LOW RANK SYMMETRIC FUSION CATEGORIES IN CHAR > 0

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IPAM - SYMMETRIC TENSOR CATEGORIES AND REPRESENTATION THEORY

PRELIMINARIES

Def A TENSOR CATEGORY is a category $\mathcal{C}$ with

1. abelian $\mathbb{k}$-linear structure ← ses, kernels, cokernels + Hom's are finite dim vector spaces

2. monoidal structure

   - $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ TENSOR PRODUCT $\mathbb{k}$-linear
   - $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ ASSOCIATIVITY $\forall X, Y, Z \in \mathcal{C}$

$$\begin{tikzcd}[column sep=small]
 & ((W\otimes X)\otimes Y)\otimes Z \arrow[dl] \arrow[dr] & \\
(W\otimes(X\otimes Y))\otimes Z \arrow[d] & & (W\otimes X)\otimes(Y\otimes Z) \arrow[d] \\
W\otimes((X\otimes Y)\otimes Z) \arrow[rr] & & W\otimes(X\otimes(Y\otimes Z))
\end{tikzcd}$$

   - $\mathbb{1} \in \mathcal{C}$ UNIT OBJECT + $\mathbb{1} \otimes X \xrightarrow{\sim} X$
     $X \otimes \mathbb{1} \xrightarrow{\sim} X$

$$\begin{tikzcd}
(X\otimes\mathbb{1})\otimes Y \arrow[rr] \arrow[dr] & & X\otimes(\mathbb{1}\otimes Y) \arrow[dl] \\
 & X\otimes Y &
\end{tikzcd}$$

+ DUALS $X \rightsquigarrow (X^*, ev_X, coev_X)$ $\begin{cases} ev_X : X^* \otimes X \to \mathbb{1} \\ coev_X : \mathbb{1} \to X \otimes X^* \end{cases}$

3. $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{k}$ $\mathbb{1}$ is simple

Def A FUSION CATEGORY is a tensor category
+ semisimple ← $X = \oplus$ simples
+ finitely many simples

RANK = n° of simple objects

Def A SYMMETRIC TENSOR CAT is a tensor category $\mathcal{C}$ + braiding
st $c_{YX} c_{XY} = \mathrm{id}_{X \otimes Y} \quad \forall X, Y \in \mathcal{C}$

Def A SYMMETRIC TENSOR FUNCTOR is a $\mathbb{k}$-linear exact symm monoidal functor

SYMMETRIC TENSOR CATEGORIES

char($\Bbbk$) = 0

$\mathcal{C}$ symm tensor category + finite length ← finite J-H series

Deligne '02

+ moderate growth $\Longleftrightarrow$ symm tensor functor

$\ell(X^{\otimes n}) \leq a_X^n \quad \forall X \in \mathcal{C},$ $\quad \mathcal{C} \to \mathrm{sVec}$

$a_X \in \mathbb{R}$

$\rightsquigarrow \mathcal{C} \simeq \mathrm{Rep}(G, z)$ $\quad G$ affine group scheme over svec, $z \in G$

z acts by parity.

comm Hopf alg

There are symm tensor categories NOT of moderate growth

$\mathrm{Rep}(S_t) \qquad \ell(V^{\otimes n}) \approx \exp(n \log(n))$

SYMMETRIC TENSOR CATEGORIES

char($\Bbbk$) = p

$\mathcal{C}$ symm tensor + finite length

↳ incompressible: any symm tensor functor $\mathcal{C} \to \mathcal{D}$ is an embedding

$\mathcal{C}$ cannot be constructed from group schemes in a smaller symmetric category.

Coulembier Etingof Ostrik '23 + moderate growth ⇒ symm tensor functor $\mathcal{C} \longrightarrow$ incompressible cat of moderate growth incompressible: $\begin{cases} \text{in char} = 0 & \text{Vec, sVec} \\ \text{in char} = p > 0 & \end{cases}$ ← all the moderate growth ones

$\underline{Q}$: what are the incompressible cats?

