Polynomial superfunctors

with applications to and from finite supergroup schemes

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IPAM Workshop on Symmetric Tensor Categories and Representation Theory; January 8–12, 2024

Outline of talk

1. Motivation: cohomology of finite supergroup schemes
2. Strict polynomial superfunctors
3. Applications to and from finite supergroup schemes
4. Other directions
5. Superized Troesch complexes

Some references

1. C. M. Drupieski, *Cohomological finite-generation for finite supergroup schemes*, Adv. Math. **288** (2016), 1360–1432.

2. C. M. Drupieski and J. R. Kujawa, *Superized Troesch complexes and cohomology for strict polynomial superfunctors*, J. Pure Appl. Algebra **226** (2022), no. 12, Paper No. 107136, 44pp.

3. C. M. Drupieski and J. R. Kujawa, *A survey of support theories for Lie superalgebras and finite supergroup schemes*, to appear in Contemporary Mathematics. arXiv:2208.01496

Motivation: cohomology of finite supergroup schemes

Motivating question

CFG Question

Let G be a finite supergroup scheme over a field k of characteristic $p > 2$. Is the cohomology ring

$$\mathrm{H}^{\bullet}(G, k) = \mathrm{Ext}_G^{\bullet}(k, k)$$

finitely-generated as a k-superalgebra?

Reducing the problem

Lemma

Let G be a finite supergroup scheme over k. Then

$$G \cong G^0 \rtimes \pi_0(G)$$

where G^0 is infinitesimal and $\pi_0(G)$ is an etale group scheme.

Up to field extension, $\pi_0(G)$ corresponds to an ordinary finite group.

G^0 is filtered by infinitesimal Frobenius kernels $G_1 \subset G_2 \subset \cdots \subset G^0$.

$\mathrm{Rep}(G_1) \equiv \mathrm{Rep}(V(\mathfrak{g}))$, where $V(\mathfrak{g})$ is the restricted enveloping algebra of the restricted Lie superalgebra $\mathfrak{g} = \mathrm{Lie}(G)$. So let's start here...

Toy example

First consider an ordinary (non-super) f.d. restricted Lie algebra $\mathfrak{g}$.

$V(\mathfrak{g})$ is filtered by monomial length, with associated graded algebra

$$\mathrm{gr}(V(\mathfrak{g})) \cong V(\mathfrak{g}_{ab}) \cong k[x_1, \ldots, x_m]/(x_1^p, \ldots, x_m^p).$$

Filtration gives rise to spectral sequence converging to $\mathrm{H}^\bullet(V(\mathfrak{g}), k)$.

$$E_0 = \mathrm{H}^\bullet(V(\mathfrak{g}_{ab}), k) \cong S(\mathfrak{g}^*)^{(1)} \otimes \Lambda(\mathfrak{g}^*),$$
$$E_2^{2i,j} \cong S^i(\mathfrak{g}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{g}, k), \quad E_2^{2i+1,j} = 0.$$

E_2 is finite over the subalgebra of permanent cycles $E_2^{\bullet,0} \cong S(\mathfrak{g}^*)^{(1)}$.
Then $\mathrm{H}^\bullet(V(\mathfrak{g}), k)$ is finite over a map of graded algebras

$$\Phi : S(\mathfrak{g}^*[2])^{(1)} \to \mathrm{H}^\bullet(V(\mathfrak{g}), k).$$

Complications arising from super phenomena

Now let $\mathfrak{g}$ be a finite-dimensional restricted Lie superalgebra.

$V(\mathfrak{g})$ is filtered by monomial length, with associated graded algebra

$$\mathrm{gr}(V(\mathfrak{g})) \cong V(\mathfrak{g}_{ab}) \cong k[x_1, \dots, x_m]/(x_1^p, \dots, x_m^p) \otimes \Lambda(y_1, \dots, y_n).$$

Filtration gives rise to a spectral sequence converting to $\mathrm{H}^\bullet(V(\mathfrak{g}), k)$.

$$E_0 = \mathrm{H}^\bullet(V(\mathfrak{g}_{ab}), k) \cong S(\mathfrak{g}_{\bar{0}}^*)^{(1)} \otimes \boldsymbol{\Lambda}(\mathfrak{g}^*),$$
$$E_2^{2i,j} \cong S^i(\mathfrak{g}_{\bar{0}}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{g}, k), \quad E_2^{2i+1,j} = 0.$$

Here $\boldsymbol{\Lambda}(\mathfrak{g}^*) \cong \Lambda(\mathfrak{g}_{\bar{0}}^*) \otimes S(\mathfrak{g}_{\bar{1}}^*)$ is the superexterior algebra on $\mathfrak{g}^*$.

