The Delannoy Category

Noah Snyder
Indiana University

Joint with Harman-Snowden

Slides at nsnyder1.pages.iu.edu

1. Background

HS Oligomorphic groups and tensor categories

2. Structural results

3. Delannoy path model

HSS The Delannoy Category

4. Circle version

HSS The Circular Delannoy Category

Review

G Oligomorphic Group & μ Measure on $\widehat{G}$-sets

$$\downarrow$$

$\mathrm{Perm}(G,\mu)$ Category of Permutation reps

Smooth $A(G)$-mods — Karoubi Envelope — Requires μ to be Regular

$$\mathrm{Rep}(G,\mu)$$

The Group

$$G = \mathrm{Aut}(\mathbb{R}, <)$$

Preserves < only!
Not alg. str.

Or, if you prefer, $\mathrm{Aut}(\underbrace{\mathbb{Q}, <}_{\text{Fraisse Limit of finite ordered sets}})$

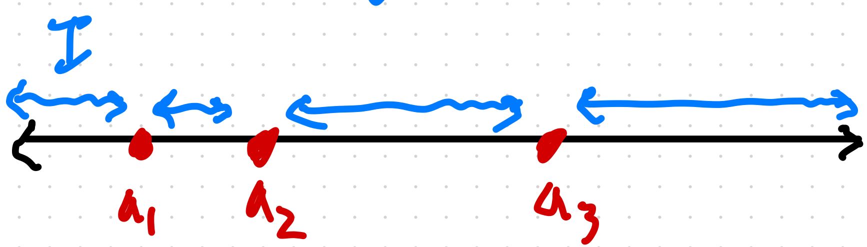
The $\widehat{G}$-Sets

Every transitive G-set is isomorphic to

$$\mathbb{R}^{(n)} := \text{Unordered distinct } n\text{-tuples}$$

stabilizer

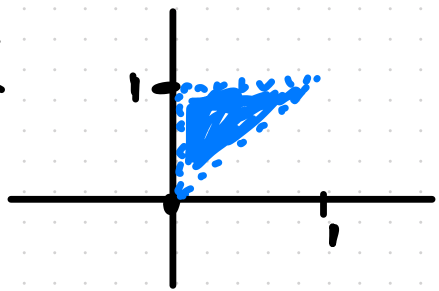
Every open subgroup is $G(a)$ for $a \in \mathbb{R}^n$



$G(a) \cong G^{\times 4}$

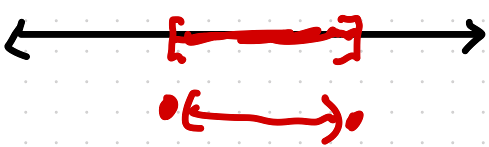
Every transitive $\hat{G}$-set is isomorphic to $I_1^{(n_1)} \times I_2^{(n_2)} \times \dots \times I_k^{(n_k)}$

Ex



$(0,1)^{(2)}$

Ex



disjoint union of three transitive $\hat{G}$ sets

The Measure

<u>Thm</u> (Harman-Snowden) There are exactly four measures for G

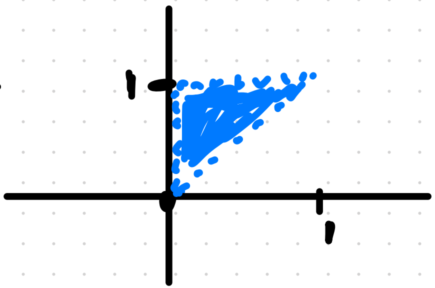
Regular!

Our μ. • Euler characteristic on $\mathbb{R}$ Schapira & Viro

• Euler characteristic on compactifications of $\mathbb{R}$ by adding $+\infty$, $-\infty$, or both

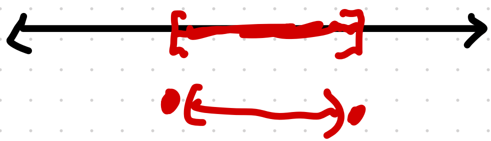
poorly behaved

Ex



$$M(X) = \overset{V}{0} - \overset{E}{0} + \overset{F}{1} = 1$$

Ex



$$M(X) = \overset{V}{2} - \overset{E}{1} = 1$$

Ex $M\left(I_1^{(n_1)} \times \dots \times I_k^{(n_k)} \right) = (-1)^{n_1 + \dots + n_k} \neq 0$

regular

Structure of $\mathbb{R}^{(1)} = \mathbb{R}$

$\mathrm{Hom}_{\mathrm{Perm}}(\mathbb{R}, \mathbb{R})$ is G-invariant Schwartz fns.

