mathcal{P}$ and $\mathcal{P} \times \mathcal{P} \to \mathcal{D}$
  - ⇝ derivable from a $\mathbb{Z}_2$-equivariant topological field theory

The $\mathcal{C}\boxtimes\mathcal{C}$-module $\mathcal{P}$

- ☞ $\mathcal{D}$-module category $\mathcal{P}$ realizable as category $A_{\mathcal{P}}$-mod of left $A_{\mathcal{P}}$-modules in $\mathcal{D}$
  - $A_{\mathcal{P}} = \bigoplus_{i\in I_{\mathcal{C}}} S_i^\vee \boxtimes S_i$ as object
  - algebra structure determined by fusion of simple objects in $\mathcal{C}$:

$$m \;=\; \bigoplus_{i,j,k\in I_{\mathcal{C}}} \sum_{\alpha=1}^{\mathrm{N}_{ij}^{\;k}}$$



$$\eta \;=\; e_{\mathbf{1}\boxtimes\mathbf{1}\prec A_{\mathcal{P}}}$$

BARMEIER-JF-RUNKEL-SCHWEIGERT 2010

BARMEIER-SCHWEIGERT 2011

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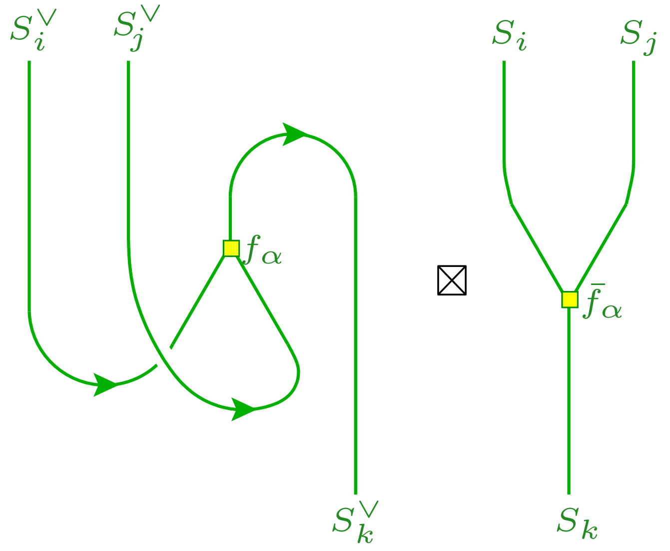
  - ↯ $A_{\mathcal{P}} = \bigoplus_{i\in I_{\mathcal{C}}} S_i^\vee \boxtimes S_i$
  - ↯ symmetric special Frobenius algebra :

$$\Delta \;=\; \bigoplus_{i,j,k\in I_{\mathcal{C}}} \frac{\dim(S_i)\,\dim(S_j)}{\mathrm{Dim}(\mathcal{C})\,\dim(S_k)} \sum_\alpha$$



$$\eta \;=\; r_{A_{\mathcal{P}}\succ \mathbf{1}\boxtimes\mathbf{1}}$$

The $\mathcal{C}\boxtimes\mathcal{C}$-module $\mathcal{P}$

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  - $A_\mathcal{P} = \bigoplus_{i\in I_\mathcal{C}} S_i^\vee \boxtimes S_i$
  - symmetric special Frobenius algebra
  - Azumaya algebra :

    braided induction functors $\alpha^\pm_{A_\mathcal{P}} : \mathcal{C} \rightarrow A_\mathcal{P}$-bimod are monoidal equivalences

$$U \longmapsto \big(A_\mathcal{P} \otimes U\,, m \otimes \mathrm{id}_U\,, (m \otimes \mathrm{id}_U) \circ (\mathrm{id}_{A_\mathcal{P}} \otimes c_{U,A_\mathcal{P}}\big)$$

resp. $\quad c^{-1}_{A_\mathcal{P},U})$

The $\mathcal{C}\boxtimes\mathcal{C}$-module $\mathcal{P}$

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  - symmetric special Frobenius Azumaya algebra
- ☞ Analogously

$$\bigoplus_{i_1,i_2,\dots,i_N\in I_{\mathcal{C}}} (S_{i_1}\boxtimes S_{i_2}\boxtimes\cdots\boxtimes S_{i_N})^{\oplus \mathrm{N}_{i_1,i_2,\dots,i_N}}$$

for $N>2$ $\qquad \mathrm{N}_{i_1,i_2,\dots,i_N} = \dim \mathrm{Hom}_{\mathcal{C}}(S_{i_1}\otimes S_{i_2}\otimes\cdots\otimes S_{i_N},\mathbf{1})$

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- ☞ For A Azumaya $\Psi_A := (\alpha_A^{+})^{-1} \circ \alpha_A^{-}$

  describes transmission of bulk Wilson lines through surface defect A-mod

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  describes transmission of bulk Wilson lines through surface defect A-mod
  - ⇝ $\alpha_{A_\mathcal{P}}^+(U\boxtimes V) \cong \alpha_{A_\mathcal{P}}^-(V\boxtimes U)$ by direct calculation

