Everything will be $\Bbbk$-linear over $\Bbbk = \overline{\Bbbk}$ of char. $\neq 2$.
Usually graded but I will often ignore for simplicity.

Basic motivation

Take your favorite abelian category
- $\mathrm{Rep}(G)$ G a reductive group
- $\bigoplus_{n \geq 0} \Bbbk S_n\text{-mod}$
- BGG category $\mathcal{O}$ for s.s. Lie alg. $\mathfrak{g}$

Study the strict monoidal category of projective endofunctors & nat. t fs
- $V \otimes -$, $V^* \otimes -$ V f.d. G-module
- Ind_H^G, Res_H^G, $H \leq G$ finite groups
- Ind_A^B, Res_A^B, $A \leq B$ Frobenius exten.

functor with left & right adjs which are $\cong$ (up to deg-shift)

One focus is to find explicit presentations using string diagrams.
Wonderful examples — Soergel bimodules, Khovanov's Heisenberg category, Khovanov-Lauda-Rouquier's Kac-Moody 2-categories ...

Diagrammatics for Frobenius extensions (Elias – Snyder – Williamson Cubes of Frobenius exts)

$A \leq B$ Frobenius extension of graded commutative algebras of degree d

So $\exists\ \mathrm{tr}_A^B : B \to A$ A-module hom of degree $-2d$
plus dual bases $\{b\}, \{b^\vee\}$ for B as free A-module
such that $\mathrm{tr}_A^B(bc^\vee) = \delta_{b,c}$.

String diagrams

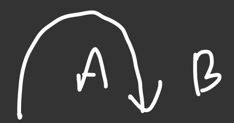
$\mathrm{ind}_A^B = B \otimes_A -$ $\qquad B \uparrow A$

$q^d\, \mathrm{res}_A^B$ $\qquad A \downarrow B$

q is downward grading shift

$(qV)_i = V_{i+1}$

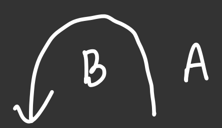
unit
deg $-d$



counit
deg d

usual adjunction (ind, res)

unit
deg d



counit
deg $-d$

second adjunction (res, ind) from Frobenius structure

$B \to B \otimes_A B$

$1 \mapsto \sum b \otimes b^\vee$

$\mathrm{tr}_A^B : B \to A$

Also have "bubbles"
a A $\qquad a \in A$
(nat. tf. defined by mult)

Singular Soergel bimodules

W a Coxeter group $\quad s_i\ (i \in N)$ simple reflections

$R = S(\mathfrak{h}^*)$ suitable realization, $\mathfrak{h}^*$ in degree 2 $\qquad R^W$ = invariants

$J \subset I \subset N$ finitary $(W_I$ finite$)$ $\qquad R^I = R^{W_I}$ for short

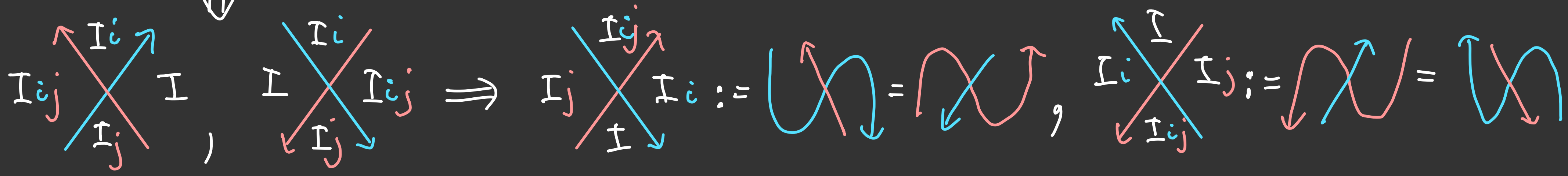
parabolic, longest elt. w_I

Demazure's theorem $R^I < R^J$ is a graded Frobenius extension, deg. $\ell(w_I) - \ell(w_J)$ with tr^I_J = product of Demazure operators according to reduced word for $w_I w_J^{-1}$

color string by $i \in N$

Transitivity of ind/res — only need $Ii \uparrow I$, $I \downarrow Ii$ — generate "one at a time"

