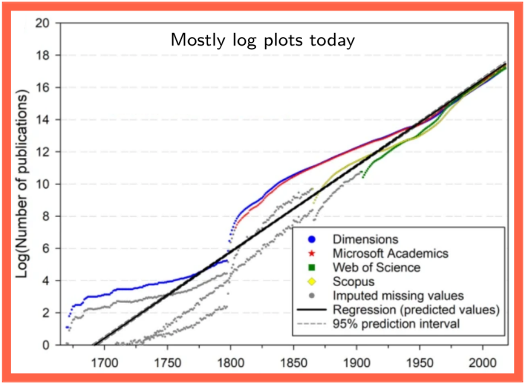
Growth and tensor products

Or: OMG exponential growth

~~Accept~~Change what you cannot ~~change~~accept



I report on work of Coulembier, Etingof, Ostrik, and many more

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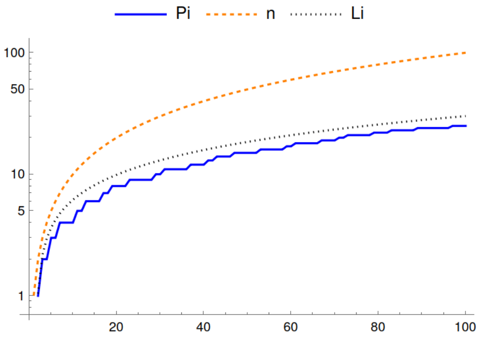
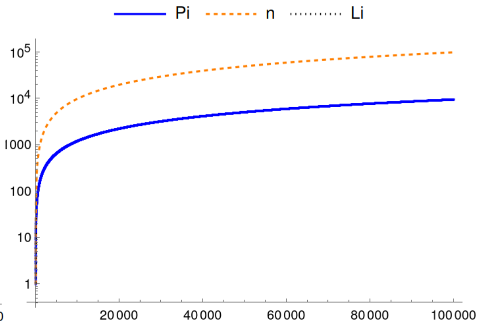
Let us not count!





- Prime number function $\pi(n)$ $= \#$ primes $\leq$ n
- Counting primes is very tricky as primes "pop up randomly"
- Question 1 What is the leading growth (of the number of primes)?
- Answer 1 There are roughly $c(n) \cdot n$ for sublinear correction term $c(n)$

Let us not count!

Seriously, counting is difficult!

Legendre ∼1808:
(for $n/(\ln n - 1.08366)$)

Limite x	Nombre y par la formule.	Nombre y par les Tables.	Limite x	Nombre y par la formule.	Nombre y par les Tables.
10000	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	350000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837			
90000	8717	8713			

Acctually, #primes<1000 =1229...

Gauss, Legendre and company counted primes up to $n = 400000$ and more

That took years (your IPhone can do that in seconds...humans have advanced!)

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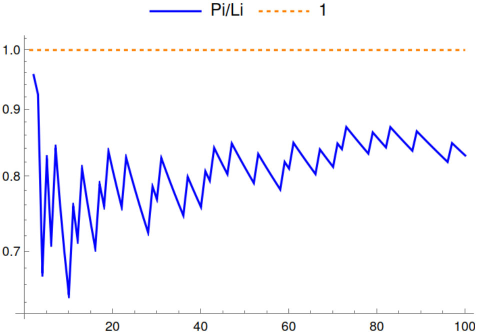
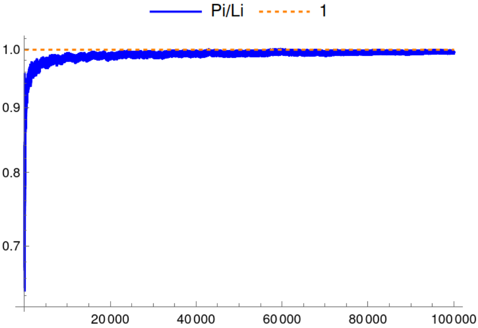
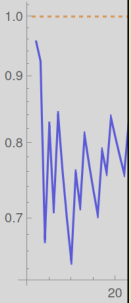




- **Asymptotically equal** $f \sim g$ if $\lim_{n\to\infty} f(n)/g(n) \to 1$
- **Logarithmic integral** $Li(x) = \int_2^x 1/\ln(t)dt$
- **Question 2** What is the growth (of the number of primes) asymptotically?
- **Answer 2** We have $\pi(n) \sim n/\log(n) \sim Li(n)$

Let us n



How good is this not counting?

Riemann ∼1859 calculates "the variance":

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Durch Einsetzung dieser Werthe in den Ausdruck von $f(x)$ erhält man

$$f(x) = Li(x) - \Sigma^\alpha \left(Li\left(x^{\frac{1}{2}+\alpha i}\right) + Li\left(x^{\frac{1}{2}-\alpha i}\right)\right)$$