    $\rightsquigarrow$ transmission of bulk Wilson lines through $\mathcal{P}$ permutes the layers

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  - $\alpha^+_{A_{\mathcal{P}}}(U \boxtimes V) \cong \alpha^-_{A_{\mathcal{P}}}(V \boxtimes U)$
- Braided induction for tensor products :

$$\begin{tikzcd}
 & & (A_1{\otimes}A_2)\text{-bimod} & & \\
 & A_1\text{-bimod} \arrow[ur, "\beta^-"'] \arrow[rr, Rightarrow, "\tilde\nu"] \arrow[u, phantom, "\Uparrow\,\nu"] & & A_2\text{-bimod} \arrow[ul, "\beta^+"] \arrow[u, phantom, "\Uparrow\,\mathrm{Id}"'] & \\
\mathcal{C} \arrow[ur, "\alpha^-_{A_1}"'] \arrow[uurr, bend left=60, "\alpha^-_{A_1\otimes A_2}"] & & \mathcal{C} \arrow[ul, "\alpha^+_{A_1}"] \arrow[ur, "\alpha^-_{A_2}"'] & & \mathcal{C} \arrow[ul, "\alpha^+_{A_2}"] \arrow[uull, bend right=60, "\alpha^+_{A_1\otimes A_2}"']
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  - $\Psi_{A_1\otimes A_2} = \Psi_{A_1} \circ \Psi_{A_2}$ as monoidal functors if $A_{1,2}$ Azumaya
  - $A_\mathcal{P} \otimes A_\mathcal{P}$ Morita equivalent to $\mathbf{1}_\mathcal{D}$

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  - ⇝ $A_{\mathcal{P}} \otimes A_{\mathcal{P}}$ Morita equivalent to $\mathbf{1}_{\mathcal{D}}$
- ☞ Fusion rules :

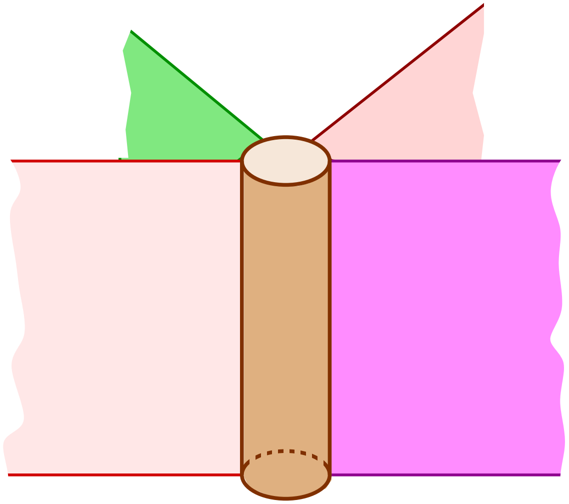
$$\mathcal{T} \boxtimes_{\mathcal{D}} \mathcal{P} \simeq \mathcal{P}$$
$$\mathcal{P} \boxtimes_{\mathcal{D}} \mathcal{P} \simeq \mathcal{T}$$

- ☞ Categories of defect Wilson lines :

$$\mathcal{F}un_{\mathcal{D}}(\mathcal{T},\mathcal{P}) \simeq (\mathbf{1}_{\mathcal{D}} \otimes A_{\mathcal{P}})\text{-mod} \cong A_{\mathcal{P}}\text{-mod} \cong \mathcal{C}$$
$$\mathcal{F}un_{\mathcal{D}}(\mathcal{P},\mathcal{T}) \simeq \mathcal{C}$$
$$\mathcal{E}nd_{\mathcal{D}}(\mathcal{T}) \simeq \mathcal{D} \simeq \mathcal{E}nd_{\mathcal{D}}(\mathcal{P})$$

Relation with extended TFT

- ☞ More general Wilson lines :



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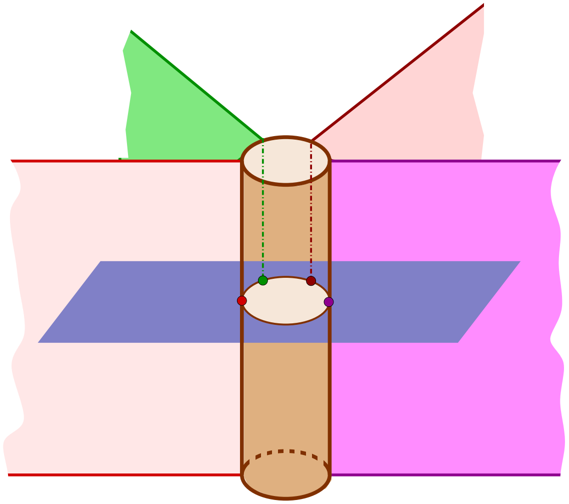