Ver_p = ss of cat of tilting modules over SL_2

$$\begin{array}{ccccccccc} \mathrm{Vec} \subset \mathrm{sVec} \subset & \mathrm{Ver}_p & \subset & \mathrm{Ver}_{p^2} & \subset \cdots & \subset & \mathrm{Ver}_{p^n} & \subset \cdots \\ & \cup & & \cup & & & \cup & \\ & \mathrm{Ver}_p^+ & \subset & \mathrm{Ver}_{p^2}^+ & \subset \cdots & \subset & \mathrm{Ver}_{p^n}^+ & \end{array}$$

incompressibles of moderate growth

CEO, conjecture

are these all the incompressibles of moderate growth?

Q: For each incompressible cat $\mathcal{C}$, which categories admit a symm tensor functor $\to \mathcal{C}$?

Answer for Verp

CEO '23

+moderate growth
+Frob exact $\iff$ symm tensor functor $\mathcal{C} \to \mathrm{Ver}_p$

Frob functor $\mathcal{C} \xrightarrow{\otimes p} \mathcal{C} \boxtimes \mathrm{Ver}_p$ is exact

SYMMETRIC FUSION CATEGORIES

- Finite length ✓
- moderate growth ✓
- Frob exact ✓

CASE $p=0$

DELIGNE

$\mathcal{C}$ symm fusion $\Rightarrow$ has a symmetric tensor functor $\mathcal{C} \to sVec$

$\mathcal{C} \simeq Rep(G, z)$ — G finite group, $z \in Z(G)$ of order ≤ 2. → CLASSIFIED

CASE $p>0$

OSTRIK '16

$\mathcal{C}$ symm fusion $\Rightarrow$ has a symmetric tensor functor $\mathcal{C} \to Ver_p$

$\mathcal{C} \simeq Rep_{Ver_p}(G, \pi)$ — G linearly reductive finite group scheme in Ver_p, π = fundamental group → NOT classified

What do we know about the classification of fusion cats when $p>0$?

$\mathcal{C}$ symmetric fusion category.

- If $\mathcal{C}$ is **TANNAKIAN** (symm tensor functor $\mathcal{C} \to \mathrm{Vec}$)

  Nagata: classified finite group schemes in Vec st $\mathrm{Rep}_{\mathrm{Vec}}(G)$ is ss.

If $p=2 \Rightarrow \mathcal{C}$ is Tannakian. $\mathrm{Ver}_2 = \mathrm{Vec}$

- If $\mathcal{C}$ is **SUPER TANNAKIAN** (symm tensor functor $\mathcal{C} \to \mathrm{sVec}$)

  Masuoka: classified finite group schemes in sVec st $\mathrm{Rep}_{\mathrm{sVec}}(G)$ is ss

If $p=3 \Rightarrow \mathcal{C}$ is super Tannakian $\mathrm{Ver}_3 = \mathrm{sVec}$

- If $\mathcal{C}$ has Rank 2 $\Rightarrow \mathcal{C} \simeq \mathrm{Ver}_5^+$ or $\mathrm{Vec}_{\mathbb{Z}_2}$, sVec

  Rank: n° of simple objects

  Etingof, Ostrik, Venkatesh

SEMISIMPLIFICATION

$\mathcal{C}$ symm tensor category

$f: X \to Y$ is NEGLIGIBLE if for any $g: Y \to X$, $\mathrm{Tr}(fg) = 0$

$\mathcal{N}$ = ideal of negligible morphisms

$\overline{\mathcal{C}}$ symm tensor categories
- objects: same as in $\mathcal{C}$
- $\mathrm{Hom}_{\overline{\mathcal{C}}}(X,Y) = \mathrm{Hom}(X,Y)/\mathcal{N}$

$\mathcal{C}$ karoubian

JANNSEN Thm
If $\dim(\mathrm{Hom}(X,Y)) < \infty$ and nilpotent endomorphisms in $\mathcal{C}$ have trace zero $\Rightarrow$ $\overline{\mathcal{C}}$ is ss

simples of $\overline{\mathcal{C}} \longleftrightarrow$ indec of $\mathcal{C}$ of nonzero dimension.

