A Chinese Remainder Theorem and Carlson Theorem for monoidal triangulated categories

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Setup

Set some notation:

- $\Bbbk$ is always an algebraically closed field, all categories are $\Bbbk$-linear.
- $\mathbf{T}$ will denote a finite tensor category ($\otimes$ bilinear, Hom spaces finite-dimensional, objects finite length, $\mathbf{1}$ simple, finitely many simples, enough projectives, duals).
- $\mathbf{K}$ will denote a monoidal triangulated category ($\otimes$ compatible with triangulated structure).
- No assumption of symmetric or braided structure (unless explicit).
- For simplicity, I will assume all categories in this talk are rigid.
- $\mathbf{1}$ denotes the tensor unit.

Support theories

A *support theory* for a finite tensor category $\mathbf{T}$ is an assignation $A \mapsto \sigma(A)$ of a (closed) subset $\sigma(A)$ of some geometric space X for each object A of $\mathbf{T}$, in a way that respects the homological and tensor-theoretical properties of $\mathbf{T}$.

Two major players:

- the *cohomological support* $W(-)$
- and the *Balmer support* $V(-)$.

The stable category

One of the typical properties under the umbrella of "respecting the homological properties of $\mathbf{T}$": *projectivity detection*, that is, $\sigma(P) = \varnothing$ if and only if P is projective.
In other words, supports for $\mathbf{T}$ are well-defined on the *stable category of* $\mathbf{T}$, denoted $\underline{\mathbf{T}}$:

- Objects are the same as in $\mathbf{T}$;
- $\mathrm{Hom}_{\underline{\mathbf{T}}}(A, B) = \mathrm{Hom}_{\mathbf{T}}(A, B) / \mathrm{PHom}_{\mathbf{T}}(A, B)$.

Using properties of dual objects, $\mathbf{T}$ is Frobenius $\Rightarrow$ $\underline{\mathbf{T}}$ is triangulated, and the tensor product is well-defined on $\underline{\mathbf{T}}$ in a way compatible with the triangulated structure. That is, $\underline{\mathbf{T}}$ is a monoidal triangulated category. Many of the results in this talk will be in this more general context.

Big picture: our goals

- Goal 1: to understand $\underline{\mathbf{T}}$, might want to classify all indecomposable objects, and understand how they tensor together. That isn't reasonable in many cases though, so a rougher goal is to classify the *thick ideals*, collections of objects $\mathbf{I}$ which are triangulated, closed under direct summands, and closed under tensor product with arbitrary objects of $\underline{\mathbf{T}}$.
- Goal 2: understand the relationship between the intrinsically defined supports of $\underline{\mathbf{T}}$: the cohomological support and the Balmer support.

Yoneda product

In an abelian category:

$$\mathrm{Ext}^i(A, B) \times \mathrm{Ext}^j(C, A) \to \mathrm{Ext}^{i+j}(C, B)$$

by sending

$$0 \to B \to M_0 \to \ldots \to M_{i-1} \to A \to 0$$
$$\times$$
$$0 \to A \to N_0 \to \ldots \to N_{j-1} \to C \to 0$$
$$\mapsto$$

Yoneda product

$$\begin{tikzcd}
 & & & & A \arrow[dr] & & & & & \\
0 \arrow[r] & B \arrow[r] & M_0 \arrow[r] & \dots \arrow[r] & M_{i-1} \arrow[u] \arrow[r] & N_0 \arrow[r] & \dots \arrow[r] & N_{j-1} \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$$\mathrm{Ext}^\bullet(A,A) \times \mathrm{Ext}^\bullet(A,A) \to \mathrm{Ext}^\bullet(A,A)$$

$$\mathrm{Ext}^\bullet(B,B) \circlearrowright \mathrm{Ext}^\bullet(A,B) \circlearrowleft \mathrm{Ext}^\bullet(A,A)$$