- ☞ Via extended TFT $\mathbf{tft}^{\mathcal{D}}_{3,2,1}$ assign categories :

$$\begin{array}{ccc} \mathcal{C}ob\mathit{ord}_{3,2,1} & \longrightarrow & 2\text{-}\mathcal{V}ect \\ M & \longmapsto & \mathbf{tft}^{\mathcal{D}}_{3,2,1}(M) \end{array}$$

1-manifold category

e.g. circle : $\mathbf{tft}^{\mathcal{D}}_{3,2,1}(\mathbb{S}) = \mathcal{D}$

Relation with extended TFT

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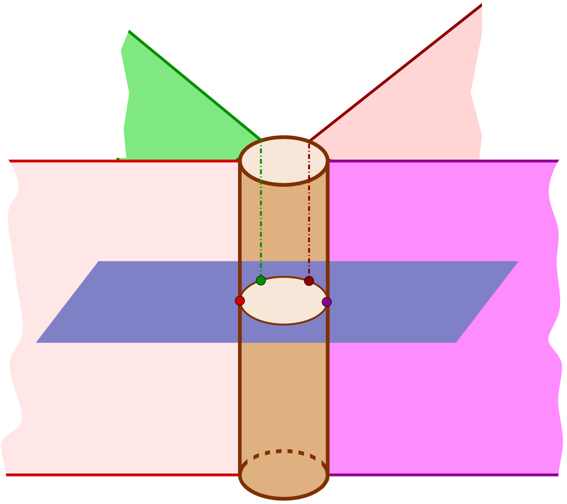


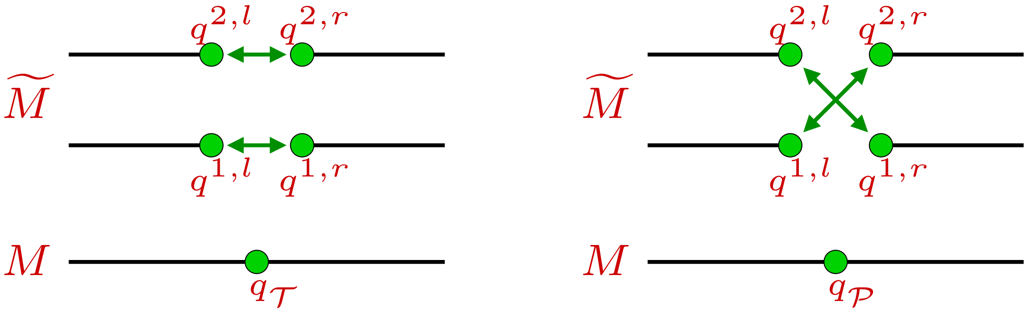
- Via extended TFT $\mathbf{tft}^{\mathcal{D}}_{3,2,1}$ assign categories : $\mathcal{C}obord_{3,2,1} \longrightarrow 2\text{-}\mathcal{V}ect$

$$M \longmapsto \mathbf{tft}^{\mathcal{D}}_{3,2,1}(M)$$

- Circle with defect points : use cover functor

$M \longmapsto$ two-sheeted cover $\widetilde{M}$

locally :



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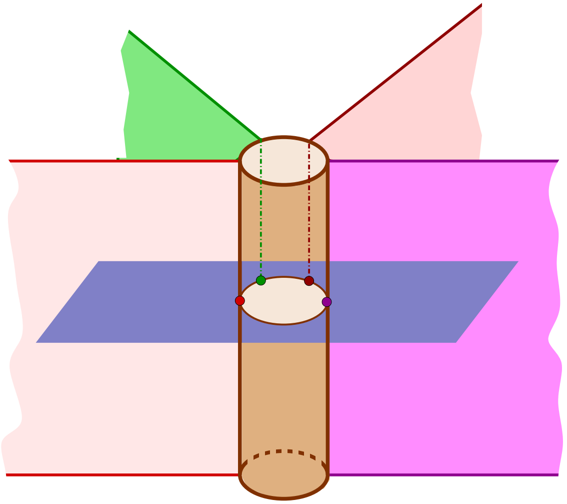

