New Constructions of Exceptional Simple Lie Superalgebras in Low Characteristic Using Tensor Categories

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January 11, 2024

IPAM Workshop on Symmetric Tensor Categories and Representation Theory @ UCLA

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$$B \circ (1_{\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}}) = 0;$$
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- A Lie algebra as you know it is an operadic Lie algebra in $\mathrm{Vec}_{\mathbb{K}}$ $(\mathrm{char}\ \mathbb{K} \neq 2)$. A Lie superalgebra as you know it is an operadic Lie algebra in $\mathrm{sVec}_{\mathbb{K}}$ $(\mathrm{char}\ \mathbb{K} \neq 2, 3)$

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- In general might not satisfy $\mathrm{gr}\ U(\mathfrak{g}) = S(\mathfrak{g})$ (PBW Theorem).

Examples of Lie Algebras in STCs

- $\mathfrak{gl}(X) = X \otimes X^*$ with bracket B given by

$$B = 1_X \otimes ev_{X^*,X} \otimes 1_{X^*} \circ \left(1_{\mathfrak{gl}(X)\otimes\mathfrak{gl}(X)} - c_{\mathfrak{gl}(X),\mathfrak{gl}(X)}\right)$$

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- Similarly, get $\mathfrak{o}(X, \beta)$ with a symmetric form $\beta = \beta \circ c_{X,X}$ identifying $\wedge^2(X)$.
- $\mathfrak{gl}(X)$ is always PBW; the others are PBW at least for any Frobenius-exact $\mathcal{C}$.

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- A Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ in $\operatorname{Rep} \mathbb{K}[t]/(t^p)$ is an ordinary Lie algebra equipped with a nilpotent derivation of degree at most p:

$$t.[x, y] = [t.x, y] + [x, t.y]$$

(so that $[\cdot, \cdot]$ is a morphism in the category).

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- Representation theory of an affine group scheme G over Ver_p is controlled by underlying ordinary group scheme G_0 and its Lie algebra $Lie(G)$.

Proposition: $\mathrm{sVec}_{\mathbb{K}}$ is a full subcategory of Ver_p if $p > 2$.

Proof.

If $J_{p-1} \in \mathrm{Rep}\,\mathbb{K}[t]/(t^p)$ has basis $\{v, t.v, \ldots, t^{p-2}.v\}$, then can show $J_{p-1} \otimes J_{p-1} = J_1 \oplus (p-2)J_p$ with J_1 spanned by

$$w = v \otimes (t^{p-2}.v) - t.v \otimes (t^{p-3}.v) + \cdots - (t^{p-2}.v) \otimes v.$$

Because p is odd, $c_{J_{p-1},J_{p-1}}(w) = -w$. After semisimplification, we get $L_{p-1} \otimes L_{p-1} = L_1$ and $c_{L_{p-1},L_{p-1}}$ is multiplication by -1. Hence L_{p-1} tensor generates $\mathrm{sVec}_{\mathbb{K}}$. $\square$

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2. Semisimplification functor being symmetric monoidal (not exact!) is a window from ordinary rep theory to super and Ver_p rep theory.

Examples

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3. If $\mathfrak{g}$ has non-degenerate Killing form, then so does $\overline{\mathfrak{g}}$, meaning it is the direct sum of simple Lie algebras. This lets us construct simple Lie algebras (for instance, in Ver_p).
4. Semisimplification is not always PBW. Consider free Lie algebra in characteristic 3 on generators x, y modulo elements of degree 4, equipped with derivation d given by $d(x) = y, d(y) = 0$. As a Lie algebra in $\mathrm{Rep}\,\mathbb{K}[t]/(t^3)$ it semisimplifies to a "Lie superalgebra" spanned by $\{z, [z, z], [z, [z, z]]\}$.