Cohomology ring of the unit

There are two ring homomorphisms

$$\mathrm{Ext}^\bullet(\mathbf{1},\mathbf{1}) \to \mathrm{Ext}^\bullet(A,A)$$

defined by (recalling that $-\otimes-$ is biexact)

$$0 \to \mathbf{1} \to M_0 \to \dots \to M_{i-1} \to \mathbf{1} \to 0$$

$$\mapsto$$

$$0 \to \mathbf{1}\otimes A \to M_0 \otimes A \to \dots \to M_{i-1}\otimes A \to \mathbf{1}\otimes A \to 0,$$

and

$$0 \to A\otimes \mathbf{1} \to A\otimes M_0 \to \dots \to A\otimes M_{i-1} \to A\otimes \mathbf{1} \to 0.$$

Fix one for the sake of defining support varieties. Let's use $-\otimes A$.

Cohomological support

Let $I(A)$ denote the annihilator of $\mathrm{Ext}^\bullet(A, A)$ in $\mathrm{Ext}^\bullet(\mathbf{1}, \mathbf{1})$. Then the *cohomological support variety of* A is defined as

$$W(A) := \{\mathfrak{p} \in \operatorname{Proj} \mathrm{Ext}^\bullet(\mathbf{1}, \mathbf{1}) : I(A) \subseteq \mathfrak{p}\}.$$

Originally comes out of modular representation theory for finite groups, much of the original theory due to Quillen, Dade, Carlson, Avrunin-Scott, Alperin-Evens, ...

Thick ideals

- Let $\mathbf{K}$ be a monoidal triangulated category (e.g. $\underline{\mathbf{T}}$).
- A *thick ideal* of $\mathbf{K}$ is a subcategory $\mathbf{I}$ such that
  - $A \in \mathbf{I}$ if and only if $\Sigma A \in \mathbf{I}$.
  - If $A \to B \to C \to \Sigma A$ is a distinguished triangle with A and $B \in \mathbf{I}$, then $C \in \mathbf{I}$.
  - If $A \oplus B \in \mathbf{I}$ then A and B are in $\mathbf{I}$.
  - If $A \in \mathbf{I}$, then so are $A \otimes B$ and $B \otimes A$.

(Thick ideal $=$ kernel of an exact monoidal functor between triangulated categories).

The Balmer spectrum

Developed by Balmer (2005) in the commutative case. We use natural noncommutative analogues of Balmer's definitions:

- *The Balmer spectrum* $\text{Spc}(\mathbf{K})$ of a monoidal triangulated cateogry $\mathbf{K}$ is defined as the collection of prime ideals of $\underline{\mathbf{T}}$: thick ideals $\mathbf{P}$ such that if $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$, then $\mathbf{I}$ or $\mathbf{J}$ is contained in $\mathbf{P}$, over all thick ideals $\mathbf{I}$ and $\mathbf{J}$.
- Equivalently, these are thick ideals $\mathbf{P}$ such that $A \otimes \mathbf{K} \otimes B \subseteq \mathbf{P}$, then A or $B \in \mathbf{P}$.
- $\mathbf{K}$ braided, then prime = completely prime := $\{A \otimes B \in \mathbf{P} \Rightarrow A \text{ or } B \in \mathbf{P}\}$.

Philosophy

Two guiding philosophies:

- Treat a monoidal triangulated category like a ring.
- Categorical properties of $\mathbf{K}$ should reflect topological properties in $\operatorname{Spc} \mathbf{K}$.

The Balmer support

- The topology on $\operatorname{Spc}\mathbf{K}$ is generated by the *Balmer supports of objects*:

$$V(A) := \{\mathbf{P} \in \operatorname{Spc}\mathbf{K} : A \notin \mathbf{P}\}.$$

- E.g. a general closed set looks like

$$V(\mathcal{S}) := \{\mathbf{P} \in \operatorname{Spc}\mathbf{K} : \mathcal{S} \cap \mathbf{P} = \varnothing\}$$

for some collection of objects $\mathcal{S}$.