$\Downarrow$



$SSBim_W$ — strict q-complete graded Karoubian 2-category (a 2-full sub-2-category of Bim)

objects R^I $\forall$ finitary I

1-mors generated by

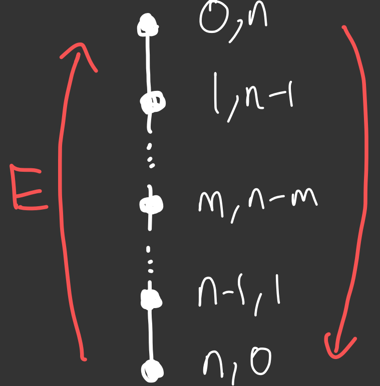
$I_i \uparrow I \ , \ I \downarrow I_i$

Williamson's theorem $K_0(SSBim_W)$ is the Hecke algebroid of W

This gives very rich combinatorics, and we have the graphical calculus (full set of relations remains open).

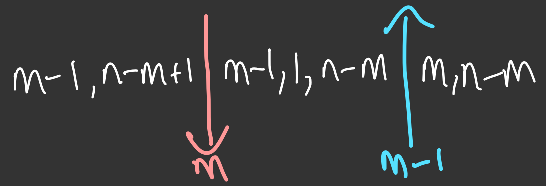
Grassmannian bimodules $GBim_n$

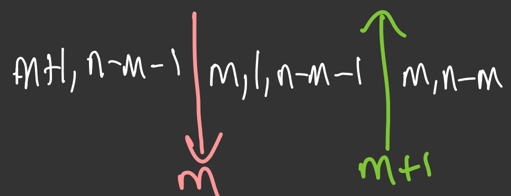
Take $W = S_n \circlearrowright R = \Bbbk[x_1, \dots, x_n]$
s_i flips x_i, x_{i+1} as usual



- Only take objects $\leftrightarrow R^{m,n-m} = R^{S_m \times S_{n-m}}$ $(0 \le m \le n)$
- 1-mors generated by special bimodules E, F

We see $\mathfrak{sl}_2$!!!

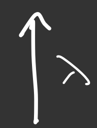
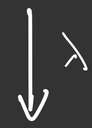
$E =$ 

$F =$ 

Omit strings of colors 0,n

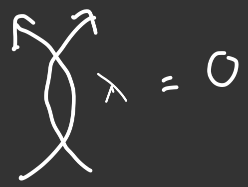
The 2-category $\mathcal{U}(\mathfrak{sl}_2)$ (Lauda, Rouquier)

strict graded 2-category

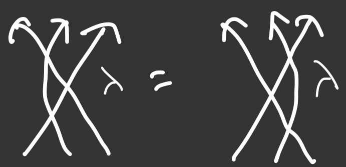
- objects $\lambda \in \mathbb{Z}$
- 1-mors generated by   $E 1_\lambda = 1_{\lambda+2} E$, $F 1_\lambda = 1_{\lambda-2} F$
- 2-mors generated by

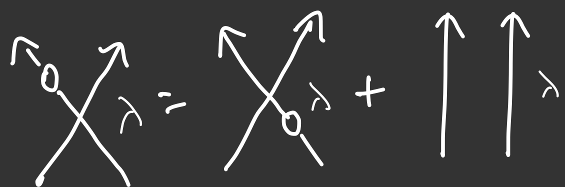
deg 2 deg −2 deg 1−λ deg 1+λ

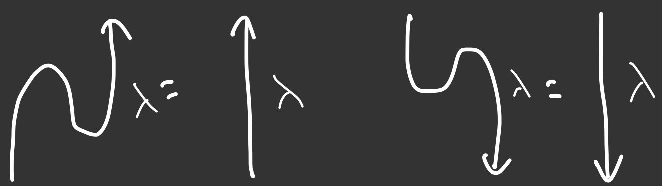
Relations (Rouquier's)

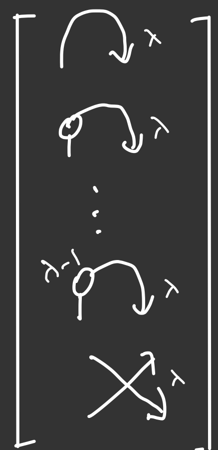


nil Hecke algebra!