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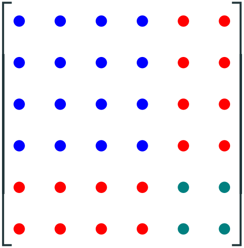
- It decomposes as $\mathfrak{gl}_6 = 16J_1 \oplus 8J_2 \oplus (J_1 \oplus J_3)$:



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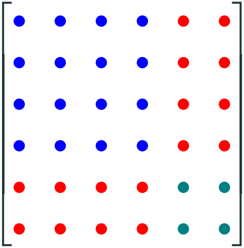

- Therefore, its semisimplification is $\mathfrak{gl}(4|1) = 16L_1 \oplus 8L_2 \oplus L_1$.

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- Constructed using the *Elduque Supermagic Square*, a super analog of the *Freudenthal Magic Square*
- Associates a Lie superalgebra to two unital composition algebras.
- We saw several of these in the previous talk and saw how semisimplification plays a role in their construction at a conceptual level. Will present an alternative construction here.

The Result

Theorem (K). These Lie superalgebras can constructed by semisimplifying an exceptional Lie algebra in characteristic 3 equipped with a nilpotent derivation of degree at most 3.

Kac-Moody Lie Superalgebra

- The setup: $A \in \mathrm{Mat}_n(\mathbb{Z})$ such that diagonal entries are either 2 or 0; if $a_{ii} = 2$, declare i to be an even index, if $a_{ii} = 0$, declare i to be an odd index. Define the Lie superalgebra $\tilde{\mathfrak{g}}(A)$ over $\mathbb{K}$ to be the free Lie superalgebra on generators $\{e_i, f_i, h_i\}_{1 \leq i \leq n}$ subject to the relations:

$$[e_i, f_j] = \delta_{ij} h_i; \quad [h, e_j] = a_{ij} e_j; \quad [h, f_j] = -a_{ij} f_j; \quad [h_i, h_j] = 0,$$

and let $\mathfrak{g}(A)$ be $\tilde{\mathfrak{g}}(A)/I$, where I is the maximal ideal trivially intersecting $\mathfrak{h} = \mathbb{K}h_1 \oplus \cdots \oplus \mathbb{K}h_n$.

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- The Elduque and Cunha Lie superalgebras are of this form (or "related").

Semisimplification in Action

The 133-dimensional simple exceptional Lie algebra $\mathfrak{e}_7$ can be written $\mathfrak{e}_7 = \mathfrak{g}(\hat{A})$, where

$$\hat{A} = \left[\begin{array}{ccccc|cc} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right].$$

The generator e_7 is ad-nilpotent of degree 3, so can view $\mathfrak{e}_7$ as an Lie algebra in $\operatorname{Rep} \alpha_3$ w.r.t. $\operatorname{ad} e_7$.

Semisimplification in Action

Its semisimplification is a finite-dimensional simple exceptional Eldque and Cunha Lie superalgebra $\mathfrak{g}(A)$ of superdimension $(66|32)$, where

$$A = \left[\begin{array}{ccccc|c} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right].$$

Idea: the copy of J_2 spanned by e_6 and $[e_6, e_7]$ in $\mathfrak{e}_7$ became an odd generator (resp. f) in the semisimplification. Demonstration.

Other Comments

- Can do this more generally by adding commuting Chevalley generators. For instance, semisimplifying $\mathfrak{e}_7$ with respect to $e_1 + e_7$ gives another Elduque and Cunha Lie superalgebra. We can get most of them this way by looking at the right Cartan matrix and comparing dimensions.

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- A few of them, however, cannot be determined by looking at Cartan matrix alone; these must be manually determined. For instance, there is the Elduque Lie superalgebra in characteristic 5. This can be constructed by semisimplifying $\mathfrak{e}_8$ with respect to $e_2 + e_3 + e_4$.