- We have a related map $\Phi_V(\mathcal{S}) = \bigcup_{A \in \mathcal{S}} V(A)$. This gives a specialization-closed subset of $\operatorname{Spc}\mathbf{K}$.

Classification of thick ideals

In many examples, determining the Balmer spectrum of $\mathbf{K}$ = classifying thick ideals of $\mathbf{K}$, that is, the thick ideals are in bijection with Thomason-closed subsets of $\operatorname{Spc}\mathbf{K}$:

1. When $\mathbf{K}$ is symmetric (Balmer, 2005);
2. When all prime ideals of $\mathbf{K}$ are completely prime (Mallick–Ray, 2023);
3. When $\operatorname{Spc}\mathbf{K}$ is Noetherian (Rowe, 2023);
4. When $\mathbf{K}$ has a thick generator (e.g. $\mathbf{K} = \underline{\mathbf{T}}$) (Nakano–V.–Yakimov, 2023).

Some examples where the thick ideals are NOT in bijection with Thomason-closed subsets of $\operatorname{Spc}\mathbf{K}$ have also been produced (Huang-V., 2023). If we weren't assuming rigidity, only classifies semiprime (in symmetric case, radical) thick ideals.

Classification of thick ideals

To be more specific, the bijection between thick ideals of $\mathbf{K}$ and Thomason-closed sets in $\operatorname{Spc}\mathbf{K}$ is given by

$$\mathbf{I} \longmapsto \bigcup_{A \in \mathbf{I}} V(A)$$

Ideals of $\mathbf{K}$ — Th.-closed subsets of $\operatorname{Spc}\mathbf{K}$

$$\{A \in \mathbf{K} : V(A) \subseteq S\} \longleftarrow S$$

This descends to a bijection between closed subsets and principal ideals, that is, every closed subset can be given as $V(A)$ for some object A.

Example: derived categories of schemes

- Let X be a topologically Noetherian scheme.
- $D^{\text{perf}}(X)$ is the perfect derived category of coherent sheaves on X (perfect means quasi-isomorphic to a bounded complex of projective coherent sheaves).
- Then $D^{\text{perf}}(X)$ is a (commutative) monoidal triangulated category under derived tensor product, and it is a theorem of Thomason and Balmer that

$$\operatorname{Spc}(D^{\text{perf}}(X)) \cong X.$$

- In other words, X can be reconstructed from its derived category, *if you remember the tensor product.* As a triangulated category, though, $D^{\text{perf}}(X)$ *doesn't* remember X.

Example: finite group (schemes)

$$\begin{tikzcd}
 & \underline{\mathrm{mod}}(\Bbbk G) \arrow[r, "\text{Balmer support}"] \arrow[dl, "\text{cohomological support}"'] \arrow[d, "\text{rank / }\pi\text{ support}"] & \operatorname{Spc} \underline{\mathrm{mod}}(\Bbbk G) \\
\operatorname{Proj} \operatorname{Ext}^\bullet(\Bbbk, \Bbbk) & \Pi(G) &
\end{tikzcd}$$

Example: finite group (schemes)

$$\begin{tikzcd} & \underline{\mathrm{mod}}(\Bbbk G) \arrow[dl] \arrow[d] \arrow[r] & \operatorname{Spc} \underline{\mathrm{mod}}(\Bbbk G) \\ \operatorname{Proj} \operatorname{Ext}^\bullet(\Bbbk, \Bbbk) \arrow[r, "\cong"] & \Pi(G) & \end{tikzcd}$$

- Due to Avrunin–Scott, Friedlander-Pevtsova

Example: finite group (schemes)

$$\begin{tikzcd}
 & \underline{\mathrm{mod}}(\mathbb{k}G) \arrow[r] \arrow[dl] \arrow[d] & \operatorname{Spc}\underline{\mathrm{mod}}(\mathbb{k}G) \\
\operatorname{Proj}\operatorname{Ext}^{\bullet}(\mathbb{k},\mathbb{k}) \arrow[r, "\cong"] & \Pi(G) \arrow[ur, "\cong"] &
\end{tikzcd}$$

- Due to Avrunin–Scott, Friedlander-Pevtsova
- and Benson–Carlson–Rickard, Friedlander–Pevtsova, Balmer.