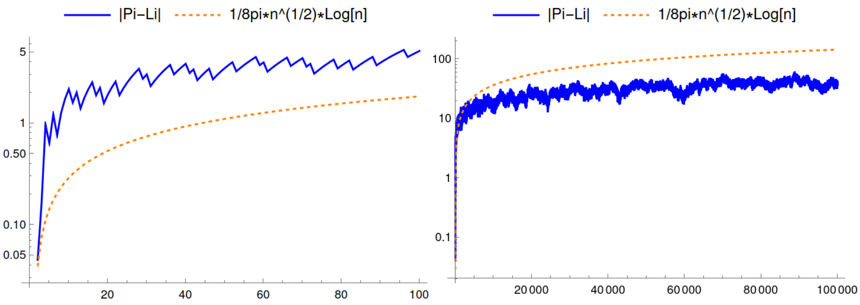
$$+ \int_x^\infty \frac{1}{x^2-1} \frac{dx}{x \log x} + \log \xi(0),$$

wenn in Σ^α für α sämmtliche positiven (oder einen positiven reellen Theil enthaltenden) Wurzeln der Gleichung $\xi(\alpha) = 0$, ihrer Grösse nach geordnet, gesetzt werden. Es lässt sich, mit Hülfe einer genaueren Discussion der Function ξ, leicht zeigen, dass bei dieser Anordnung der Werth der Reihe

f is essentially the prime counting function π

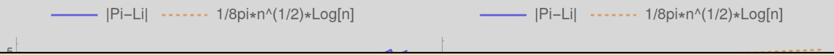
- Asy
- Log
- Que ... tically?
- Ans

Let us not count!



- Asymptotically equal does not imply that the difference is good
- $|f(n) - g(n)|$ is a measurement of how good the approximation is
- Question 3 What is variance from the expected value $(Li(n))$?
- Conjectural answer 3 We have $|\pi(n) - Li(n)| \in O(n^{1/2} \log n)$ or $|\pi(n) - Li(n)| \leq \frac{1}{8\pi} n^{1/2} \log n$ (for $n \geq 2657$)

Let us not count!



What to expect from not counting

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Let us not count!

- Γ = something that has a tensor product (more details later)
- $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- Problem Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Examples of what Γ could be

Any finite group, monoid, semigroup

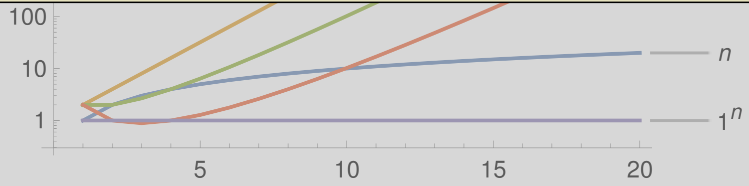
Symmetric groups, alternating groups, cyclic groups, the monster, $GL_N(\mathbb{F}_{p^k})$, ...

Actually any group, monoid, semigroup

$GL_N(\mathbb{C})$, $GL_N(\mathbb{R})$, $GL_N(\overline{\mathbb{F}_{p^k}})$, symplectic, orthogonal, braid groups, Thompson groups, ...

Super versions

$GL_{M|N}$, $OSP_{M|2N}$, periplectic, queer, ...



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Examples (that we will touch later)

Up to some slight change of setting we could also include:

Fusion categories or even finite additive Krull–Schmidt monoidal categories

Proj$(G, \mathbb{K})$, **Inj**$(G, \mathbb{K})$, semisimpl. of quantum group reps, Soergel bimodules of finite type, ...

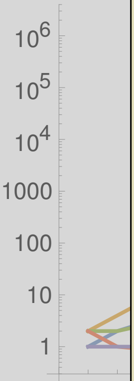
General additive Krull–Schmidt monoidal categories up to one condition (given later)

Rep(GL_n) and friends, quantum group reps, Soergel bimodules of affine type, ...

Most importantly, your favorite example might be included on this list

- Problem Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Let us not count!



$\mathrm{im}\ V)^n$

$\frac{\mathrm{im}\ V)^n}{n}$

$\frac{\mathrm{im}\ V)^n}{n^2}$

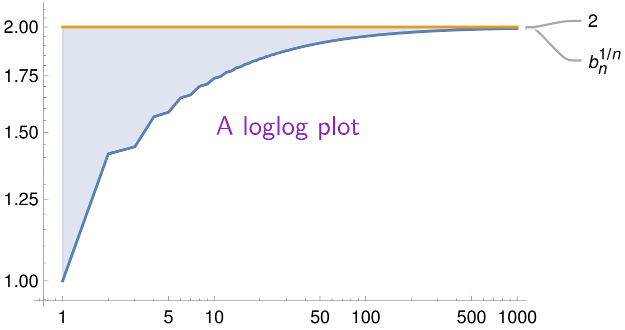
Let us pause for a second...the setting is way to general!

Decomposing $V^{\otimes n}$ for an arbitrary group is not happening

Better: Let us answer a not counting question!

- Γ = som
- $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- Problem Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Leading growth for "groups"



- $b_n = b_n^{\Gamma, V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$, then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $2 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 1.99265$

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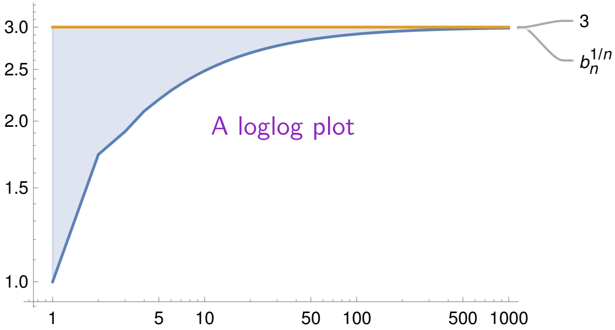

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- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathrm{Sym}\,\mathbb{C}^2$, then

$$\{1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953\}, \quad b_n \text{ for } n = 0, ..., 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $3 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 2.9875$

Lead

Observation 1

Whatever is true for SL_2 over $\mathbb{C}$ is true in general, right?