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$M \longmapsto$ two-sheeted cover $\widetilde{M}$

$$\mathbf{tft}^{\mathbb{Z}_2;\mathcal{D}}_{3,2,1}(\mathbb{S}_{n_\mathcal{T},n_\mathcal{P}})$$

$$= \mathbf{tft}^{\mathcal{C}}_{3,2,1}(\widetilde{\mathbb{S}}_{n_\mathcal{T},n_\mathcal{P}}) = \begin{cases} \mathbf{tft}_\mathcal{C}(\mathbb{S} \sqcup \mathbb{S}) \simeq \mathbf{tft}_\mathcal{C}(\mathbb{S}) \boxtimes \mathbf{tft}_\mathcal{C}(\mathbb{S}) = \mathcal{C} \boxtimes \mathcal{C} \ = \ \mathcal{D} & \text{for } n_\mathcal{P} \text{ even} \\ \mathbf{tft}_\mathcal{C}(\mathbb{S}) \ = \ \mathcal{C} & \text{for } n_\mathcal{P} \text{ odd} \end{cases}$$

reproducing the previous results for categories of defect Wilson lines

Relation with extended TFT

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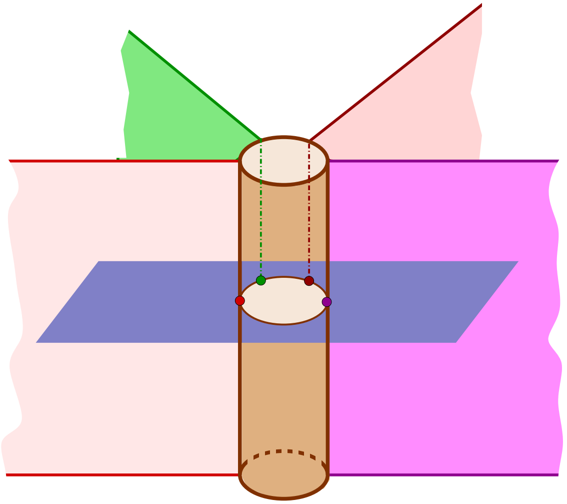
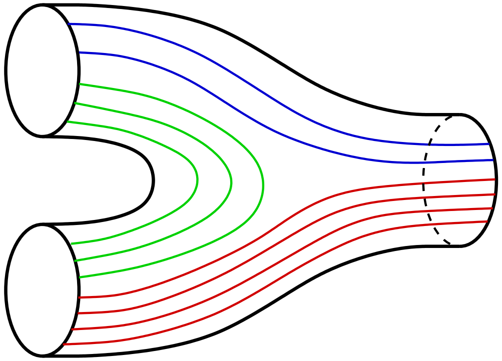


- ☞ Via extended TFT $\mathbf{tft}^{\mathcal{D}}_{3,2,1}$ assign functors to 2-manifolds
- ☞ General surfaces with Wilson lines :



Relation with extended TFT

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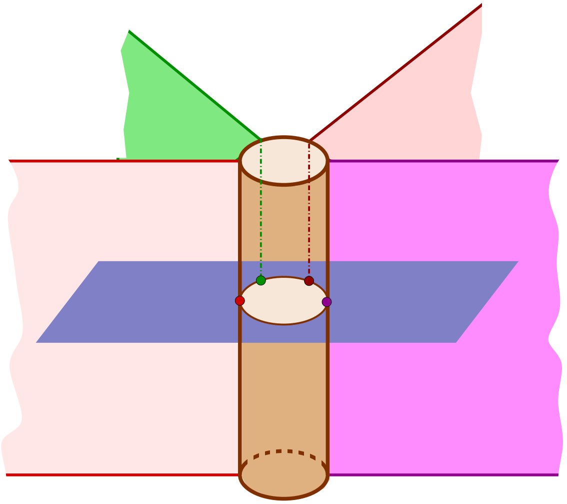

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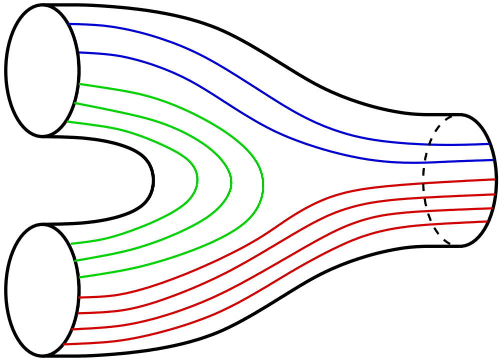


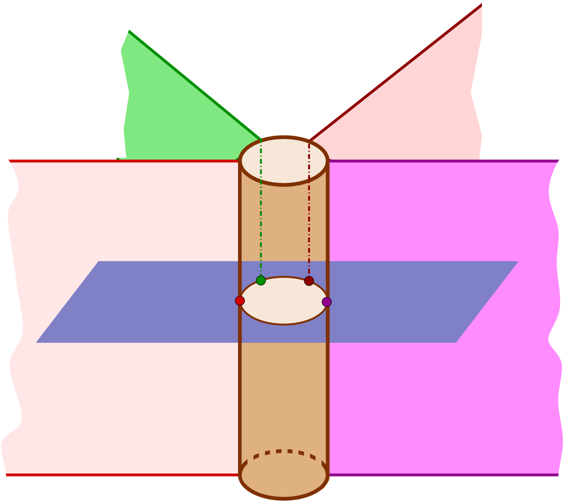
functor $\mathbf{tft}^{\mathcal{D}}_{3,2,1}(\partial_-\Sigma \xrightarrow{\Sigma} \partial_+\Sigma)$

e.g. pair of pants

$\mathrm{Y} \longmapsto \boxtimes : \mathcal{D}\times\mathcal{D}\rightarrow\mathcal{D}$