VERLINDE CATEGORIES

Ver_p

$p>0$

$\mathbb{Z}_p$ cyclic group of order p with generator σ

$$\Bbbk[\mathbb{Z}_p] = \Bbbk[\sigma]/(\sigma^p - 1) = \Bbbk[\sigma]/(\sigma-1)^p$$

INDECOMPOSABLES OF $\mathrm{Rep}_{\Bbbk}(\mathbb{Z}_p)$

$$\tilde{L}_s := \Bbbk[\sigma]/(1-\sigma)^s \quad 1 \leq s \leq p$$

SYMMETRIC FUSION CATEGORY

$$\mathrm{Ver}_p = \mathrm{Rep}_{\Bbbk}(\mathbb{Z}_p)/\text{negligible morphisms}$$

SIMPLES images $\mathbb{1} = L_1, \ldots, L_{p-1}$ of indecomposables.

FUSION RULES

$$L_r \otimes L_s = \sum_{i=1}^{\min(r,s,p-r,p-s)} L_{|r-s|+2i-1}$$

Ver_p

$p > 0$

$\mathrm{Ver}_2 = \mathrm{Vec}$ $\quad\quad$ $\mathrm{Ver}_3 = \mathrm{sVec}$

$p \geq 5$

Ver_p^+ = abelian subcategory gen by L_i for i odd

↳ it is a fusion subcategory

$\mathrm{rank} = \frac{p-1}{2}$

Ver_p has only 4 fusion subcategories: Ver_p, Ver_p^+, sVec, Vec

sVec gen by L_{p-1}

$L_{p-1}^2 = L_1$.

$$\mathrm{Ver}_p \simeq \mathrm{Ver}_p^+ \boxtimes \mathrm{sVec}$$

EXAMPLE

Ver_7 : simples $L_1, L_2, L_3, L_4, L_5, L_6$ $\quad$ $L_i^2 = \sum_{j=1}^{\min(i,p-i)} L_{2j-1}$

$$L_2 \otimes L_3 = \sum_{i=1}^{\min(2,3,5,4)} L_{|2-3|+2i-1} = L_2 \oplus L_4$$

Ver_{p^n}
$p>0$

Venkatesh
Benson-Etingof-Ostrik
Coulembier

in SL_2,

$\mathcal{N} \supset I_2 \supset I_3 \supset \ldots$ tensor ideals

$\mathcal{C}$ = cat of tilting modules over SL_2

quotient Tilt SL_2 + abelian envelope

$\mathrm{Ver}_p \subset \underbrace{\mathrm{Ver}_{p^2} \subset \mathrm{Ver}_{p^3} \subset \ldots}_{\text{not ss.}}$

ss

incompressible.
moderate growth.

more examples

GELFAND-KAZHDAN
GEORGIEV-MATHIEU

G ss group e.g. $G = SL_n$

T = tilting G-modules

$\mathrm{Ver}(G) := \tilde{T}$ is a ss tensor category

if $p \geq$ Coxeter no of $G \Rightarrow$ finitely many simples

$\mathrm{Ver}(SL_2) = \mathrm{Ver}_p$.

GROTHENDIECK RING

$\mathcal{C}$ fusion

$K(\mathcal{C})$ = free abelian group gen by $[X]$, $X \in \mathcal{C}$ simple.

MULTIPLICATION $\quad [X][Y] = \sum_{Z \text{ simple}} N_{XY}^{Z} [Z]$, where $\quad X \otimes Y = \bigoplus_{Z \text{ simple}} N_{XY}^{Z} Z$.

GROTHENDIECK RING OF $\mathcal{C}$.

EXAMPLES

- $\mathrm{Vec}_G \rightsquigarrow K(\mathcal{C}) \simeq \mathbb{k}[G]$
- $\mathrm{Rep}(G) \rightsquigarrow K(\mathcal{C}) \simeq$ group of characters

Thm (BENSON, ETINGOF, OSTRIK) $\quad K(\mathrm{Ver}_p^+) \otimes \mathbb{Q} \cong \mathbb{Q}(z+z^{-1})$ $\quad z$ p-th root of unity.

$\cong$ as $\mathbb{Q}$-algebras.

Recall $\quad \mathrm{Ver}_p \cong \mathrm{Ver}_p^+ \boxtimes s\mathrm{Vec}$

$\rightsquigarrow K(\mathrm{Ver}_p) \otimes \mathbb{Q} \cong \mathbb{Q}(z+z^{-1}) \oplus \mathbb{Q}(z+z^{-1})$.