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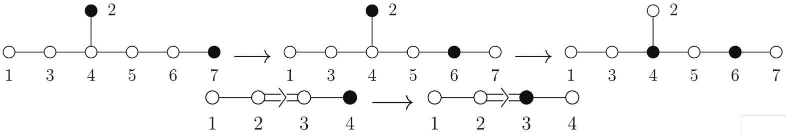
- If e and e' lie in the same nilpotent orbit, then the semisimplifications of $\mathfrak{g}(A)$ w.r.t. e and e' are isomorphic. This gives us large class of realizations (next slide).

Other Comments

Here are some examples of legal swaps:



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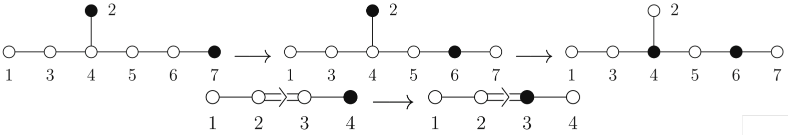

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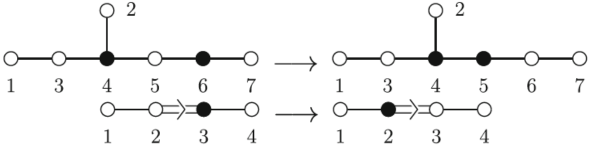

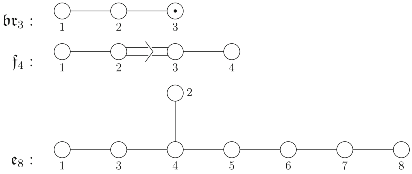

Lie algebra	Nilpotent element	Lie superalgebra
$\mathfrak{br}_3$	e_1, e_2	$\mathfrak{brj}_{2;3}$
$\mathfrak{f}_4$	e_1	see ($\star$) below
	e_4	$\mathfrak{g}(1,6)$
	$e_1 + e_4$	see ($\star$) below
$\mathfrak{e}_6^{(1)}$	e_1, e_2, e_6	$\mathfrak{g}(2,6)^{(1)}$
	$e_1 + e_2,\ e_2 + e_6,\ e_1 + e_6$	$\mathfrak{g}(3,3)^{(1)}$
	$e_1 + e_2 + e_6$	$\mathfrak{g}(2,3)^{(1)}$
$\mathfrak{e}_7$	e_1, e_2, e_7	$\mathfrak{g}(4,6)$
	$e_1 + e_2,\ e_2 + e_7,\ e_1 + e_7$	$\mathfrak{el}(5;3)$
	$e_1 + e_2 + e_7$	$\mathfrak{g}(4,3)$
	$e_2 + e_5 + e_7$	$\mathfrak{f}_4$; see ($\star\star$) below
	$e_1 + e_2 + e_5 + e_7$	$\mathfrak{g}(1,6)$
$\mathfrak{e}_8$	e_1, e_2, e_8	$\mathfrak{g}(8,6)$
	$e_1 + e_2,\ e_2 + e_8,\ e_1 + e_8$	$\mathfrak{g}(6,6)$
	$e_1 + e_2 + e_8$	$\mathfrak{g}(8,3)$
	$e_1 + e_2 + e_6 + e_8$	$\mathfrak{g}(3,6)$

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- What other simple Lie superalgebras can be obtained this way? What about simple Lie algebras in Ver_p?
- Semisimplify other algebraic objects (like distribution algebras of affine group schemes). What happens?

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- Polynomial Functors for STCs

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- Polynomial Functors for STCs
- Deligne's Theorem analog in characteristic p

Some Open Problems

- Classification of simple algebraic groups and Lie algebras in Ver_p
- Notions of reductive groups and root systems in Ver_p, and associated representation theory (some progress made for $GL(X)$, minor progress for $O(X, \beta)$ and $Sp(X, \beta)$)
- Finite-generation of cohomology of finite group schemes for Ver_p in characteristic p
- Polynomial Functors for STCs
- Deligne's Theorem analog in characteristic p
- More generally: what theorems that extend from vector spaces to supervector spaces extend to the Verlinde setting? What new things do we get along the way?

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