Relation with extended TFT

- More general Wilson lines :



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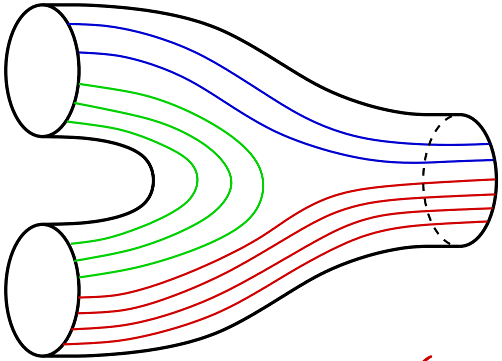

functor $\mathbf{tft}^{\mathcal{D}}_{3,2,1}(\partial_-\Sigma \xrightarrow{\Sigma} \partial_+\Sigma)$

e.g. pair of pants

$\mathrm{Y} \longmapsto \boxtimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$

- General case :

e.g. via cover functor : pair of pants $\mathrm{Y}_{n_1,n_2,n_3} \longmapsto \begin{cases} \boxtimes & \text{for } n_1+n_2 \text{ even} \\ \triangleleft & \text{for } n_1+n_2 \text{ odd} \end{cases}$

(n_1,n_2,n_3) $\mathcal{P}$-defects on $\partial\mathrm{Y}$

Spaces of conformal blocks

- ☞ Surface without defect lines with $\partial_+\Sigma = \emptyset$ and $g_\Sigma = 0$ and $\pi_0(\partial\Sigma) = m$ gives functor

$$\begin{aligned} \mathcal{D}^{\boxtimes m} &\longrightarrow \mathcal{V}ect \\ U_1 \boxtimes \cdots \boxtimes U_m &\longmapsto \mathrm{Hom}_{\mathcal{D}}(U_1 \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} U_m, \mathbf{1}_{\mathcal{D}}) \end{aligned}$$

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  gives functor

$$\begin{aligned} \mathcal{D}^{\boxtimes m} &\longrightarrow \mathcal{V}ect \\ U_1\boxtimes\cdots\boxtimes U_m &\longmapsto \mathrm{Hom}_{\mathcal{D}}(U_1\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}} U_m, \mathbf{1}_{\mathcal{D}}) \\ &= \text{space of conformal blocks} \\ &= \text{space of ground states of topologial phase} \end{aligned}$$

  - ⇝ generalizes to higher genus
  - ⇝ dimension computed by Verlinde formula

Spaces of conformal blocks

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$$\begin{array}{rcl} \mathcal{D}^{\boxtimes m} & \longrightarrow & \mathcal{V}ect \\ U_1 \boxtimes \cdots \boxtimes U_m & \longmapsto & \mathrm{Hom}_{\mathcal{D}}(U_1 \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} U_m, \mathbf{1}_{\mathcal{D}}) \end{array}$$

- ☞ General surface :

  m_0 boundary circles ○ with even number of $\mathcal{P}$-defects

  m_1 boundary circles ⬭ with odd number of $\mathcal{P}$-defects

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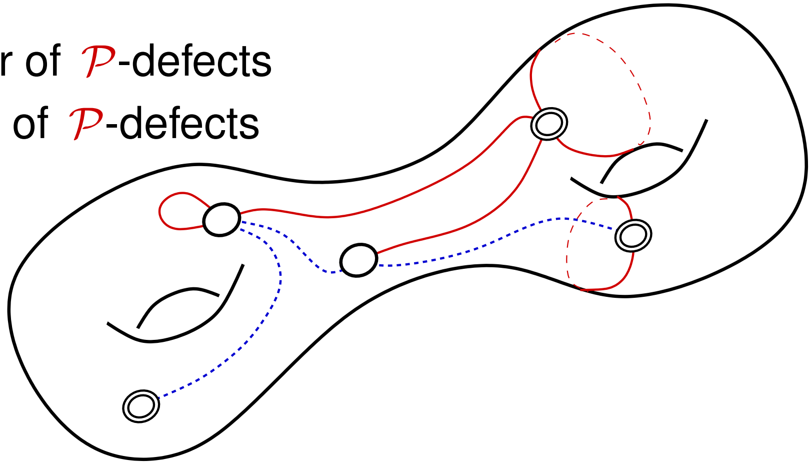

gives functor $\mathcal{D}^{\boxtimes m_0} \boxtimes \mathcal{C}^{\boxtimes m_1} \longrightarrow \mathcal{V}ect$