EXAMPLES

$K(\mathrm{Ver}_2) \otimes \mathbb{Q} \cong \mathbb{Q}$ $\qquad K(\mathrm{Ver}_3) \otimes \mathbb{Q} \cong \mathbb{Q}^{\oplus 2}$ $\qquad K(\mathrm{Ver}_5) \otimes \mathbb{Q} \cong \mathbb{Q}(\sqrt{5})^{\oplus 2}$

Thm Let $p \geq 5$. If $\mathcal{C}$ is not super Tannakian symm fusion cat, then

$$\mathrm{Rank}(\mathcal{C}) \geq \frac{p-1}{2}$$

Does not hold for $\mathcal{C}$ Tannakian, e.g. $\mathrm{Rep}(\mathbb{Z}_2)$, $\mathrm{Rep}(S_3)$.

Note: Ver_p^+ has rank $\frac{p-1}{2}$ For $p=5$, $\mathrm{Rank}(\mathcal{C}) = \frac{5-1}{2} \Leftrightarrow \mathcal{C} = \mathrm{Ver}_5^+$

- Is it true for $p>5$ that $\mathrm{Rank}(\mathcal{C}) = \frac{p-1}{2}$ $\Rightarrow$ $\mathcal{C} \simeq \mathrm{Ver}_p^+$?
  $\mathcal{C}$ symm fusion, non Tannakian
  $p=7$ ✓
- There is an example of non-Tannakian symm fusion with rank $\frac{p+3}{2}$ $\Rightarrow$ Are there any with $\mathrm{Rank} = \frac{p+1}{2}$?

Thm Let $p \geq 5$, $\mathcal{C}$ symm fusion cat with Verlinde fiber functor $F: \mathcal{C} \to \mathrm{Ver}_p$. If F is surjective, then $\mathrm{Rank}(\mathcal{C}) \geq p-1$.

THE SECOND ADAMS OPERATION

$\mathcal{C}$ symmetric tensor cat

$S_n \circlearrowright X^{\otimes n}$ by $S_n \to \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})$

$(i\ i{+}1) \longmapsto \mathrm{id}_{X^{\otimes(i-1)}} \otimes c_{X,X} \otimes \mathrm{id}_{X^{\otimes(n-i-1)}}$

$S^n(X)$ = max quotient of $X^{\otimes n}$ on which the action of S_n is trivial

SYMMETRIC nth POWER

$\Lambda^n(X)$ = max quotient of $X^{\otimes n}$ on which the action of S_2 factors through the sign Rep

EXTERIOR nth POWER

Def The 2nd ADAMS OPERATION $\Psi_2 : k(\mathcal{C}) \longrightarrow k(\mathcal{C})$ is defined by

$$\Psi_2(X) = S^2(X) - \Lambda^2(X) \quad \text{for all } X \in k(\mathcal{C})$$

$\psi_2(X) = S^2(X) - \Lambda^2(X)$ for all $X \in K(\mathcal{C})$

$\psi_2 : K(\mathcal{C}) \to K(\mathcal{C})$ is a **RING ENDOMORPHISM**

Note. ψ_n is defined for n st $p \nmid n$

EXAMPLE. $p \nmid |G|$

$\mathcal{C} = \mathrm{Rep}(G)$

Identify $K(\mathcal{C}) \simeq$ group of characters χ

Then $\psi_n : K(\mathcal{C}) \to K(\mathcal{C})$ maps $\chi(g) \longmapsto \chi(g^n)$ $\forall g \in G$.

EXAMPLE

$\mathcal{C} = \mathrm{Vec}_G$ $\quad K(\mathcal{C}) \simeq \mathbb{K}[G]$, $\psi_n(g) = g^n$.