$: EF1_\lambda \to 1_\lambda^{\oplus \lambda} \oplus FE 1_\lambda$ for $\lambda \geq 0$

for $\lambda \leq 0$

$: 1_\lambda^{\oplus(-\lambda)} \oplus EF1_\lambda \to FE1_\lambda$

are invertible

Theorem (Lauda, Khovanov–Lauda) $K_0(\mathcal{U}(sl_2)) \cong {}_{\mathbb{Z}[q,q^{-1}]}\dot{U}(sl_2)$

split Grothendieck category of additive Karoubi envelope "f.g. projectives"

Lusztig's modified (idempotented) integral form for $U_q(sl_2)$

Key steps in proof

Over $\mathbb{Q}(q)$

1. Construct obvious homomorphism so $e1_\lambda \mapsto [E1\!\!1_\lambda]$, $f1_\lambda \mapsto [F1\!\!1_\lambda]$
2. Check Hom-form on LHS = Lusztig form on RHS
3. Construct primitive homogeneous idempotents matching canonical basis

proved using 2 and "almost orthogonality" of canonical basis

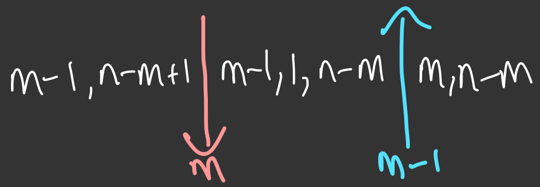
Proof of 2 is hardest here, need basis for 2-morphism spaces in $\mathcal{U}(sl_2)$. This remains the delicate place for other KM 2-cats. It has been solved in general by Webster by "new approach" (still writing)

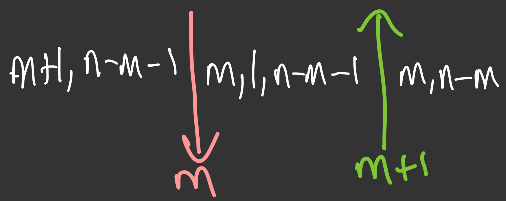
Lauda's approach to basis theorem uses:

Theorem (Lauda) There's a 2-functor

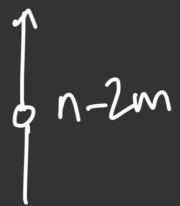
$$\mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{G}Bim_n^0 \quad (n \geq 0)$$

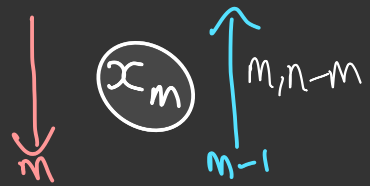
$0 \leq m \leq n$:

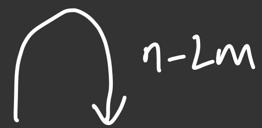
$E1_{n-2m} \mapsto$ 

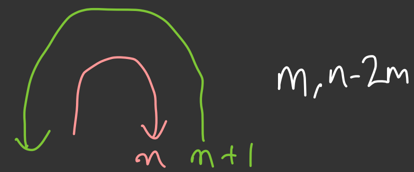
$F1_{n-2m} \mapsto$ 

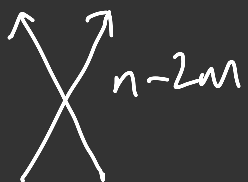
all other $E1_\lambda, F1_\lambda \mapsto 0$