So let us come back to the general setting:
Γ = affine semigroup superscheme
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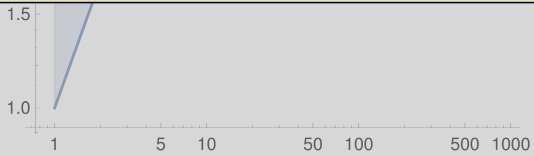

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Observation 2

$$b_n b_m \leq b_{n+m} \Rightarrow$$
$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n}$$
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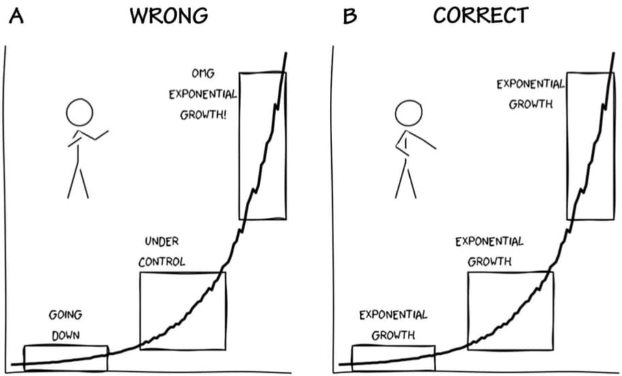
Observation 3

$$1 \leq \beta \leq \dim_{\mathbb{K}} V$$

$\beta = 1 \Leftrightarrow V^{\otimes n}$ for $n \gg 0$ is 'one block'

$\beta = \dim_{\mathbb{K}} V \Leftrightarrow$ summands of $V^{\otimes n}$ for $n \gg 0$ are 'essentially one-dimensional'

Leading growth for "groups"



Coulembier–Ostrik ∼2023 We have

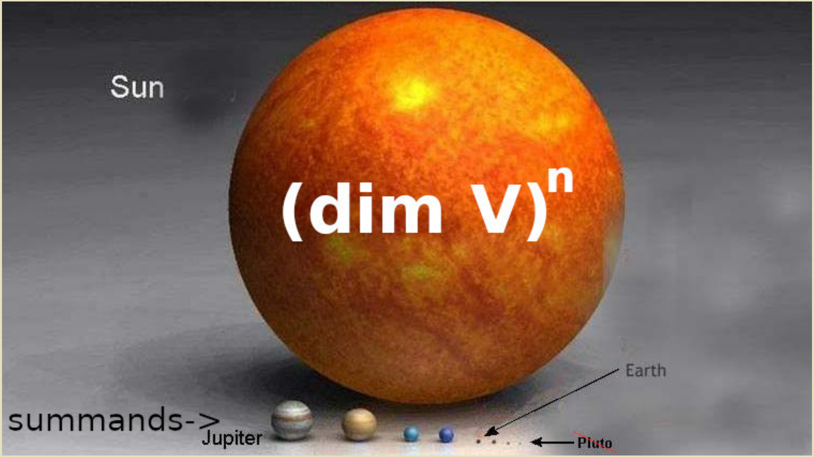
$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

Lea

Exponential growth is scary

Roughly what this shows is "$b_n \sim c(n) \cdot (\dim_{\mathbb{K}} V)^n$" for subexponential $c(n)$

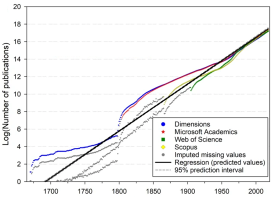
In other words, compared to the size of the exponential growth of $(\dim_{\mathbb{K}} V)^n$ all indecomposable summands are 'essentially one-dimensional'



Cou

On the next slide there is a formula of the form

$$\underbrace{b_n}_{b(n)} \sim \underbrace{c(n) \cdot (\dim_{\mathbb{K}} V)^n}_{a(n)} \text{''}$$



We will explore the formula by examples
so no need to memorize it

The take away messages are:

The formula is completely explicit and works in quite some generality specified later

It only depends on eigenvalues and eigenvectors associated to a matrix

The assumptions on the next slide are not necessary
but make the formula look nicer

The recurrent case – everything goes

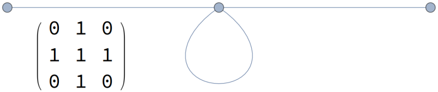
- Take a finite based $\mathbb{R}_{\geq 0}$-algebra R with basis $C = \{c_0, ..., c_{r-1}, ...\}$
- Assume that R is the Grothendieck ring of our starting category
- For $a_i \in \mathbb{R}_{\geq 0}$, the action matrix M of $c = a_0 \cdot c_0 + ... + a_{r-1} \cdot c_{r-1} \in R$ is the matrix of left multiplication of c on C
- Assume that M has a leading eigenvalue λ of multiplicity one; all other eigenvalues of the same absolute value are $\exp(k2\pi i/h)\lambda$ for some h
- Denote the right and left eigenvectors of M for λ and $\exp(k2\pi i/h)\lambda$ by v_i and w_i, normalized such that $w_i^T v_i = 1$
- Let $v_i w_i^T[1]$ denote taking the sum of the first column of the matrix $v_i w_i^T$
- The formula $b(n) \sim a(n)$ we are looking for is ($\zeta = \exp(2\pi i/h)$)

$$b(n) \sim \big(v_0 w_0^T[1] \cdot 1 + v_1 w_1^T[1] \cdot \zeta^n + v_2 w_2^T[1] \cdot (\zeta^2)^n + ... + v_{h-1} w_{h-1}^T[1] \cdot (\zeta^{h-1})^n\big) \cdot \lambda^n$$