G abelian

EXAMPLE In Verp,

$$\boxed{\Psi_2(L_i) = -\Psi_2(L_{p-i})} \leftsquigarrow \Lambda^2(L_i) = L_{p-1}^2 \otimes S^2(L_{p-i}) = S^2(L_{p-i})$$

Vers simples L_1, L_2, L_3, L_4

$$L_3^2 = L_1 + L_3 \quad \Rightarrow \quad \Psi_2(L_3) = \varepsilon_1 L_1 + \varepsilon_3 L_3, \quad \varepsilon_i \in \{\pm 1\}$$

$$\begin{aligned} &\Vert \\ &S^2(L_3) + \Lambda^2(L_3) \end{aligned}$$

$$\Psi_2(L_3)^2 = L_1 + 2\varepsilon_1\varepsilon_3 L_3 + L_3^2 = 2L_1 + (2\varepsilon_1\varepsilon_3 + 1) L_3$$

$$\Vert$$

$$\Psi_2(L_3^2) = \Psi_2(L_1 + L_3) = L_1 + \varepsilon_1 L_1 + \varepsilon_3 L_3$$

$$\Rightarrow \begin{cases} 1 + \varepsilon_1 = 2 \rightsquigarrow \varepsilon_1 = 1 \\ 2\varepsilon_3 + 1 = \varepsilon_3 \rightsquigarrow \varepsilon_3 = -1 \end{cases} \qquad \Psi_2(L_3) = L_1 - L_3$$

SECOND ADAMS OPERATION IN Ver_p

$\psi_2 : K(\mathrm{Ver}_p) \to K(\mathrm{Ver}_p)$ is given by

$$\boxed{\psi_2(L_t) = \sum_{s=1}^{\min(t,p-t)} (-1)^{s+1} L_{2s-1}} \quad t \text{ odd},\ 1 \leq t \leq p-1$$

COR $X \in \mathrm{Ver}_p$ is fixed by $\psi_2 \iff X = mL_1,\ m \in \mathbb{N}$.

Remark. This is only true for actual objects in Ver_p ($\mathbb{Z}_{\geq 0}$ linear combinations of $L_1, \dots, L_{p-1}$)

Example: $X = L_5 - L_7 + L_9 - L_{15} \in K(\mathrm{Ver}_{17})$, $\psi_2(X) = X$.

Thm Let $p>2$, $\mathcal{C}$ non-super-Tannakian symm fusion.
If $\psi_2^a = \psi_2^b$ for some $a,b \in \mathbb{Z}_{\geq 0}$, then $2^a \equiv \pm 2^b \mod p$.

Cor Let $p>2$ and $\mathcal{C}$ symm fusion category
If $\psi_2^a = \psi_2^{a-1}$ for some $a \geq 1$ $\Rightarrow$ $\mathcal{C}$ super Tannakian

classified

Cor Let $p>2$ and $\mathcal{C} \neq Vec$ symm fusion. Then $\psi_2 \neq id$.

$\mathcal{C}$ being of moderate growth is necessary.
Delannoy category $\rightsquigarrow$ $\psi_2 = id$.
SS
Harman, Snowden, Snyder

APPLICATIONS

classification of rank 3 symm fusion

classification of rank 4 symm fusion st 2 self dual simples

CLASSIFICATION $\mathcal{C}$ symm fusion

ETINGOF
OSTRIK
VENKATESH

RANK 2

- $p>0,\ p\neq 5$, $\mathrm{Rep}(\mathbb{Z}_2)$, sVec
- $p=5$, $\mathrm{Rep}(\mathbb{Z}_2)$ OR Ver_5^+, sVec

RANK 3

- $p=2$, $\mathrm{Rep}(\mathbb{Z}_3)$
- $p=3$, $\mathrm{Vec}_{\mathbb{Z}_3}^{\mathbb{Z}_2}$, $\mathrm{Vec}_{\mathbb{Z}_3}$
- $p=7$, $\mathrm{Rep}(S_3)$, $\mathrm{Rep}(\mathbb{Z}_3)$, Ver_7^+
- $p=5$ OR $p>7$, $\mathrm{Rep}(S_3)$, $\mathrm{Rep}(\mathbb{Z}_3)$

RANK 4 MNSD.

- $p=2$, $\mathrm{Vec}_{\mathbb{Z}_4}^{\mathbb{Z}_3}$, $\mathcal{C}(\mathbb{Z}_4, q)$ $q: \mathbb{Z}_4 \to \mathbb{k}^\times$ group map
- $p=3$, $\mathcal{C}(\mathbb{Z}_4, q)$
- $p>3$, $\mathrm{Rep}(A_4)$, $\mathcal{C}(\mathbb{Z}_4, q)$.

NOTE Rank 4 non super Tannakian symm fusion $\Rightarrow$ $p=5$ or 7

THANKS!