- expressible as a composite of functors in pair-of-pants decomposition of Σ
- glue $\mathbb{Z}_2$-covers of pairs of pants $\rightsquigarrow$ branched twofold cover $\widetilde{\Sigma}$
- compatible with gluing of surfaces with defects
- $\mathbf{tft}^{\mathbb{Z}_2;\mathcal{D}}_{3,2,1}(\Sigma) = \mathbf{tft}^{\mathcal{C}}_{3,2,1}(\widetilde{\Sigma})$

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  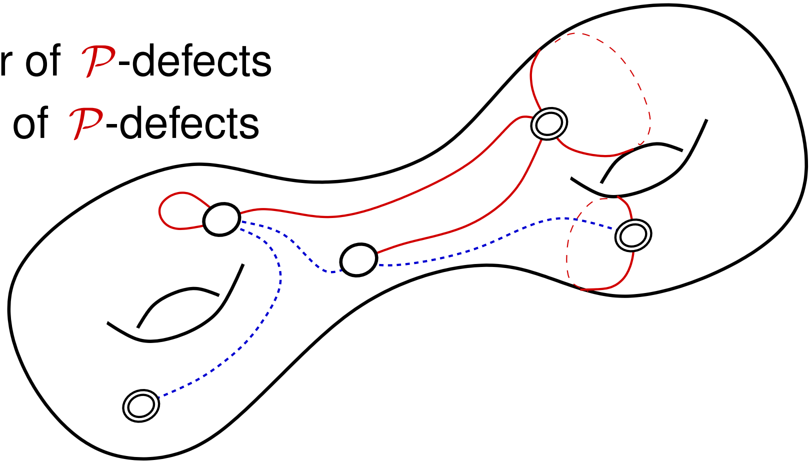

  gives functor $\mathcal{D}^{\boxtimes m_0} \boxtimes \mathcal{C}^{\boxtimes m_1} \longrightarrow \mathcal{V}ect$

- Generalized Verlinde formula via ordinary Verlinde formula for $\mathbf{tft}^{\mathcal{C}}_{3,2,1}(\widetilde{\Sigma})$
  - boundary circle with even number of $\mathcal{P}$-defects labeled by $U \boxtimes \tilde{U} \in \mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}$

    (pre-image on $\widetilde{\Sigma}$ consisting of two circles)
  - boundary circle with odd number of $\mathcal{P}$-defects labeled by $V \in \mathcal{C}$

    (pre-image on $\widetilde{\Sigma}$ consisting of one circle)

Spaces of conformal blocks

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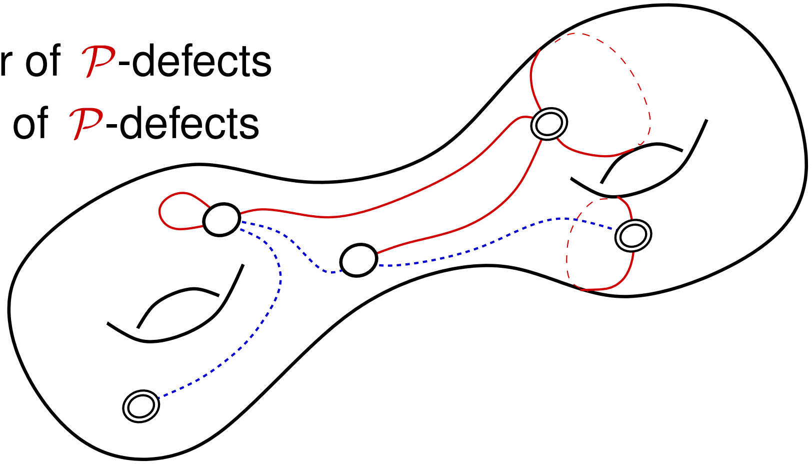

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☞ Generalized Verlinde formula via ordinary Verlinde formula for $\mathbf{tft}^{\mathcal{C}}_{3,2,1}(\widetilde{\Sigma})$

⤳ boundary circle with even number of $\mathcal{P}$-defects labeled by simple $U_i \boxtimes \tilde{U}_i \in \mathcal{D}$

⤳ boundary circle with odd number of $\mathcal{P}$-defects labeled by simple $V_j \in \mathcal{C}$

$$\boxed{\dim_{\mathbb{C}}\big(\mathbf{tft}^{\mathcal{D}}(\Sigma;\, \{U_i \boxtimes \tilde{U}_i\}, \{V_j\})\big) = \sum_{n \in I_{\mathcal{C}}} (S_{0,n})^{2\chi - m_1} \prod_{i=1}^{m_0} \frac{S_{U_i,n}}{S_{0,n}} \frac{S_{\tilde{U}_i,n}}{S_{0,n}} \prod_{j=1}^{m_1} \frac{S_{V_j,n}}{S_{0,n}}}$$