Example: finite group (schemes)

Why is this such a fantastic theorem?

- Many examples (e.g. $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ in char. 3) cannot classify the indecomposable objects.
- But can try to classify indecomposables up to a notion of equivalence: say A and B are equivalent if you can generate one from the other by taking kernels, cokernels, extensions, direct summands, and tensor product with arbitrary modules. This is the same as classifying thick ideals in the stable category.
- The Benson–Carlson–Rickard, Friedlander–Pevtsova theorems say you can classify modules up to this notion of equivalence.
- Gives computational tool: given two indecomposables A and B, you can generate B from A via these processes if $W(B) \subseteq W(A)$.

The categorical center of the cohomology ring

When the category $\mathbf{T}$ is not braided, there are known cases where cohomological and Balmer support do not coincide.

Definition (Nakano-V.-Yakimov)

The *categorical center* $C^\bullet$ of the cohomology ring of $\underline{\mathbf{T}}$ is the subalgebra generated by all homogenous $g \in \mathrm{Ext}^\bullet(\mathbf{1}, \mathbf{1}) = \bigoplus_{i \geq 0} \mathrm{Hom}(\mathbf{1}, \Sigma^i \mathbf{1}) := R^\bullet_{\underline{\mathbf{T}}}$ such that the diagram

$$\begin{tikzcd}
\mathbf{1} \otimes M \arrow[r, "\cong"] \arrow[d, "g \otimes \mathrm{id}_M"] & M & M \otimes \mathbf{1} \arrow[l, "\cong"'] \arrow[d, "\mathrm{id}_M \otimes g"] \\
\Sigma^i \mathbf{1} \otimes M \arrow[r, "\cong"] & \Sigma^i M & M \otimes \Sigma^i \mathbf{1} \arrow[l, "\cong"']
\end{tikzcd}$$

commutes for all simple objects M.

The central cohomological support

Define

$$W_C(A) := \{\mathfrak{p} \in \operatorname{Proj} C^\bullet : \operatorname{Ann}(\operatorname{Ext}^\bullet(A, A)) \subseteq \mathfrak{p}\}.$$

Conjecture (Nakano-V.-Yakimov)

For any finite tensor category $\mathbf{T}$*, the conditions of the previous theorem are satisfied. In particular, central cohomological support classifies the thick ideals of* $\underline{\mathbf{T}}$*, and* $\operatorname{Proj} C^\bullet \cong \operatorname{Spc} \underline{\mathbf{T}}$.

Examples where the conjecture is known

1. Finite groups: Benson–Carlson–Rickard (1997);
2. Finite group schemes (finite-dimensional cocommutative Hopf algebras): Friedlander–Pevtsova (2007);
3. Finite-dimensional commutative Hopf algebras: Negron–Pevtsova (2022);
4. Benson–Witherspoon and Plavnik–Witherspoon smash coproduct Hopf algebras (which generalize (3)): Nakano–V.–Yakimov (2022);
5. Drinfeld centers of (4): V. (2023);
6. General crossed product categories (generalization of (4)): Huang–V. (2023).

Obstacles

There are three major obstacles to proving Conjecture 1.

- Etingof–Ostrik finite generation conjecture (this allows us to conclude for instance that each $\mathrm{Ext}^\bullet(A, A)$ is finitely-generated over $C^\bullet$, using a theorem of Negron–Plavnik);
- The tensor product property: we need to know that the central support W_C satisfies

$$\bigcup_B W_C(A \otimes B \otimes C) = W_C(A) \cap W_C(C);$$

- Extension: we need to know that there is a well-behaved extension of the support theory W_C to the category of *all* (i.e., infinite-dimensional) modules. (Will say a little more about this context later).