- The convergence is geometric with ratio $|\lambda^{sec}/\lambda|$

The recurrent case – everything goes

Symmetric group S_3, $\mathbb{K} = \mathbb{C}$, V=standard rep

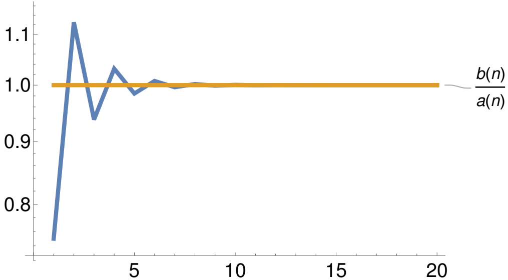


$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Example $\lambda = 2$, others$=0, -1$, $v = w = 1/\sqrt{6}(1,2,1)$, $vw^T = \begin{pmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & 2/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{pmatrix}$ and

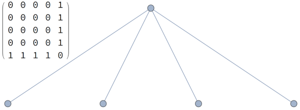
$$a(n) = \frac{2}{3} \cdot 2^n$$

Symmetric Group S3



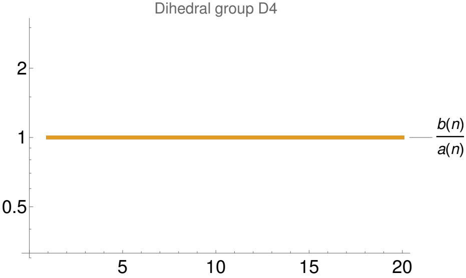
The recurrent case – everything goes

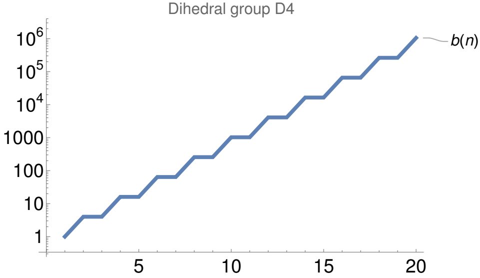
Dihedral group D_4 of order 8, $\mathbb{K} = \mathbb{C}$, V=defining rotation rep



Example $\lambda = 2$, others$=-2, 0, 0, 0$, $v_\lambda = w_\lambda = 1/\sqrt{8}(1, 1, 1, 1, 2)$
$v_{-2} = w_{-2} = 1/\sqrt{8}(-1, -1, -1, -1, 2)$ and

$$a(n) = \left(\tfrac{3}{4} + \tfrac{1}{4}(-1)^n\right) \cdot 2^n$$





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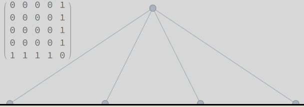
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

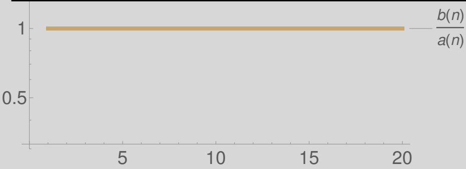


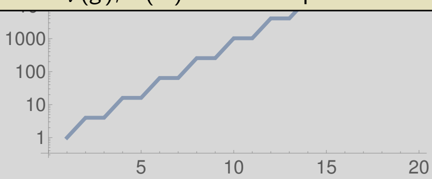
Example (general finite group, $\mathbb{K} = \mathbb{C}$, V=any faithful G-rep)

In this case we have a general formula:

$$a(n) = \left(\frac{1}{\#G} \sum_{g \in Z_V(G)} \Big(\sum_{L \in \mathrm{S}(G)} \omega_L(g) \dim_{\mathbb{C}} L \Big) \cdot \omega_V(g)^n \right) \cdot (\dim_{\mathbb{C}} V)^n$$

$Z_v(G)$=elements g acting by a scalar $w_V(g)$; $S(G)$=set of simples





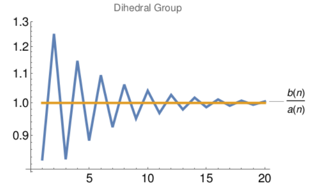
Example (continued)

Symmetric group S_m $a(n) = \left(\sum_{k=0}^{m/2} 1/((m-2k)!k!2^k)\right) \cdot (\dim_{\mathbb{C}} V)^n$

Dihedral group D_m of order $2m$

$$a(n) = \begin{cases} \frac{m+1}{2m} \cdot 2^n & \text{if } m \text{ is odd,} \\ \frac{m+2}{2m} \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is odd,} \\ \left(\frac{(m+2)}{2m} \cdot 1 + \frac{1}{m} \cdot (-1)^n\right) \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is even.} \end{cases}$$

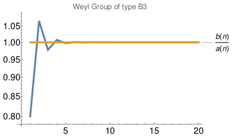
$m = 5$:



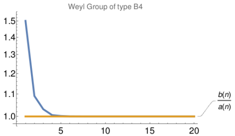
Complex reflection group $G(d, 1, m)$

$\begin{cases} d = 1, \\ m = 3 \end{cases} : a(n) = \frac{2}{3} \cdot 3^n$, $\begin{cases} d = 2, \\ m = 3 \end{cases} : a(n) = \frac{5}{12} \cdot 3^n$, $\begin{cases} d = 2, \\ m = 4 \end{cases} : a(n) = \left(\frac{19}{96} \cdot 1 + \frac{1}{32} \cdot (-1)^n\right) \cdot 4^n$

$\begin{cases} d = 2, \\ m = 3 \end{cases}$:



, $\begin{cases} d = 2, \\ m = 4 \end{cases}$:



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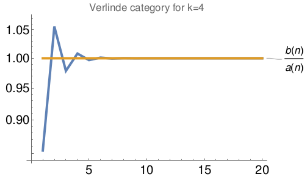
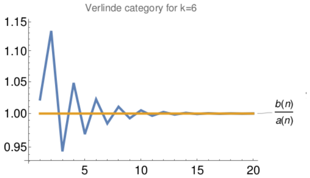
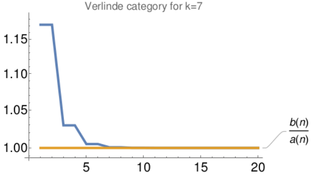
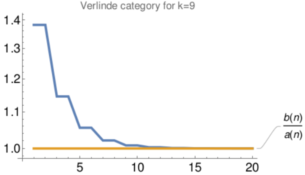
The recurrent case – everything goes

$k = 4$: 

, $k = 6$: 

$k = 7$: 

, $k = 9$: 

Example For the SL_2 Verlinde category over $\mathbb{C}$ at level *k* and *V*=gen. object:

$$a(n) = \begin{cases} \frac{[1]_q+\ldots+[k]_q}{[1]_q^2+\ldots+[k]_q^2} \cdot \big(2\cos(\pi/(k+1))\big)^n & \text{if } k \text{ is even,} \\ \Big(\frac{[1]_q+\ldots+[k]_q}{[1]_q^2+\ldots+[k]_q^2} \cdot 1 + \frac{[1]_q-[2]_q+\ldots-[k-1]_q+[k]_q}{[1]_q^2+\ldots+[k]_q^2} \cdot (-1)^n\Big) \cdot \big(2\cos(\pi/(k+1))\big)^n & \text{if } k \text{ is odd.} \end{cases}$$

The recurren…

$k = 4:$

$k = 7:$

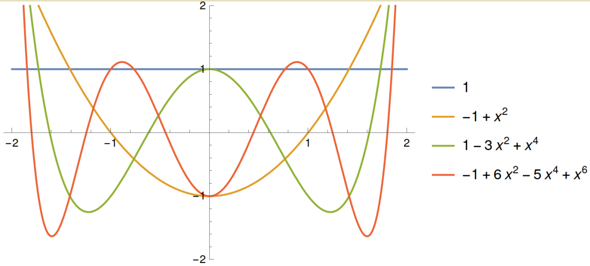
Example For… …n. object:

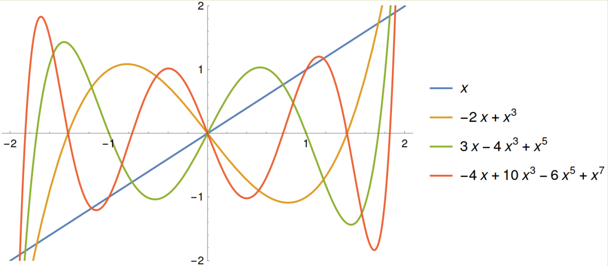
$a(n) = \begin{cases} \frac{[1]_q + \dots}{[1]_q^2 + \dots} & \text{if } k \text{ is even,} \\ \left(\frac{[1]_q + \dots}{[1]_q^2 + \dots}\right)^n & \text{if } k \text{ is odd.} \end{cases}$

Example (continued)

The growth rate in this case is not in $\mathbb{N}$

but rather the leading root of the Chebyshev polynomial:



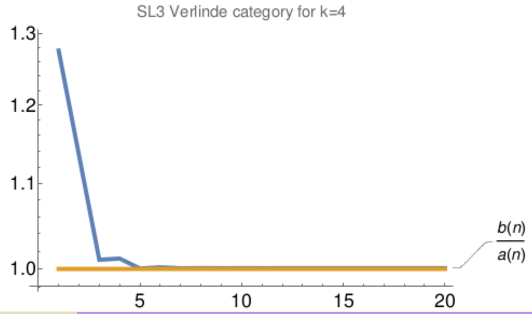


Example (continued)

Here is the SL_3 Verlinde category over $\mathbb{C}$ at level $k = 4$ and V=gen. object:

$$k = 4\colon a(n) = \frac{1}{7}\Big(2 + 2\cos\left(\tfrac{3\pi}{7}\right)\Big) \cdot \Big(1 + 2\cos\left(\tfrac{2\pi}{7}\right)\Big)^n,$$

$k = 4$:



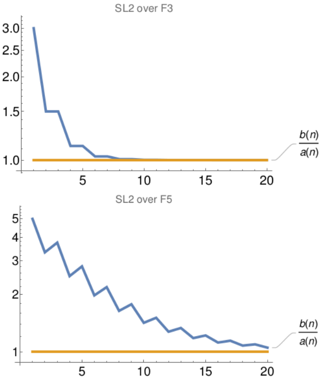
.

