New Constructions of Exceptional Simple Lie Superalgebras in Low Characteristic Using Tensor Categories

Arun Kannan (MIT)
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IPAM Workshop on Symmetric Tensor Categories and Representation Theory © UCLA
• An (operadic) Lie algebra in an STC $\mathcal{C}$ is an object $g \in \mathcal{C}$ and a morphism $B : g \otimes g \to g$ such that

$$B \circ (1_{g \otimes g} + c_{g,g}) = 0;$$

$$B \circ (B \otimes 1_g) \circ (1_{g \otimes 3} + (123)_{g \otimes 3} + (132)_{g \otimes 3}) = 0.$$
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- A Lie algebra as you know it is an operadic Lie algebra in $\text{Vec}_K$ ($\text{char } K \neq 2$). A Lie superalgebra as you know it is an operadic Lie algebra in $s\text{Vec}_K$ ($\text{char } K \neq 2, 3$)
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• In general might not satisfy $\text{gr } U(g) = S(g)$ (PBW Theorem).
Examples of Lie Algebras in STCs

- $\mathfrak{gl}(X) = X \otimes X^*$ with bracket $B$ given by

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- $\mathfrak{gl}(X)$ is always PBW; the others are PBW at least for any Frobenius-exact $\mathcal{C}$. 

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- A Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ in $\text{Rep } \mathbb{K}[t]/(t^p)$ is an ordinary Lie algebra equipped with a nilpotent derivation of degree at most $p$:

$$t.[x, y] = [t.x, y] + [x, t.y]$$

(so that $[\cdot, \cdot]$ is a morphism in the category).
The Verlinde Category $\text{Ver}_p$

- The Verlinde category $\text{Ver}_p$ is the semisimplification of $\text{Rep} \mathbb{K}[t]/(t^p)$.

- $\text{Ver}_p$ is a counter-example to Deligne's theorem in positive characteristic ($p \neq 2, 3$) and plays a role in generalizing it.

- Representation theory of an affine group scheme $G$ over $\text{Ver}_p$ is controlled by underlying ordinary group scheme $G_0$ and its Lie algebra $\text{Lie}(G)$. 

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- Simple objects: $L_1, \ldots, L_{p-1}$, images of $J_1, \ldots, J_{p-1}$ (resp.). The indecomposable $J_p$ goes to zero as $\dim J_p = p = 0$. 

Tensor product rule ("truncated Clebsch-Gordan rule"): $L_n \otimes L_m = \min(n, m, p-n, p-m) \bigoplus_{i=1}^{\min(n,m)} L_{|n-m|+2i-1}$. 

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Proposition: $\text{sVec}_K$ is a full subcategory of $\text{Ver}_p$ if $p > 2$.

Proof.

If $J_{p-1} \in \text{Rep } K[t]/(t^p)$ has basis $\{v, t.v, \ldots, t^{p-2}.v\}$, then can show $J_{p-1} \otimes J_{p-1} = J_1 \oplus (p-2)J_p$ with $J_1$ spanned by

$$w = v \otimes (t^{p-2}.v) - t.v \otimes (t^{p-3}.v) + \cdots - (t^{p-2}.v) \otimes v.$$

Because $p$ is odd, $c_{J_{p-1}, J_{p-1}}(w) = -w$. After semisimplification, we get $L_{p-1} \otimes L_{p-1} = L_1$ and $c_{L_{p-1}, L_{p-1}}$ is multiplication by $-1$. Hence $L_{p-1}$ tensor generates $\text{sVec}_K$. \qed
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2. Semisimplification functor being symmetric monoidal (not exact!) is a window from ordinary rep theory to super and $\text{Ver}_p$ rep theory.
Examples

1. If $\mathfrak{g}$ is a Lie algebra in $\mathcal{C}$ and $V$ a module over $\mathfrak{g}$, then $\overline{\mathfrak{g}}$ is a Lie algebra in $\overline{\mathcal{C}}$ and $\overline{V}$ is a module over $\overline{\mathfrak{g}}$. 
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2. \( \overline{\mathfrak{gl}(X)} = \mathfrak{gl}(\overline{X}), \overline{\mathfrak{sl}(X)} = \mathfrak{sl}(\overline{X}), \overline{\mathfrak{sp}(X, \beta)} = \mathfrak{sp}(\overline{X}, \overline{\beta}), \overline{\mathfrak{o}(X, \beta)} = \mathfrak{o}(\overline{X}, \overline{\beta}) \)
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4. Semisimplification is not always PBW. Consider free Lie algebra in characteristic 3 on generators $x, y$ modulo elements of degree 4, equipped with derivation $d$ given by $d(x) = y, d(y) = 0$. As a Lie algebra in $\text{Rep} \mathbb{K}[t]/(t^3)$ it semisimplifies to a “Lie superalgebra” spanned by $\{z, [z, z], [z, [z, z]]\}$. 
Example: $\mathfrak{gl}_6$

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- Since

$$e_{56}^3 = 0 \implies (\mathrm{ad} e_{56})^3 = 0$$

$(\mathfrak{gl}_6, \mathrm{ad} e_{56})$ is a Lie algebra in Rep $\alpha_3$. 

\[ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \]

- Therefore, its semisimplification is $\mathfrak{gl}(4|1) = 16L_1 \oplus 8L_2 \oplus L_1$. 