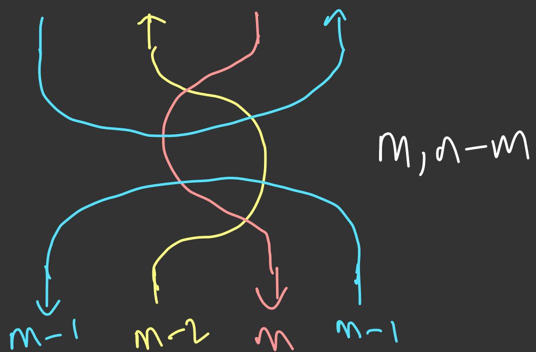


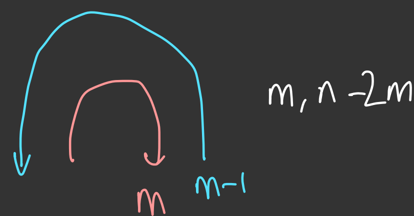
$\mapsto$ 



$\mapsto$ 

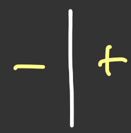


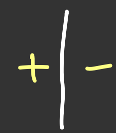
$\mapsto$ 

$\mapsto$ 

The nil-Brauer category

Fix $t \in \{0,1\}$. Then $\mathcal{NB}_t$ is the strict graded 2-category with objects $+, -$, generating 1-morphisms $B\mathbb{1}_+ = \mathbb{1}_- B$ and $B\mathbb{1}_- = \mathbb{1}_+ B$





and generating 2-morphisms

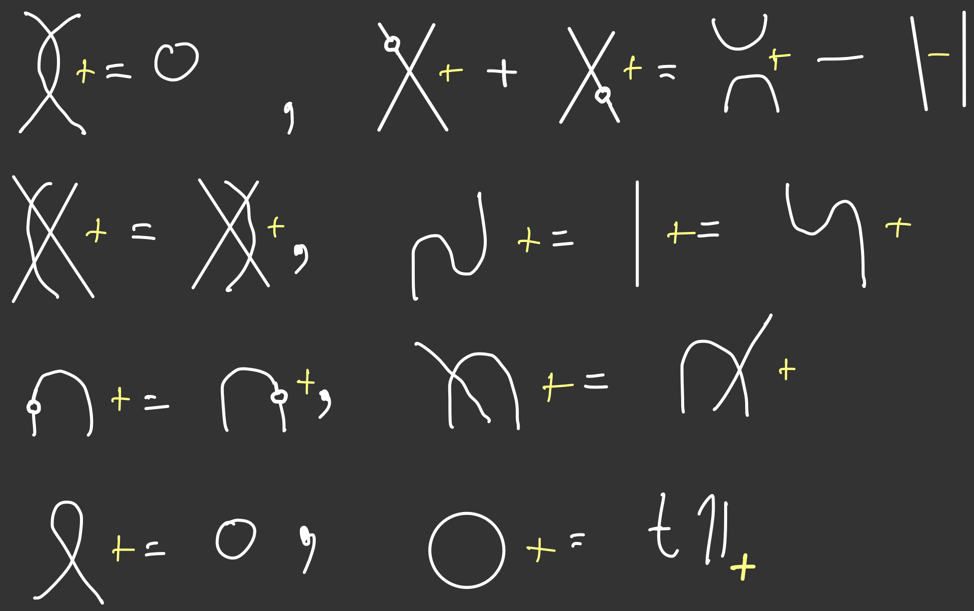
deg 2  $+ : B\mathbb{1}_+ \to B\mathbb{1}_+$

deg −2  $+ : B^2\mathbb{1}_+ \to B\mathbb{1}_+$

deg 0  $+ : B^2\mathbb{1}_+ \to \mathbb{1}_+$

deg 0  $+ : \mathbb{1}_+ \to B^2\mathbb{1}_+$

Relations



and another set of gens & rels with $+$ replaced by $-$ (graph auto.)

