Support varieties and the tensor product property

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The setting

$\mathcal{C}$ - finite tensor category over a field k with $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}) \cong k$

$\mathcal{M}$ - exact module category over $\mathcal{C}$: $\begin{array}{l} \mathcal{C} \times \mathcal{M} \to \mathcal{M} \\ (X, M) \mapsto X * M \end{array}$

Assume $\mathcal{M}$ has finitely many simple objects

$$\mathrm{H}^*(\mathcal{C}) := \mathrm{Ext}^*_{\mathcal{C}}(\mathbf{1},\mathbf{1}) = \bigoplus_{n\geq 0} \mathrm{Ext}^n_{\mathcal{C}}(\mathbf{1},\mathbf{1})$$

$$\mathrm{H}^*(M) := \mathrm{Ext}^*_{\mathcal{M}}(M, M) \text{ for objects } M \text{ of } \mathcal{M}$$

Note:

- $\mathrm{H}^*(\mathcal{C})$ is a graded commutative algebra under cup product
- $\mathrm{H}^*(M)$ is an $\mathrm{H}^*(\mathcal{C})$-module via $- * M$

Conjecture $\mathrm{H}^*(\mathcal{C})$ is a finitely generated k-algebra and $\mathrm{H}^*(M)$ is a finitely generated $\mathrm{H}^*(\mathcal{C})$-module for all objects M of $\mathcal{M}$.

Varieties for tensor categories

From now on let $\mathcal{C}$ be a finite tensor category for which
$\mathsf{H}^*(\mathcal{C}) := \mathsf{Ext}^*_{\mathcal{C}}(\mathbf{1},\mathbf{1})$ is finitely generated, and
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See also Buan-Krause-Snashall-Solberg 2020, Nakano-Vashaw-Yakimov 2022 for tensor triangulated categories and relation to Balmer spectrum

Some standard properties of support varieties
(Bergh-Plavnik-W 2021 and arXiv 2023)

(1) $\mathrm{cx}(M) = \dim \mathcal{V}(M)$, where the **complexity** $\mathrm{cx}(M)$ of an object M of $\mathcal{M}$ is the rate of growth of a minimal projective resolution P of M, $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, as measured by length of P_n

(2) $\mathcal{V}(M \oplus N) = \mathcal{V}(M) \cup \mathcal{V}(N)$

(3) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence, then $\mathcal{V}(M_i) \subseteq \mathcal{V}(M_j) \cup \mathcal{V}(M_l)$ whenever $\{i, j, l\} = \{1, 2, 3\}$

Module product property

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Remark This is known to be an equality when $\mathcal{C} = \mathcal{M} = A$-mod for a cocommutative Hopf algebra A (Friedlander-Pevtsova 2005) and for some quantum groups A (Nakano-Vashaw-Yakimov 2022, Negron-Pevtsova 2023)

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In general, it is known to be an equality when X is projective or when $X = L_\zeta$ ("Carlson's L_ζ objects"), i.e. $L_\zeta := \mathrm{ker}(\zeta)$ for nonzero $\zeta \in \mathrm{Hom}_{\mathcal{C}}(\Omega^n(\mathbf{1}), \mathbf{1}) \cong \mathrm{H}^n(\mathcal{C})$.

It is known **not** to be an equality for some modules of some noncocommutative Hopf algebras (Benson-W 2014, Plavnik-W 2018, Bergh-Plavnik-W arXiv 2023).

Module product property: reduction to complexity 1

Theorem (Bergh-Plavnik-W) Let $\mathcal{C}$ be a *braided* finite tensor category with finitely generated cohomology etc., $\mathcal{M}$ an exact module category. TFAE:

(i) $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$ for all objects X, M.

(ii) $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$ for all objects X, M of complexity 1.

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Remark
$\mathcal{V}(X * M) \subseteq \mathcal{V}(X) \cap \mathcal{V}(M)$ follows from defns of actions and braiding.

Module product property: reduction to complexity 1

Thm TFAE: (i) $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$ for all X, M.
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Idea of proof that (ii) implies (i): Assume $\mathrm{cx}(X) \geq 1$ and $\mathrm{cx}(M) \geq 1$.
Induction on $\mathrm{cx}(X) + \mathrm{cx}(M)$; (ii) is case $\mathrm{cx}(X) + \mathrm{cx}(M) = 2$.

Case $\mathrm{cx}(M) \geq 2$: Reduce to case $\mathcal{V}(M) = Z(\mathfrak{p})$ for a minimal prime $\mathfrak{p}$.
Reduce complexity: For each $\mathfrak{m} \in \mathcal{V}(M)$, $\exists$ a SES

$$0 \to W_{\mathfrak{m}} \to \Omega^n(M) \oplus P \to M \to 0$$

with P projective, $\mathfrak{m} \in \mathcal{V}(W_{\mathfrak{m}})$, and $\mathrm{cx}(W_{\mathfrak{m}}) = \mathrm{cx}(M) - 1$;
it follows that $\mathcal{V}(W_{\mathfrak{m}}) \subseteq \mathcal{V}(M)$ and (a) $\mathcal{V}(M) = \cup_{\mathfrak{m}} \mathcal{V}(W_{\mathfrak{m}})$.
By induction, (b) $\mathcal{V}(X * W_{\mathfrak{m}}) = \mathcal{V}(X) \cap \mathcal{V}(W_{\mathfrak{m}})$.