E_2 is finite over a map of graded superalgebras

$$S(\mathfrak{g}_{\bar{0}}^*[2])^{(1)} \otimes S(\mathfrak{g}_{\bar{1}}^*[p])^{(1)} \to E_2,$$

but it's not clear at the outset if the image of this map consists of permanent cycles, hence if $\mathrm{H}^\bullet(V(\mathfrak{g}), k)$ is finitely-generated.

Problem

Given an arbitrary infinitesimal supergroup G, need to cook up a nice subalgebra of cohomology classes over which $\mathrm{H}^\bullet(G, k)$ finite.

Enter strict polynomial functors

Classical polynomial functors

A classical polynomial functor is a functor on the category of (f.d.) vector spaces that acts on morphisms via a polynomial map.

Example: the second symmetric power

Suppose V has basis $\{u, v\}$ and W has basis $\{x, y\}$.

Then $S^2(V)$ has basis $\{u^2, uv, v^2\}$ and $S^2(W)$ has basis $\{x^2, xy, y^2\}$.

Let $\phi : V \to W$ be a linear map with associated matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

The associated matrix for $S^2(\phi)$ is then

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & (ad + cb) & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

Classical polynomial functors

Example: Frobenius twist of a vector space

Let $k = \mathbb{F}_2$, and let $I^{(1)} : V \to V^{(1)}$ be the Frobenius twist functor.

Identify $V^{(1)}$ with the subspace of $S^2(V)$ spanned by $\{u^2, v^2\}$.

Then $\phi : V \to W$ induces the map $\phi^{(1)} : V^{(1)} \to W^{(1)}$ with matrix

$$\phi^{(1)} = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then $I^{(1)} : V \mapsto V^{(1)}$ is isomorphic to the identity functor!

But do we want it to be...?

“Strict” polynomial functors address this ambiguity by requiring that fixed polynomials defining the action on morphisms be built into the definition of the functor.

The category $\Gamma^d\mathcal{V}$

Let $\mathcal{V}$ be the category of finite-dimensional k-vector superspaces.

Given $V \in \mathcal{V}$, the symmetric group Σ_d acts on $V^{\otimes d}$ by signed place permutations.

$$V \otimes W \cong W \otimes V, \quad v \otimes w \mapsto (-1)^{\overline{v}\cdot\overline{w}} w \otimes v$$

Given $V \in \mathcal{V}$, set $\Gamma^d(V) = (V^{\otimes d})^{\Sigma_d}$.

The category $\Gamma^d\mathcal{V}$

Let $\Gamma^d\mathcal{V}$ be the category whose objects are the same as those in $\mathcal{V}$, but in which spaces of morphisms are defined by

$$\mathrm{Hom}_{\Gamma^d\mathcal{V}}(V, W) = \Gamma^d \,\mathrm{Hom}_k(V, W) \cong \mathrm{Hom}_{k\Sigma_d}(V^{\otimes d}, W^{\otimes d}),$$

and composition is that of $k\Sigma_d$-module homomorphisms.

Strict polynomial superfunctors, after Pirashvili

The category $\mathcal{P}_d$ (Axtell 2013)

The category $\boldsymbol{\mathcal{P}}_d$ of degree-d homogeneous strict polynomial superfunctors is the category of even linear functors

$$F : \boldsymbol{\Gamma}^d \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}},$$

i.e., functors such that for all $V, W \in \boldsymbol{\mathcal{V}}$, the function

$$F_{V,W} : \boldsymbol{\Gamma}^d \operatorname{Hom}_k(V, W) = (\operatorname{Hom}_k(V, W)^{\otimes d})^{\Sigma_d} \to \operatorname{Hom}_k(F(V), F(W))$$

is an even linear map.