The <u>split $\imath$-quantum group of rank one</u> $U^\imath_q(\mathfrak{sl}_2)$ is the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $\boxed{b = f + qk^{-1}e}$ $\begin{pmatrix}\text{Letzter} \\ \text{QSP } (\mathfrak{sl}_2, \mathfrak{so}_2)\end{pmatrix}$

Bao and Wang introduced $\imath$-canonical basis $b^{(n)}$ $(n \geq 0)$, which in this "easy" case can be defined recursively by

$$b^{(0)} = 1 \qquad b\, b^{(n)} = \begin{cases} [n+1]\, b^{(n+1)} & \text{if } n \equiv t\ (2) \\ [n+1]\, b^{(n+1)} + [n]\, b^{(n-1)} & \text{if } n \not\equiv t\ (2) \end{cases}$$

$\longleftarrow$ Depends on choice of $t \in \{0,1\}$

It spans a $\mathbb{Z}[q,q^{-1}]$-form ${}_{\mathbb{Z}[q,q^{-1}]}U^\imath_t$.

<u>Theorem</u> (B.-Wang-Webster) $K_0(\mathcal{NB}_t) \cong \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2\left({}_{\mathbb{Z}[q,q^{-1}]}U^\imath_t\right) \right\}$ with α, δ even β, γ odd

Indecomposables $\longleftrightarrow$ $\imath$-canonical basis, Hom-form $\longleftrightarrow$ Bao-Wang-Lusztig form

The strategy for the proof is similar to Lauda's from before, but steps are much harder.

① Construct obvious homomorphism so $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \mapsto [B 1\!\!1_+]$, $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mapsto [B 1\!\!1_-]$

② Check Hom-form = BWL-form

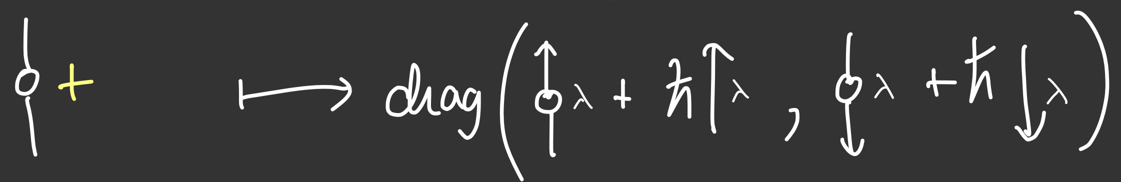
Requires basis theorem for 2-morphism spaces. We prove this by "new approach" based on 2-functor

grading completion of 2-cat $\mathcal{U}(\mathfrak{sl}_2)$ for $\lambda \equiv t \ (2)$

$$\mathcal{NB}_t \longrightarrow \mathrm{Add}\left(\widehat{\mathcal{U}}(\mathfrak{sl}_2; t)\right)$$

$$B 1\!\!1_+ \longmapsto \bigoplus_{\lambda \equiv t \ (4)} E 1\!\!1_\lambda \oplus F 1\!\!1_\lambda$$

invertible!

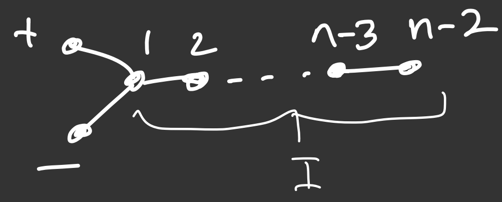


$\hbar \neq 0$

...

③ Construct primitive idempotents satisfying same recurrence as $b^{(n)}$.