Koornwinder polynomials make their appearance

$\cdot (-1)^n\Big) \cdot (2\cos(\pi/(k+1)))^n$ if k is odd.

The recurrent case – everything goes

$p = 3:$



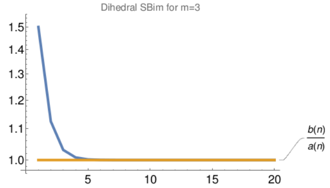
$p = 5:$

Example For $\mathrm{SL}_2(\mathbb{F}_p)$, $\mathbb{K} = \mathbb{F}_p$ and $V = \mathbb{F}_p^2$ we get:

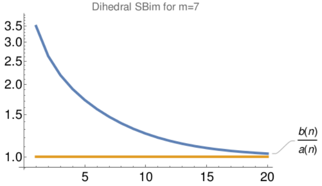
$$a(n) = \left(\frac{1}{2p-2} \cdot 1 + \frac{1}{2p^2 - 2p} \cdot (-1)^n \right) \cdot 2^n$$

The recurrent case – everything goes

$m = 3$:



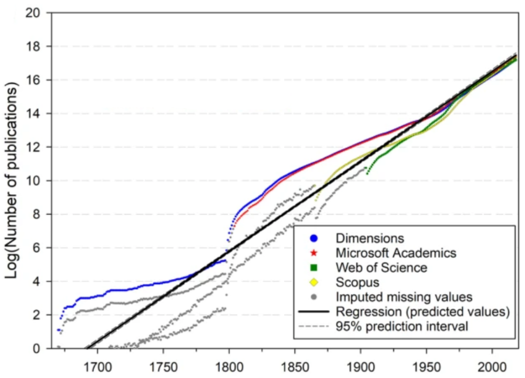
$m = 7$:



Example For dihedral Soergel bimodules of D_m, $\mathbb{K} = \mathbb{C}$ and $V = B_{st}$ we get:

$$a(n) = \frac{1}{2m} \cdot 4^n$$

Observe that the growth of $b(n)$ is always exponential



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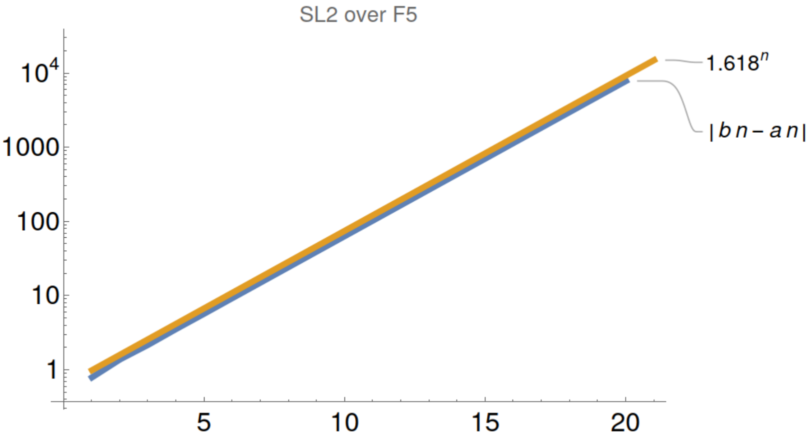
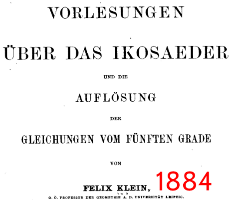
The recurrent case – everything goes



- The variance is given by $(\lambda_{sec})^n$ (second largest EV)
- Example Above for $\mathrm{SL}_2(\mathbb{F}_5)$, $\mathbb{K} = \mathbb{F}_5$ and $V = \mathbb{F}_5^2$, λ_{sec}=golden ratio