Basis $\mathbb{1}_{x<y}$, $\mathbb{1}_{x=y}$, $\mathbb{1}_{x>y}$

Compose $f \circ g = \int_{\mathbb{R}} f(x,y)\, g(y,z)\, dy$

uses μ

	$\mathbb{1}_{x<y}$	$\mathbb{1}_{x=y}$	$\mathbb{1}_{x>y}$
$\mathbb{1}_{y<z}$	$-\mathbb{1}_{x<z}$	$\mathbb{1}_{x<z}$	$-\mathbb{1} = -\mathbb{1}_{x>z} - \mathbb{1}_{x=z} - \mathbb{1}_{x<z}$
$\mathbb{1}_{y=z}$	$\mathbb{1}_{x<z}$	$\mathbb{1}_{x=z}$	$\mathbb{1}_{x>y}$
$\mathbb{1}_{y>z}$	$-\mathbb{1}$	$\mathbb{1}_{x>z}$	$-\mathbb{1}_{x>z}$

Karoubian Description

Perp. Projections: $\mathbb{1}$, $\mathbb{1}_{x \leq y}$, $\mathbb{1}_{x \geq y}$

$$\mathbb{R}^{(1)} \cong \mathbb{1} \oplus L_{\bullet} \oplus L_{\circ}$$

$\mathrm{Rep}(G, \mu)$ description

$\mathcal{C}(\mathbb{R}) \longleftarrow$ Schwartz functions

$L_\bullet$ Submodule gen by char. functions of $(\longleftrightarrow]$

(Left continuous finitary functions)

$L_\circ$ Submodule gen by char functions of $[\longleftrightarrow)$

$\mathbb{1}$ Submodule of constant functions

Novel Properties (HS)

- Doesn't come from (super)groups <u>or</u> from Deligne interpolation.

- Semisimple in all characteristics

- In characteristic p it's the first example of a semisimple pre-Tannakian cat tha does not have finite growth.

Questions

- Simple objects?
- Fusion rules?
- Branching rules?
- Schur functors?

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Simple Objects

$$\mathcal{C}(\mathbb{R}^{(n)}) \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \mathcal{C}(\mathbb{R}^{(n-1)})$$

joint kernel is the "new stuff"

Multiplicity Free!

<u>Goal</u>: Decompose the "new stuff" into simple objects.

Def A **weight** is a word in $\{\bullet, \circ\}$

Def A half-open interval has type

$\bullet$ if it is of the form $(\, , \,]$

$\circ$ if it is of the form $[\, , \,)$

A tuple $I = (I_1, \ldots, I_n)$ of half-open is ordered if $I_1 < I_2 < \ldots < I_n$

and has type the word of types of I_k

Def Let L_λ be the $\underline{G}$-submodule of $\mathcal{C}(\mathbb{R}^{(n)})$ generated by the $\mathbb{1}_I$ where I is an ordered tuple of half-open intervals of type λ.

$$\mathbb{1}_I(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } x_i \in I_i \\ 0 & \text{else} \end{cases}$$

Thm

- L_λ is simple
- $L_\lambda \cong L_\mu$ iff $\lambda = \mu$
- Every simple obj is isom. to some L_λ
- $$J_+ \operatorname{Ker}\left(\mathcal{C}(\mathbb{R}^n) \begin{smallmatrix} \longrightarrow \\ \vdots \\ \longrightarrow \end{smallmatrix} \mathcal{C}(\mathbb{R}^{n-1}) \right) \cong \bigoplus_{\lambda : \operatorname{len}(\lambda) = n} L_\lambda$$

Fusion Rules

Thm Fusion rules given by "ruffle product"

$$L_{\bullet} \otimes L_{\bullet\circ} \cong L_{\bullet\bullet\circ} \oplus L_{\bullet\bullet\circ} \oplus L_{\bullet\circ\bullet} \quad \text{"shuffles"}$$

$$\underbrace{\oplus L_{\bullet\circ}}_{\bullet \,\&\, \bullet \text{ collide}} \oplus \underbrace{L_{\bullet\circ} \oplus L_{\bullet\bullet} \oplus L_{\bullet}}_{\bullet \,\&\, \circ \text{ collide}} \quad \text{"ruffles"}$$

Other Structure

$$\mathrm{Res}^{G}_{G\times G}\left(L_{\bullet\bullet\circ}\right) \cong L_{\bullet\bullet\circ}\boxtimes \mathbb{1} \oplus L_{\bullet\bullet}\boxtimes \mathbb{1} \oplus L_{\bullet\bullet}\boxtimes L_{\circ} \oplus L_{\bullet}\boxtimes L_{\circ} \oplus L_{\bullet}\boxtimes L_{\bullet\circ} \oplus \mathbb{1}\boxtimes L_{\bullet\circ} \oplus \mathbb{1}\boxtimes L_{\bullet\bullet\circ}$$

cut

$$\mathrm{Ind}^{G\times G}_{G}\left(L_{\bullet}\boxtimes L_{\circ}\right) \cong L_{\bullet\bullet\circ} \oplus L_{\bullet\circ\circ} \oplus L_{\bullet\circ}$$

insert

Grothendieck Ring

K has basis a_λ

Hopf algebra via:

$$\mathrm{res}: K \longrightarrow K \otimes K$$
$$\dim: K \longrightarrow \mathbb{Z}$$
$$a_\lambda \longmapsto (-1)^{\mathrm{len}(\lambda)}$$