Spaces of conformal blocks

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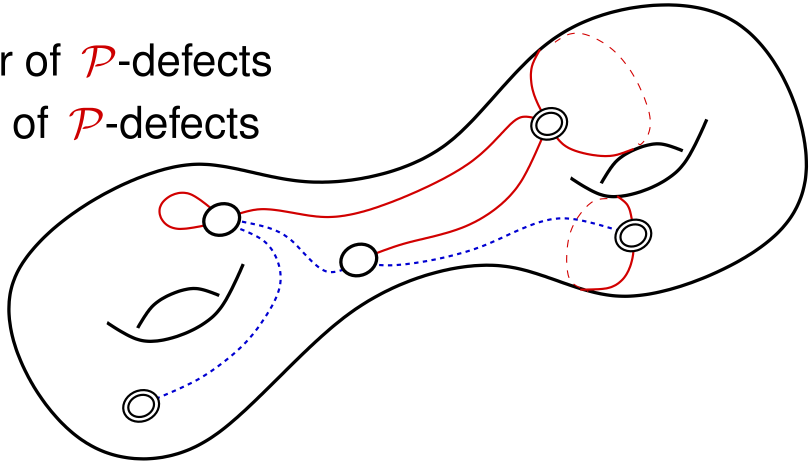

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  e.g. $$\dim_{\mathbb{C}}\big(\mathbf{tft}^{\mathcal{D}}(S^2; \emptyset , \{V, V, \dots , V\})\big) = \sum_{n \in I_{\mathcal{C}}} (S_{0,n})^{4-2m_1} (S_{V,n})^{m_1}$$

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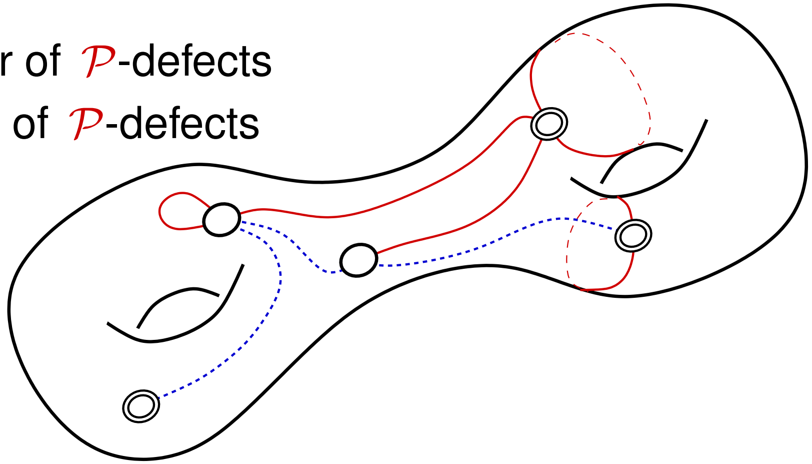

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    modular S-matrix of $\mathcal{C}$
  - depends on *genon type* V

APPENDIX

Defects in Dijkgraaf-Witten theories

- ☞ Dijkgraaf-Witten theories
  - input data : finite group G and cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$
  - $\mathcal{C} = D^\omega(G)$-mod $\simeq \mathcal{Z}(\mathcal{V}ect(G)^\omega)$ Turaev-Viro type
  - ω gives holonomy on closed three-manifolds $\rightsquigarrow$ topological bulk Lagrangian
  - two-step gauge-theoretic construction :

$$\mathcal{C}obord_{3,2,1} \xrightarrow{\widetilde{\mathrm{Bun}}} \mathcal{S}pan\mathcal{G}rp \xrightarrow{[-,\mathcal{V}ect]^\tau} 2\text{-}\mathcal{V}ect$$ twisted linearization

FREED 1995

MORTON 2013

groupoid cocycle $\tau \in H^2(G/\!/G, \mathbb{C}^\times)$ obtained by transgression

WILLERTON 2008

Defects in Dijkgraaf-Witten theories

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- ☞ extends to TFT with boundaries and defects
  via (bi)relative manifolds and (bi)relative bundles

Defects in Dijkgraaf-Witten theories

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☞ extends to TFT with boundaries and defects

- ↝ category of relative bundles for smooth map $j\colon Y \to X$
  and group homomorphism $\iota\colon H \to G$

objects : G-bundle $P_G \to X$ and H-bundle $P_H \to Y$

with isomorphism $\alpha\colon \mathrm{Ind}_H^G(P_H) \xrightarrow{\cong} j^* P_G$

morphisms : bundle morphisms

$$P_G \xrightarrow{\varphi_G} P'_G$$
$$P_H \xrightarrow{\varphi_H} P'_H$$ s.t.

$$\begin{tikzcd}
\mathrm{Ind}_H^G(P_H) \arrow[r, "\alpha"] \arrow[d, "\mathrm{Ind}_H^G(\varphi_H)"'] & j^* P_G \arrow[d, "j^*\varphi_G"] \\
\mathrm{Ind}_H^G(P'_H) \arrow[r, "\alpha'"] & j^* P'_G
\end{tikzcd}$$