A surjective map

Under a weaker finite-generation condition, though, we can at least get a surjective map in one direction:

Theorem (Nakano–V.–Yakimov)

Suppose each $\operatorname{Ext}^\bullet(A, A)$ *is finitely-generated as a* $C^\bullet$*-module. Then there exists a surjective map*

$$\begin{aligned} \operatorname{Spc} \underline{\mathbf{T}} &\xrightarrow{\rho} \operatorname{Proj} C^\bullet \\ \mathbf{P} &\mapsto \langle g \in C^\bullet : \operatorname{cone}(g) \notin \mathbf{P} \rangle. \end{aligned}$$

The Carlson Connectedness Theorem

Theorem (Carlson (1984), Bergh–Plavnik–Witherspoon (2021))

Suppose that $\mathbf{T}$ *satisfies the finite-generation conjecture. If* A *is an indecomposable object in* $\mathbf{T}$*, then* $W(A)$ *(cohomological support) is connected.*

Original result due to Carlson in the finite group case (here know $\mathbf{T}$ satisfies finite-generation by work of Golod, Venkov, Evens), and generalized to finite tensor categories by Bergh–Plavnik–Witherspoon. There is also a version in triangulated categories due to Buan–Krause–Snashall–Solberg, also for a version of cohomological support, but requiring a number of technical conditions.

Summary

- We have conjectured that for finite tensor categories, Balmer support and (a version of) cohomological support coincide;
- This conjecture is hard to prove, although under weaker conditions we have a surjective map;
- Cohomological support of an indecomposable module is connected (under finite-generation).

A generalized Carlson theorem

Theorem (Nakano–V.–Yakimov)

Let $\mathbf{K}$ *be a monoidal triangulated category generated (as a thick subcategory) by a single object, which is the compact part of a compactly generated monoidal triangulated category (e.g.* $\underline{\mathbf{T}}$*, this setting explained below). If* A *is an indecomposable object in* $\mathbf{K}$*, then* $V(A)$ *(Balmer support) is connected.*

No requirements on finite generation or tensor product property.
Remainder of the talk: how to prove (a generalized Chinese remainder theorem)

Compactly generated triangulated categories

An object in a triangulated category is called *compact* when $\mathrm{Hom}(A, -)$ commutes with coproducts. Say $\widehat{\mathbf{K}}$ is *compactly generated* if:

- There exist arbitrary set-indexed coproducts.
- The compact objects generate the category by taking set-indexed coproducts and closing as a triangulated category.

E.g.: $\underline{\mathrm{mod}}(H)$ is the compact part of the compactly generated $\underline{\mathrm{Mod}}(H)$, for a finite-dimensional Hopf algebra H. More generally, if $\mathbf{T}$ is a finite tensor category, then $\underline{\mathbf{T}}$ is the compact part of the compactly generated category $\underline{\mathrm{Ind}(\mathbf{T})}$.

Compactly generated triangulated categories

What's the payoff for having a compactly generated triangulated category $\widehat{\mathbf{K}}$?

- $\mathrm{Loc}(\mathbf{I})$ denotes the localizing category generated by $\mathbf{I}$: close under set-indexed coproducts, shifts, and cones.
- Brown Representability (triangulated version due to Neeman):

$$\mathrm{Loc}(\mathbf{I}) \underset{i^!}{\overset{i_*}{\rightleftarrows}} \widehat{\mathbf{K}} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \widehat{\mathbf{K}}/\,\mathrm{Loc}(\mathbf{I}).$$

Compactly generated triangulated categories

- Obtain two functors $\widehat{\mathbf{K}} \to \widehat{\mathbf{K}}$ via $L_{\mathbf{I}} = j_* j^*$ and $\Gamma_{\mathbf{I}} = i_* i^!$.
- Note: $\Gamma_{\mathbf{I}}$ takes values in $\mathrm{Loc}(\mathbf{I})$.
- $L_{\mathbf{I}}$ takes values in $\mathrm{Loc}(\mathbf{I})^{\perp}$ (objects which admit no maps from objects in $\mathrm{Loc}(\mathbf{I})$).
- For any object A, there is a distinguished triangle

$$\Gamma_{\mathbf{I}} A \to A \to L_{\mathbf{I}} A \to \Sigma \Gamma_{\mathbf{I}} A.$$