If $\phi : V \to W$ is an even linear map, then $\phi^{\otimes d} \in \boldsymbol{\Gamma}^d \operatorname{Hom}_k(V, W)$. Thus, each $F \in \boldsymbol{\mathcal{P}}_d$ restricts to an ordinary functor $F : \boldsymbol{\mathcal{V}}_{\mathrm{ev}} \to \boldsymbol{\mathcal{V}}_{\mathrm{ev}}$ on the underlying even subcategory of $\boldsymbol{\mathcal{V}}$, with action on morphisms defined by $\phi \mapsto F(\phi^{\otimes d})$.

Connection to polynomial maps

For $V \in \mathcal{V}$, the supersymmetric power $S^d(V)$ is defined by

$$S^d(V) = (V^{\otimes d})_{\Sigma_d} \quad \text{(coinvariants)}.$$

Let $A = \mathrm{Hom}_k(V, W)$ and $B = \mathrm{Hom}_k(F(V), F(W))$.

Then $F_{V,W}$ is required to be an even element in

$$\begin{aligned}
\mathrm{Hom}_k([A^{\otimes d}]^{\Sigma_d}, B) &\cong B \otimes ([A^{\otimes d}]^{\Sigma_d})^* \\
&\cong B \otimes ([A^{\otimes d}]^*)_{\Sigma_d} \\
&\cong B \otimes ([(A^*)^{\otimes d}])_{\Sigma_d} \\
&\cong B \otimes S^d(A^*) \\
&= \mathrm{Pol}_d(A, B)
\end{aligned}$$

Examples of strict polynomial superfunctors

- $\mathbf{\Pi} \in \mathcal{P}_1$ — parity change functor
- $I^{(r)} \in \mathcal{P}_{p^r}$ for $r \geq 1$ — Frobenius twist functor
- $\mathbf{\Gamma}^d : V \mapsto (V^{\otimes d})^{\Sigma_d}$ — $\mathbf{\Gamma}(V) \cong \Gamma(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$
- $\mathbf{A}^d : V \mapsto [\mathrm{sgn} \otimes (V^{\otimes d})]^{\Sigma_d}$ — $\mathbf{A}(V) \cong \Lambda(V_{\overline{0}}) \otimes \Gamma(V_{\overline{1}})$
- $\mathbf{S}^d : V \mapsto (V^{\otimes d})_{\Sigma_d}$ — $\mathbf{S}(V) \cong S(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$
- $\mathbf{\Lambda}^d : V \mapsto [\mathrm{sgn} \otimes (V^{\otimes d})]_{\Sigma_d}$ — $\mathbf{\Lambda}(V) \cong \Lambda(V_{\overline{0}}) \otimes S(V_{\overline{1}})$
- $\mathbf{A}^d \simeq \mathbf{\Gamma}^d(\mathrm{Hom}_k(k^{0|1}, -))$ — isomorphism of superdegree $\overline{d}$
- $\mathbf{\Lambda}^d \simeq \mathbf{S}^d(k^{0|1} \otimes -)$ — isomorphism of superdegree $\overline{d}$

Duality $F \mapsto F^{\#}$ on $\mathcal{P}_d$ defined by $F^{\#}(V) = F(V^*)^*$.

- $(\mathbf{\Gamma}^d)^{\#} \cong \mathbf{S}^d$
- $(\mathbf{A}^d)^{\#} \cong \mathbf{\Lambda}^d$

(Non)examples of strict polynomial superfunctors

Non-examples

- $I_0 : V \mapsto V_{\bar{0}}$ projection onto even subspace
- $I_1 : V \mapsto V_{\bar{1}}$ projection onto odd subspace

are not natural with respect to odd linear maps:

$$\begin{tikzcd} k^{1|0} \arrow[r, "\simeq"] \arrow[d, "I_0"] & k^{0|1} \arrow[r, "\simeq"] \arrow[d, "I_0"] & k^{1|0} \arrow[d, "I_0"] \\ k^{1|0} \arrow[r] & 0 \arrow[r] & k^{1|0} \end{tikzcd}$$

However, for $r \geq 1$ the Frobenius twist functor does decompose:

$$I^{(r)} = I_0^{(r)} \oplus I_1^{(r)} \quad \text{where} \quad I_0^{(r)}(V) = V_{\bar{0}}^{(r)} \quad \text{and} \quad I_1^{(r)}(V) = V_{\bar{1}}^{(r)}.$$

Power maps induce embeddings $I_0^{(r)} \hookrightarrow \boldsymbol{S}^{p^r}$ and $I_1^{(r)} \hookrightarrow \boldsymbol{\Lambda}^{p^r}$.