Defects in Dijkgraaf-Witten theories

- Dijkgraaf-Witten theories
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- extends to TFT with boundaries and defects
- Example : category for circle $\mathbb{S}$ with one defect point p
  - to interval $\mathbb{S}\backslash\{p\}$ assign group G with cocycle ω
  - to p assign homomorphism $\imath\colon H \to G \times G$ with cochain $\theta \in C^2(H,\mathbb{C}^\times)$
  - $\widetilde{\mathrm{Bun}}$ gives action groupoid $G \backslash\!\backslash G \times G /\!\!/_{\imath} H$
  - twisted linearization gives $\big[\,G \backslash\!\backslash G \times G /\!\!/_{\imath} H, \mathcal{V}ect\,\big]^{\tau_{\omega,\theta}}$

  find $\tau_{\omega,\theta}((\gamma_1,\gamma_2);(g,h),(g',h')) = [\theta(h',h)]^{-1}$

$$\omega(g',g,\gamma_1)\,[\omega(g',g\gamma_1\imath_1(h)^{-1},\imath_1(h))]^{-1}\omega(g'g\gamma_1\imath_1(h)^{-1}\imath_1(h')^{-1},\imath_1(h'),\imath_1(h))$$
$$[\omega(g',g,\gamma_2)]^{-1}\omega(g',g\gamma_2\imath_2(h)^{-1},\imath_2(h))\,[\omega(g'g\gamma_2\imath_2(h)^{-1}\imath_2(h')^{-1},\imath_2(h'),\imath_2(h))]^{-1}$$

Defects in Dijkgraaf-Witten theories

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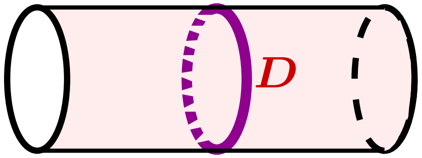
  - ↝ $\widetilde{\mathrm{Bun}}$ gives action groupoid $G\,\backslash\!\!\backslash\, G\times G /\!\!/_{\imath-} H$
  - ↝ twisted linearization gives $\big[\, G\,\backslash\!\!\backslash\, G\times G /\!\!/_{\imath-} H, \mathcal{V}ect \,\big]^{\tau_{\omega,\theta}}$
  - ↝ thus equivalent to category of $G\times G$-graded vector spaces $\bigoplus_{g_1,g_2\in G} V_{(g_1,g_2)}$
    with $\tau_{\omega,\theta}$-twisted $G\times H$-action $\pi_{g,h} :\ V_{(g_1,g_2)} \to V_{(gg_1,gg_2)\,\imath(h)^{-1}}$
  - ↝ equivalent to category of $A_{G_{\mathrm{diag}}}$-$A_{H,\theta}$-bimodules in $\mathcal{V}ect(G)^\omega \boxtimes \mathcal{V}ect(G)^{\omega^{-1}}$

Further developments



☞ A few other available results :

- transmission functors for invertible defects realize bijection

  invertible $\mathcal{A}$-bimodule categories $\longleftrightarrow$ braided auto-equivalences of $\mathcal{Z}(\mathcal{A})$

- gauge-theoretic description of symmetries of abelian Dijkgraaf-Witten theories

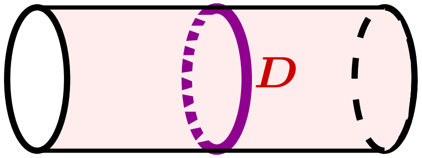
  $\mathrm{O}_q(A \oplus A^*)$ generated by

  $\varphi \oplus (\varphi^*)^{-1}$ with $\varphi \in \mathrm{Aut}(A)$

  $(g,\chi) \mapsto (g, \chi + \beta(g,-))$ with β alternating bicharacter (B-*field*)

  electric-magnetic dualities

Further developments



- ☞ A few other available results :
  - transmission functors for invertible defects realize bijection
    invertible $\mathcal{A}$-bimodule categories $\longleftrightarrow$ braided auto-equivalences of $\mathcal{Z}(\mathcal{A})$
  - gauge-theoretic description of symmetries of abelian Dijkgraaf-Witten theories
  - simplicial constructions à la TV/BW CRANE-YETTER 2014
  - deconfining of twist defects SEE Z. WANG'S TALK
  - interpretation of categories arising as $\mathbf{tft}_{3,2,1}^{\mathcal{Z}(\mathcal{A})}(\mathbb{S})$ as category-valued trace $\bar{\otimes}$
    for 1-morphisms in the tricategory of finite tensor categories
- ☞ Among next steps :
  - formulation of Dijkgraaf-Witten results in terms of relative Deligne product and $\bar{\otimes}$
    so as to extend to all Turaev-Viro TFTs