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The Elduque and Cunha Lie Superalgebras

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- Constructed using the *Elduque Supermagic Square*, a super analog of the *Freudenthal Magic Square*
- Associates a Lie superalgebra to two unital composition algebras.
- We saw several of these in the previous talk and saw how semisimplification plays a role in their construction at a conceptual level. Will present an alternative construction here.
Theorem (K). These Lie superalgebras can be constructed by semisimplifying an exceptional Lie algebra in characteristic 3 equipped with a nilpotent derivation of degree at most 3.
The setup: \( A \in \text{Mat}_n(\mathbb{Z}) \) such that diagonal entries are either 2 or 0; if \( a_{ii} = 2 \), declare \( i \) to be an even index, if \( a_{ii} = 0 \), declare \( i \) to be an odd index. Define the Lie superalgebra \( \tilde{g}(A) \) over \( \mathbb{K} \) to be the free Lie superalgebra on generators \( \{e_i, f_i, h_i\}_{1 \leq i \leq n} \) subject to the relations:

\[
[e_i, f_j] = \delta_{ij} h_i; \quad [h, e_j] = a_{ij} e_j; \quad [h, f_j] = -a_{ij} f_j; \quad [h_i, h_j] = 0,
\]

and let \( g(A) \) be \( \tilde{g}(A)/I \), where \( I \) is the maximal ideal trivially intersecting \( h = \mathbb{K} h_1 \oplus \cdots \oplus \mathbb{K} h_n \).
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The Elduque and Cunha Lie superalgebras are of this form (or “related”).
The 133-dimensional simple exceptional Lie algebra $\mathfrak{e}_7$ can be written $\mathfrak{e}_7 = g(\hat{A})$, where

$$
\hat{A} = \begin{bmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}.
$$

The generator $e_7$ is ad-nilpotent of degree 3, so can view $\mathfrak{e}_7$ as an Lie algebra in $\text{Rep} \alpha_3$ w.r.t. $\text{ad} \ e_7$. 
Its semisimplification is a finite-dimensional simple exceptional Eldque and Cunha Lie superalgebra $g(A)$ of superdimension $(66|32)$, where

$$A = \begin{bmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
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\end{bmatrix}.$$

Idea: the copy of $J_2$ spanned by $e_6$ and $[e_6, e_7]$ in $e_7$ became an odd generator (resp. $f$) in the semisimplification. Demonstration.
• Can do this more generally by adding commuting Chevalley generators. For instance, semisimplifying $e_7$ with respect to $e_1 + e_7$ gives another Elduque and Cunha Lie superalgebra. We can get most of them this way by looking at the right Cartan matrix and comparing dimensions.

• A few of them, however, cannot be determined by looking at Cartan matrix alone; these must be manually determined. For instance, there is the Elduque Lie superalgebra in characteristic 5. This can be constructed by semisimplifying $e_8$ with respect to $e_2 + e_3 + e_4$.

• If $e$ and $e'$ lie in the same nilpotent orbit, then the semisimplifications of $g(A)$ w.r.t. $e$ and $e'$ are isomorphic. This gives us large class of realizations (next slide).
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Here are some examples of legal swaps:
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Here are some examples of illegal swaps:
**Summary of Results**

<table>
<thead>
<tr>
<th>Lie algebra</th>
<th>Nilpotent element</th>
<th>Lie superalgebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{b}_3$</td>
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<td>$\mathfrak{br}(2;3)$</td>
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<td>see (⋆) below</td>
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- Semisimplify other algebraic objects (like distribution algebras of affine group schemes). What happens?
Some Open Problems

- Classification of simple algebraic groups and Lie algebras in $\text{Ver}_p$

- Notions of reductive groups and root systems in $\text{Ver}_p$, and associated representation theory (some progress made for $\text{GL}(X)$, minor progress for $\text{O}(X,\beta)$ and $\text{Sp}(X,\beta)$).

- Finite-generation of cohomology of finite group schemes for $\text{Ver}_p$ in characteristic $p$.

- Polynomial Functors for STCs.

- Deligne's Theorem analog in characteristic $p$.

- More generally: what theorems that extend from vector spaces to supervector spaces extend to the Verlinde setting? What new things do we get along the way?
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