2024 Symmetric Tensor Categories and Representation Theory
IPAM

Growth in tensor powers

Victor Ostrik

University of Oregon

vostrik@uoregon.edu

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arxiv: 2107.02372, 2301.00885, 2301.09804
(jt with Kevin Coulembier, Pavel Etingof)

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Warning: counterexamples for comodules over Hopf algebras

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• Open True/False question: is $\gamma(V) = \gamma'(V)$ for all V?

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Exercise. Compute $\gamma(J_2)$ and $\gamma(J_4)$ (of course $\gamma(J_1) = 1$ and $\gamma(J_5) = 0$)

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More generally, (possibly decomposable) W is negligible if every indecomposable summand is negligible

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Other counts: non-negligible summands

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Question: is $\delta(V) = \delta'(V)$ for any V?

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This is $\mathbb{Z}/2\mathbb{Z}-$equivariantization of $\mathrm{Vec}_{\mathbb{Z}/3\mathbb{Z}}$ = semisimple reduction of $\mathrm{Rep}(S_3)$ to characteristic 3.

Comments on proof, 3

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Example

3. $\mathcal{C} = \mathrm{Rep}(\mathbb{Z}/p\mathbb{Z})$.
Simple objects of $\bar{\mathcal{C}}$: $\bar{J}_1 = \mathbb{1} =: L_1, \bar{J}_2 =: L_2, \ldots, \bar{J}_{p-1} =: L_{p-1}$

$$L_m \otimes L_n = \bigoplus_{i=1}^{\min(m,n,p-m,p-n)} L_{|m-n|+2i-1}$$

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By *Nagata's theorem* S is an extension of finite group $\pi_0(S)$ by a diagonalizable group. Thus there is a uniform bound $\mathrm{FPdim}(F(L)) = \dim(F(L)) \leq |\pi_0(S)|$ for any simple object $L \in \mathcal{D}$.

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Corollary: $\delta(V)$ is finitely computable (finitely many $d_n(V)$ are required)

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Step 5: The subgroup $N_1 \cdot D \subset S$ is abelian of index $\leq J(d)3^{d/3}$. Best possible bound for $J(d)$ in the literature: $J(d) \leq 3^{d-1}$ (G. Robinson).

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Step 5: The subgroup $N_1 \cdot D \subset S$ is abelian of index $\leq J(d)3^{d/3}$. Best possible bound for $J(d)$ in the literature: $J(d) \leq 3^{d-1}$ (G. Robinson).

$$\boxed{a(d) = 3^{4/3d-1}}$$

Benson's conjecture

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Assume $p = 2$ and Γ is a finite 2-group.

Conjecture (D. Benson): Any object of $\overline{\mathrm{Rep}(\Gamma)}$ is invertible.

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Assume $p = 2$ and Γ is a finite 2-group.

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True when Γ is cyclic or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Thanks for listening!