Combining (a) and (b): $\mathcal{V}(X) \cap \mathcal{V}(M) \overset{(a)}{=} \mathcal{V}(X) \cap (\cup_{\mathfrak{m}} \mathcal{V}(W_{\mathfrak{m}}))$

$$\overset{(b)}{=} \cup_{\mathfrak{m}} \mathcal{V}(X * W_{\mathfrak{m}}) \overset{(c)}{\subseteq} \mathcal{V}(X * M),$$

(c) by properties of $*$ and varieties.

Detecting projectivity

Thm TFAE: (i) $V(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$ for all X, M.
(ii) $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$ for all X, M of complexity 1.

Further reflection yields a third equivalent condition:
(iii) For all indecomposable periodic X, M with $\mathcal{V}(X) = \mathcal{V}(M)$, the object $X * M$ is **not** projective.

This allows checking the above module product property for a potentially more limited collection of objects. For some algebras, there are representation theoretic techniques for this.

We will see this method in action next, for symmetric tensor categories over algebraically closed fields of characteristic 0

Symmetric tensor categories

k - alg closed, char 0

$\mathcal{C}$ - symmetric finite tensor category

Deligne's classification

$\mathcal{C}$ is equivalent to category of modules for the Hopf algebra $\bigwedge(V) \rtimes G$, where

- G is a finite group acting on a fin dim vector space V,
- G contains a subgroup of order 2 with generator g acting as multiplication by -1 on V
- $\Delta(v) = v \otimes 1 + g \otimes v$ for all $v \in V$, $\Delta(h) = h \otimes h$ for all $h \in G$

Theorem (Bergh-Plavnik-W arXiv 2023) Let $\mathcal{C}$ be a symmetric finite tensor category over alg closed field k of char 0. Then for all $X, Y \in \mathcal{C}$, $\mathcal{V}(X \otimes Y) = \mathcal{V}(X) \cap \mathcal{V}(Y)$.

Idea of proof:

(1) Use Deligne's classification: take $\mathcal{C} = \bigwedge(V) \rtimes G$-mod, with key case being $\bigwedge(V) \rtimes \mathbb{Z}_2$

(2) Apply reduction to complexity 1: Suffices to check that two indec periodic modules having **same** variety have tensor product **nonprojective**

(3) Apply techniques of Pevtsova-W 2009: analog of shifted cyclic subgroups in $\bigwedge(V)$ detect projectivity and give rise to induced modules that are periodic - boils down to checking restriction to copies of Sweedler 4-dimensional Hopf algebra as Hopf subalgebras of $\bigwedge(V) \rtimes \mathbb{Z}_2$

Summary of open questions

Let $\mathcal{C}$ be a finite tensor category, $\mathcal{M}$ an exact module category.

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(Known to be true for many classes of examples, unknown in general.)

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(Known to be true for many classes of examples, unknown in general.)

A positive answer opens the door to support varieties with good properties.

(2) Is $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$?
(Known to be true in some settings; known not to be true in others; unknown in general; for $\mathcal{C}$ braided, reduced to objects of complexity 1.)

Module product: negative answers

$$\mathcal{V}(X * M) \stackrel{?}{=} \mathcal{V}(X) \cap \mathcal{V}(M)$$

This is known **not** to be an equality for many classes of examples (Benson-W 2014, Plavnik-W 2018, Bergh-Plavnik-W arXiv 2023). In fact:

Theorem (Bergh-Plavnik-W arXiv 2023) Let k be a perfect field, let $\mathcal{C}$ be a nonsemisimple finite tensor category with finitely generated cohomology etc. Then $\mathcal{C}$ embeds as a finite tensor category into a finite tensor category also having finitely generated cohomology, but **not** the support variety tensor product property.

Module product: negative answers details

G - finite group acting on $\mathcal{C}$ by autoequivalences

Crossed product category $\mathcal{C} \rtimes G$: objects $\oplus_{g\in G}(M_g,\ g)$ with $M_g \in \mathcal{C}$, morphisms componentwise, tensor product given by
$(M,\ g) \otimes (N,\ h) = (M \otimes g(N),\ gh)$, unit object $(\mathbf{1}, 1_G)$.
As a k-linear abelian category, $\mathcal{C} \rtimes G$ is the Deligne product $\mathcal{C} \boxtimes \mathrm{Vec}_G$.

Cohomology $\mathrm{H}^*(\mathcal{C} \rtimes G) \cong \mathrm{H}^*(\mathcal{C})$, support varieties "same" as for $\mathcal{C}$, so e.g. if $\mathcal{C}$ has the support variety tensor product property, then
$\mathcal{V}((M,\ g) \otimes (N,\ h)) = \mathcal{V}(M) \cap \mathcal{V}(g(N))$

Theorem (Bergh-Plavnik-W arXiv 2023)
Let k be a perfect field and $\mathcal{C}$ nonsemisimple. Let $G = \mathbb{Z}_2$ with nonidentity element interchanging factors in $\mathcal{C} \boxtimes \mathcal{C}$. Then $(\mathcal{C} \boxtimes \mathcal{C}) \rtimes G$ does **not** have the support variety tensor product property.