The Chinese Remainder Theorem

Theorem

Let $I_1, \ldots, I_n$ be a collection of pairwise coprime 2-sided ideals of a ring R. Then there is an isomorphism

$$R/I \xrightarrow{\cong} (R/I_1) \times \ldots \times (R/I_n).$$

Here coprime means that $\langle I_i, I_j \rangle = R$ for $i \neq j$.

The Chinese Remainder Theorem for monoidal triangulated categories

Theorem (Nakano–V.–Yakimov)

Let $\mathbf{K}$ be a small $M\Delta C$ which is the compact part of an associated big $M\Delta C$, $\widehat{\mathbf{K}}$, and $\mathbf{I}_1, \ldots, \mathbf{I}_n$ be a collection of thick ideals of $\mathbf{K}$ that are pairwise coprime. We have the equivalences of monoidal triangulated categories:

$$\widehat{\mathbf{K}}/\operatorname{Loc}(\mathbf{I}_1 \cap \ldots \cap \mathbf{I}_n) \cong \widehat{\mathbf{K}}/\operatorname{Loc}(\mathbf{I}_1) \oplus \ldots \oplus \widehat{\mathbf{K}}/\operatorname{Loc}(\mathbf{I}_n) \quad \text{and}$$
$$(\mathbf{K}/(\mathbf{I}_1 \cap \ldots \cap \mathbf{I}_n))^\natural \cong (\mathbf{K}/\mathbf{I}_1)^\natural \oplus \ldots \oplus (\mathbf{K}/\mathbf{I}_n)^\natural,$$

where Loc *refers to localizing ideal and* $()^\natural$ *to idempotent completion.*

Here coprime means that $\langle \mathbf{I}_i, \mathbf{I}_j \rangle = \mathbf{K}$ for $i \neq j$.

Proof sketch

Sketch of proof of Chinese remainder theorem:

- If $\mathbf{I}$ and $\mathbf{J}$ are coprime, then $L_{\mathbf{I}}\mathbf{1} \otimes L_{\mathbf{J}}\mathbf{1} \cong 0 \Rightarrow L_{\mathbf{J}}A \cong \Gamma_{\mathbf{I}}A$ for all $A \in \mathbf{K}$.
- It follows that the standard Rickard triangle

$$\Gamma_{\mathbf{I}}A \to A \to L_{\mathbf{I}}A \to \Sigma\Gamma_{\mathbf{I}}A$$

  splits for each A. Note $\Gamma_{\mathbf{I}}A \in \mathrm{Loc}(\mathbf{I})$ and $L_{\mathbf{I}}A \in \mathrm{Loc}(\mathbf{J})$.
- If $\mathbf{I}_1, \mathbf{I}_2$, and $\mathbf{I}_3$ are all pairwise coprime, then $\mathbf{I}_1 \cap \mathbf{I}_2$ is coprime to $\mathbf{I}_3$, using the classic lemma

$$\langle \mathcal{M} \rangle \otimes \langle \mathcal{N} \rangle \subseteq \langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle.$$

- If $\mathbf{I}$ is a thick ideal of $\mathbf{K}$, then $\widehat{\mathbf{K}}/\,\mathrm{Loc}(\mathbf{I}) \cong (\mathbf{K}/\mathbf{I})^{\natural}$.