Max. Isotropic Grassmannian bimodules in type D_n $(n \geq 2)$



$IGBim_n$ $\quad R = \Bbbk[x_0, x_1, \ldots, x_{n-1}]$ $\begin{cases} \alpha_+ = x_0 - x_1 \\ \alpha_- = x_0 + x_1 \end{cases}$

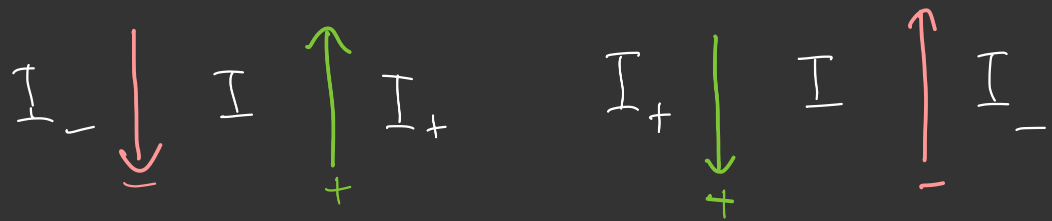
$\cap$

$SSBim_{D_n}$

Two A_{n-1} subsystems $I^\pm$ and A_{n-2} subsystem I

Just take objects R^{I+} and R^{I-}

1-mors gen'd by special bimodules:



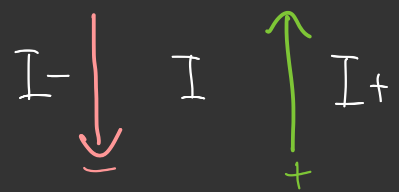
Theorem (B.-Bodish-Elias) There's a 2-functor $\mathcal{NB}_t \to IGBim_n^\circ$

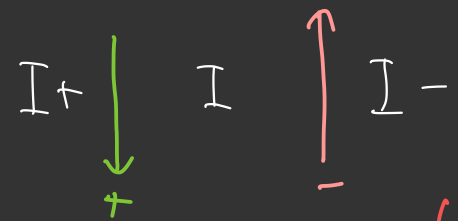
We do not yet have full picture — kernel of this functor?

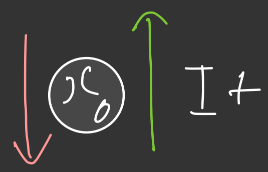
— fullness, use to reprove basis thm.

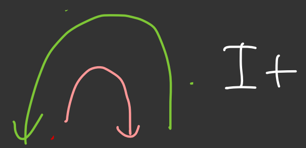
Pictures for this 2-functor:

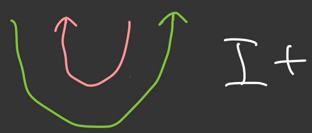
$+ \mapsto R^{I+}$ $\qquad\qquad$ $- \mapsto R^{I-}$