The recurrent case – everything goes


VORLESUNGEN

ÜBER DAS IKOSAEDER

UND DIE

AUFLÖSUNG

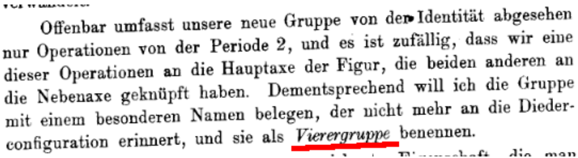
DER

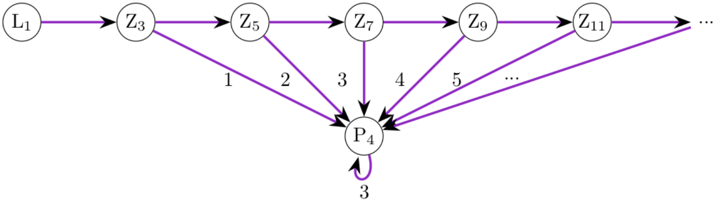
GLEICHUNGEN VOM FÜNFTEN GRADE

VON

FELIX KLEIN, 1884

O. Ö. PROFESSOR DER GEOMETRIE A. D. UNIVERSITÄT LEIPZIG.

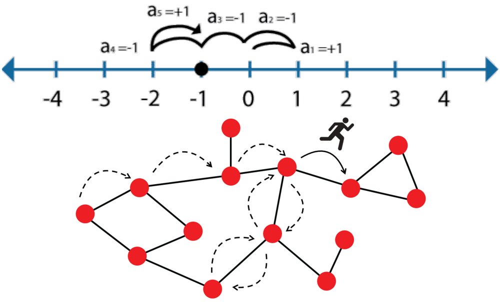

Offenbar umfasst unsere neue Gruppe von der Identität abgesehen nur Operationen von der Periode 2, und es ist zufällig, dass wir eine dieser Operationen an die Hauptaxe der Figur, die beiden anderen an die Nebenaxe geknüpft haben. Dementsprechend will ich die Gruppe mit einem besonderen Namen belegen, der nicht mehr an die Dieder-configuration erinnert, und sie als *Vierergruppe* benennen.



Example For the Klein four group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, $\mathbb{K}=\overline{\mathbb{F}_2}$ and $V=Z_3$=3d inde. we get:

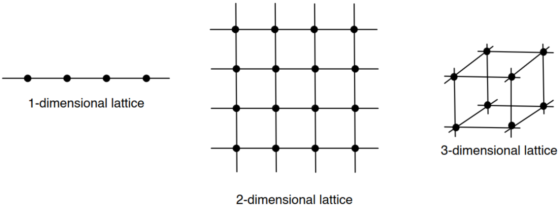
$$b_n \sim 3^n$$

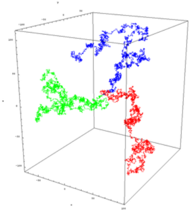
The recurrent case – everything goes



- We randomly walk on some (connected) graph = at each step choose the next step/edge randomly but equally likely "coin flip walk"
- Question How often do we visit a vertex?
- Recurrent := We will hit every point infinitely often with P(robability)=1
- Example Every (random walk on a) finite graph is recurrent

The recurrent case – everything goes





- **Pólya ~1921** $\mathbb{Z}^d$ is recurrent/transient $\Leftrightarrow$ $d \leq 2 / d > 2$
- A drunkard will find their way home, but a drunken bird may get lost forever
- Transient := We will hit every point finitely often with P(robability)=1

The recurrent case – everything goes

Pólya ∼1921

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz.

Von

Georg Pólya in Zürich.

															1			
							1						3			3		
		1				2		2			3			9			3	
1 ,	1		1 ,	1			4		1 ,	1		9			9			1
		1				2		2			3			9			3	
							1						3			3		
															1			

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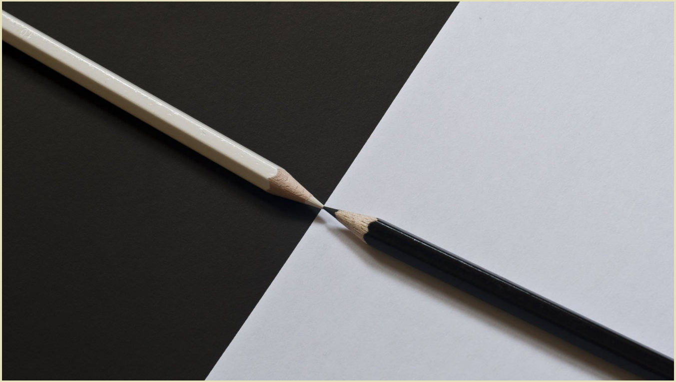
The recu

Every graph is either recurrent or transient



This is an instance of a 0-1-theorem :
a lot of properties hold with P=0 or P=1 but 0<P<1 rarely appears

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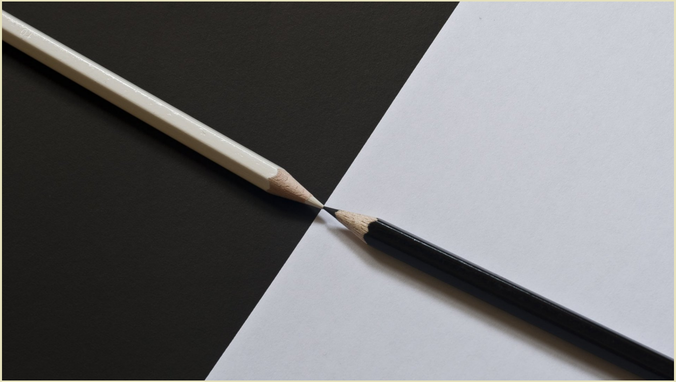


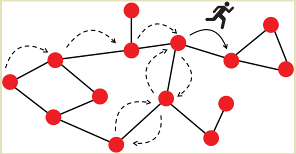
This is an instance of a 0-1-theorem :
a lot of properties hold with P=0 or P=1 but 0<P<1 rarely appears

- Po
- A ... forever
- T ... =1

Perron ∼1907, Frobenius ∼1912, Vere-Jones ∼1967, etc.