Lyndon Word

$\mathbb{Q} \otimes K$ is a polynomial algebra in a_λ

$$\mathcal{C}(\mathbb{R}^{(n)}) \cong \bigoplus_{\lambda} L_{\lambda}^{\binom{n}{\ell(\lambda)}}$$

$$L^{*}_{\bullet\bullet\circ} \cong L_{\circ\circ\bullet}$$

---

class of L_λ

Adams Operations: $\psi_i(a_\lambda) = a_\lambda$

Schur Functors: $S_\mu(V) \cong \bigoplus_{i \geq 0} (\Lambda^i V)^{c(\mu, i)}$

Ex $\mathrm{Sym}^2(V) = \Lambda^2 V \oplus V$

See paper for formula

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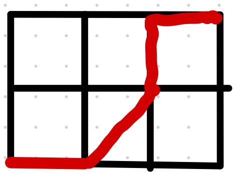
$\mathrm{Hom}\left(\mathcal{C}(\mathbb{R}^{(n)}), \mathcal{C}(\mathbb{R}^{(m)})\right)$

Basis indexed by n-by-m

Delannoy Paths



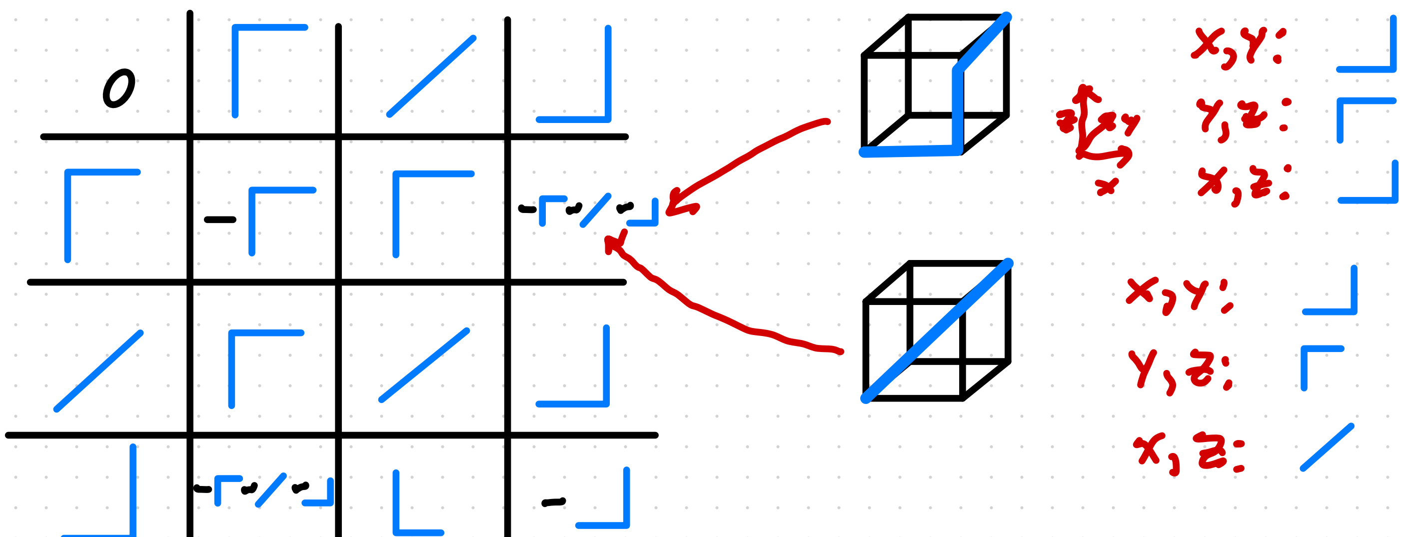
3-by-2

$$\longrightarrow \mathbb{1}_{x_1 < x_2 = y_1 < y_2 < x_2}$$

Composition: $p \circ q = \sum_{\gamma} (-1)^{\cdots}\, r$

Where γ is a 3-dim Delannoy path which projects down to p, q, and r

Ex



Can you fully describe $\underline{\mathrm{Perm}}(G,M)$ with paths?

- $\mathbb{R}^{(n)} \otimes \mathbb{R}^{(m)} \cong \bigoplus_{n\text{-by-}m \text{ path } \gamma} \mathbb{R}^{\mathrm{len}(\gamma)}$
- Associator is easy!
- $f \otimes g$ involves 4-dimensional paths ☹

  $f \otimes id$, $id \otimes g$ each only use 3-dim paths

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Circular Delannoy

$\mathrm{Aut}(S^1, \text{cyclic ordering})$

μ again is Euler char.

$\mu(G) = 0$ but $\mu(G(n)) = (-1)^n \neq 0$

Quasi-regular so $\mathrm{Rep}(G, \mu)$ non-s.s.

Main Results

- Classify simples, projectives, injectives
- Tame, so classify indecomposables.
- Compute branching rules
- Semisimplification
- Delannoy loop model

Don't Know Fusion rules, $\mathrm{char}(k) \neq 0$.