$B1_+ \mapsto$ 

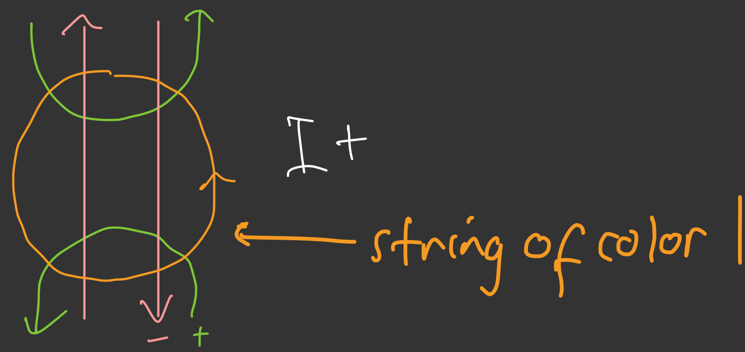
$B1_- \mapsto$ 

$\longmapsto$ 

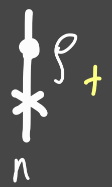
$\longmapsto$ 

$\longmapsto$ 

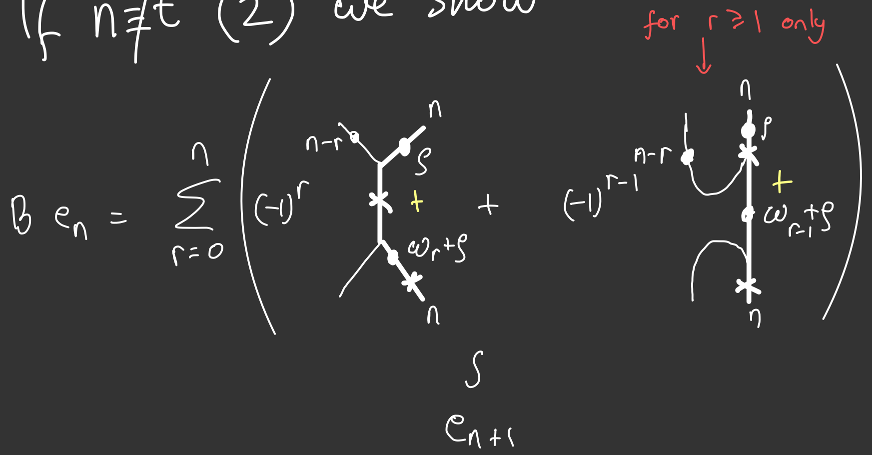
Unlike Grassmannian case earlier, this is hard to describe w/out diagrams

$\longmapsto$ 

and similar diagrams with + replaced by −
(apply graph automorphism $+ \longleftrightarrow -$)

Theorem (BWW) $e_n :=$  is a primitive homogeneous idempotent in NB_t, and every such is conjugate to one of these. Moreover we decompose $B \star e_n$ as a sum of conjugates of these.

If $n \not\equiv t$ (2) we show



sums of mut. orthogonal primitive homogeneous idemps.

If $n \equiv t$ (2) we show

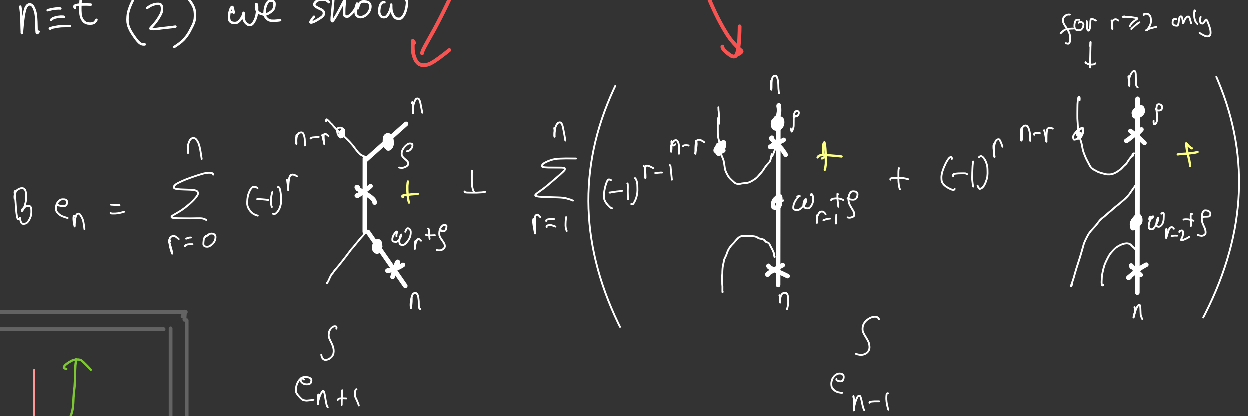
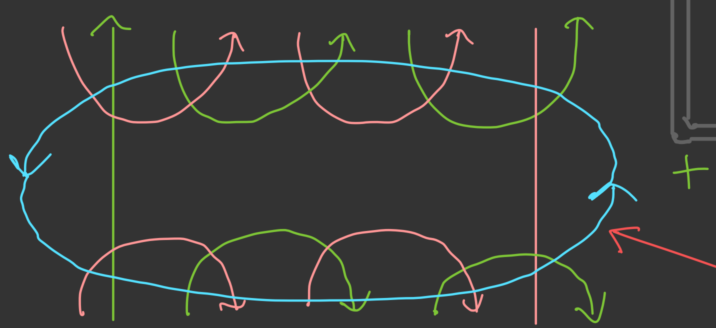


Image of  in IGBim_n is



string of color 4