Proof sketch

Sketch of proof of the Carlson connectedness theorem, given the Chinese remainder theorem:

1. Two ideals $\mathbf{I}$ and $\mathbf{J}$ are coprime $\Leftrightarrow$ the union of the supports of $\mathbf{I}$ and $\mathbf{J}$ is all of $\operatorname{Spc}\mathbf{K}$;
2. The intersection of $\mathbf{I}$ and $\mathbf{J}$ is empty $\Leftrightarrow$ the supports of $\mathbf{I}$ and $\mathbf{J}$ are disjoint.
3. If $V(A) = V(B) \sqcup V(C)$, then decomposition result says: can write $A \cong L_{\mathbf{I}}B \oplus L_{\mathbf{J}}C$ in $\widehat{\mathbf{K}}$.

Recovering previous results

- Recovers Carlson's original theorem (since Balmer support corresponds to cohomological support, by Benson–Carlson–Rickard).
- Recovers the central cohomological analogue of Bergh–Plavnik–Witherspoon generalization to finite tensor categories when the following conditions are satisfied:
  - W_C satisfies the generalized tensor product property;
  - Each $\mathrm{Ext}^\bullet(A, A)$ is finitely-generated over $C^\bullet$.
- Note: these conditions are weaker than those necessary to guarantee that Balmer support is exactly the central cohomological support.

Special case

- Special case: $\operatorname{Spc} \mathbf{K} = Z_1 \sqcup Z_2$ for some idempotent-complete monoidal triangulated category $\mathbf{K}$ which is the compact part of some $\widehat{\mathbf{K}}$.
- Then there exist objects A and B with $V(A) = Z_1$, $V(B) = Z_2$, and we have that $\mathbf{I} = \langle A \rangle$ and $\mathbf{J} = \langle B \rangle$ are coprime complementary ideals.
- By the Chinese Remainder Theorem, we have

$$\mathbf{K} = (\mathbf{K}/\mathbf{I})^{\natural} \oplus (\mathbf{K}/\mathbf{J})^{\natural}.$$

- And $\operatorname{Spc}(\mathbf{K}/\mathbf{I})^{\natural} = \operatorname{Spc} \mathbf{K}/\mathbf{I} = Z_1$, likewise $\operatorname{Spc}(\mathbf{K}/\mathbf{J})^{\natural} = Z_2$.

Examples

- Let X a scheme, $\mathbf{K} = D^{\mathrm{perf}}(X)$, $\widehat{\mathbf{K}} = D(X)$.
- Suppose X is disconnected, $X = X_1 \sqcup X_2$.
- Then

$$\begin{aligned} D(X) &\cong D(X)/\operatorname{Loc}(\mathbf{I}_1 \cap \mathbf{I}_2) \\ &\cong D(X)/\operatorname{Loc}(\mathbf{I}_1) \oplus D(X)/\operatorname{Loc}(\mathbf{I}_2) \\ &\cong D(X_2) \oplus D(X_1). \end{aligned}$$

Examples

- $\mathbf{K} = \underline{\mathrm{mod}}(\Bbbk G)$ for G a finite group and $\Bbbk$ a field of characteristic dividing the order of G, and $\widehat{\mathbf{K}} = \underline{\mathrm{Mod}}(\Bbbk G)$.
- Denote the irreducible components of $\operatorname{Spc}\mathbf{K} = \operatorname{Proj}\mathrm{H}^{\bullet}(G, \Bbbk)$ by $Z_1, ..., Z_m$, and set

$$\mathbf{I} := \{A \in \mathbf{K} : \dim V(A) < r\}$$

and

$$\mathbf{J} := \{A \in \mathbf{K} : Z_i \not\subseteq V(A) \ \ \forall \ \ i\}.$$

- Work of Carlson, Carlson–Donovan–Wheeler, Benson, and Benson–Krause described the categories $(\mathbf{K}/\mathbf{I})^{\natural}, (\mathbf{K}/\mathbf{J})^{\natural}$ in the late 90s and early 2000s as direct sums.
- But note that both $\mathbf{I}$ and $\mathbf{J}$ can be written as intersections of maximal ideals. Since maximal ideals are pairwise coprime, we can use the Chinese remainder theorem to recover these decompositions.

Conclusion

Thank you for your time!