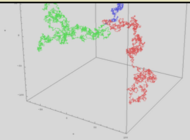
The previous eigenvalue strategy applies to recurrent settings :
For $b_n(V)$ take the fusion graph for V and check whether it is recurrent

Examples of recurrent growth problems



Easy If one has finitely many indecomposables

Coulembier–Etingof–Ostrik $\sim$**2023** V is an object of a finite tensor categories



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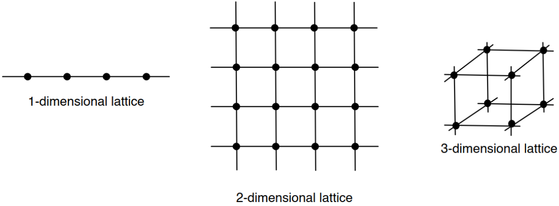
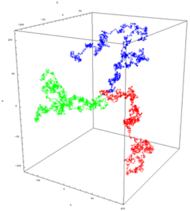
The recurrent case – everything goes





- **Pólya ∼1921** $b_n(V)$ for V a faithful Γ-rep in char zero is recurrent ⇔ Γ is virtually $\mathbb{Z}^d$ for $d \in \{0, 1, 2\}$
- Virtually means we allow extensions by finite groups

Biané ∼1993, Coulembier–Etingof–Ostrik ∼2023

showed that surprisingly (not recurrent!)

for complex fin dim simple Lie algebras ($\mathfrak{sl}_n$+friends) in char zero

one can still answer the three growth questions

2.2. THÉORÈME :

$$m(\lambda, E^{\otimes n}) = 0 \quad \text{si } \lambda \notin n\,P(E) + Q(E)$$

$$= \frac{\prod_{\alpha \in R_+} q^*(\alpha, \rho)}{\mathrm{vol}_q(\mathfrak{h}_{\mathbb{R}}/Q^\vee)} \frac{k(E)\, d(E)^n}{(2\pi)^{l/2}\, n^{m/2}}\, d(\lambda) \left(e^{-(q^*(\lambda+\rho)/2n)} + O\left(\frac{1}{n}\right)\right) \text{ sinon}$$

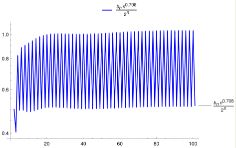
Le terme $O(1/n)$ *est uniforme en* $\lambda \in P_{++}$, *et* $\mathrm{vol}_q(\mathfrak{h}_{\mathbb{R}}/Q^\vee)$ *désigne la mesure pour dx d'un domaine fondamental du réseau* $Q^\vee$.

Exp. fac. $d(E)^n = (\dim_{\mathbb{C}} E)^n$, subexp. fac. $n^{\#\text{pos. roots}/2}$, some scalar, variance

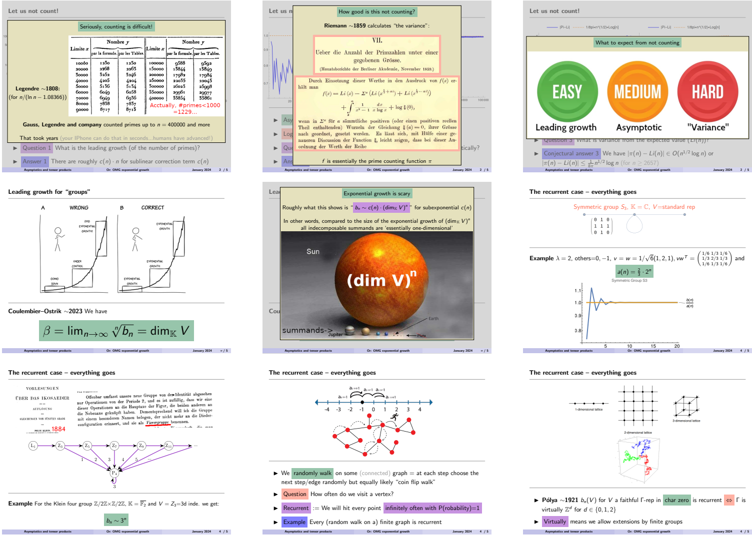
Char p is difficult, even for SL_2

The subexp. factor has transcendental power (fractals!)

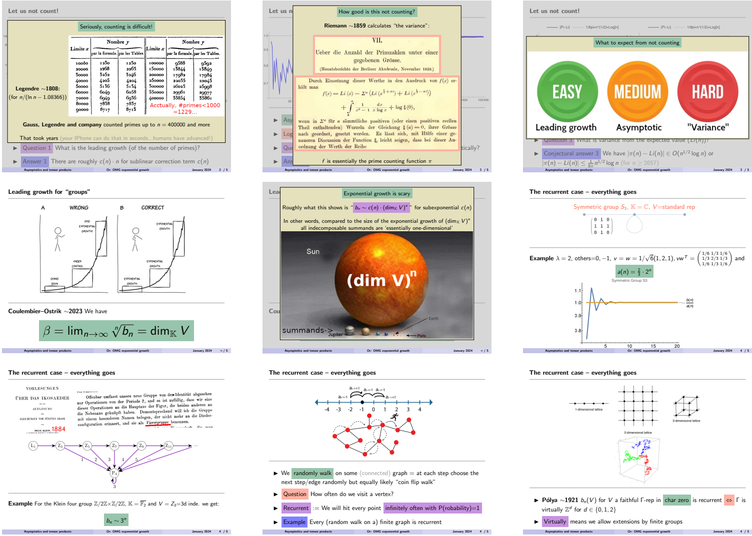
the "scalar function" is highly oscillating, etc.



- **Pólya ∼1** ... urrent ⇔ Γ is virtually $\mathbb{Z}$
- Virtually



There is still much to do...



Thanks for your attention!