Projective and injective objects

Yoneda Lemma

For all $W \in \boldsymbol{\mathcal{V}}$ and all $F \in \boldsymbol{\mathcal{P}}_d$, there are natural isomorphisms

$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(\boldsymbol{\Gamma}^d(\mathrm{Hom}_k(W,-)), F) \cong F(W),$$
$$\mathrm{Hom}_{\boldsymbol{\mathcal{P}}}(F, \boldsymbol{S}^d(W \otimes -)) \cong F^{\#}(W).$$

Consequence

Functors of the form

- $\boldsymbol{\Gamma}^d(\mathrm{Hom}_k(W,-))$ and $\boldsymbol{A}^d(\mathrm{Hom}_k(W,-))$ are projective objects
- $\boldsymbol{S}^d(W \otimes -)$ and $\boldsymbol{\Lambda}^d(W \otimes -)$ are injective objects

Connection to Schur superalgebras

Schur superalgebra

Let $V = k^{m|n}$. The Schur superalgebra $S(m|n,d)$ can be defined by

$$S(m|n,d) = \mathrm{Hom}_{k\Sigma_d}(V^{\otimes d}, V^{\otimes d}) = \mathrm{End}_{\boldsymbol{\Gamma}^d\boldsymbol{\mathcal{V}}}(V).$$

$S(m|n,d)$-smod $\simeq$ category of degree-d polynomial rep. of $GL_{m|n}$.

Evaluation on V defines a functor $\boldsymbol{\mathcal{P}}_d \to S(m|n,d)$-smod.

Theorem (Axtell 2013)

Let $V \in \boldsymbol{\mathcal{V}}$. If $V \cong k^{m|n}$ and $m, n \geq d$, then evaluation on V

$$F \mapsto F(V)$$

defines an equivalence of categories $\boldsymbol{\mathcal{P}}_d \simeq S(m|n,d)$-smod.

Applications to and from finite supergroup schemes

Fundamental Ext calculation

The sum $I^{(r)} = I_0^{(r)} \oplus I_1^{(r)}$ gives rise to the matrix decomposition

$$\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)}) = \begin{pmatrix} \mathrm{Ext}^\bullet_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}) \\ \mathrm{Ext}^\bullet_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}) \end{pmatrix}.$$

Theorem (Drupieski 2016)

$$\mathrm{Ext}^s_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}) \cong \mathrm{Ext}^s_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) \cong \begin{cases} k & \text{if } s \geq 0 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Ext}^s_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) \cong \mathrm{Ext}^s_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}) \cong \begin{cases} k & \text{if } s \geq p^r \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Restriction to the Frobenius kernel $GL_{m|n(r)}$

$\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)})$ is generated as an algebra by extension classes

$$c_r \in \mathrm{Ext}^{p^r}_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}), \qquad c_r^\Pi \in \mathrm{Ext}^{p^r}_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}),$$
$$e_i \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}), \qquad e_i^\Pi \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}),$$

for $1 \leq i \leq r$, whose restrictions to $G = GL_{m|n(r)}$ are nonzero:

$$\begin{aligned}
\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)}) &\to \mathrm{Ext}^\bullet_{GL_{m|n(r)}}((k^{m|n})^{(r)}, (k^{m|n})^{(r)}) \\
&\cong \mathrm{Ext}^\bullet_{GL_{m|n(r)}}(k, k) \otimes \mathrm{Hom}_k(k^{m|n}, k^{m|n})^{(r)} \\
&\cong \mathrm{H}^\bullet(GL_{m|n(r)}, k) \otimes \mathfrak{gl}(m|n)^{(r)} \\
&\cong \mathrm{Hom}_k(\mathfrak{gl}(m|n)^{*(r)}, \mathrm{H}^\bullet(GL_{m|n(r)}, k))
\end{aligned}$$

CFG for finite supergroup schemes

Putting together the linear maps from all generators, and extending multiplicatively, get a homomorphism of graded superalgebras

$$\phi_{GL_{m|n(r)}} : \left(\bigotimes_{i=1}^{r} S(\mathfrak{gl}(m|n)_{\bar{0}}^{*}[2p^{i-1}])^{(r)} \right) \otimes S(\mathfrak{gl}(m|n)_{\bar{1}}^{*}[p^{r}])^{(r)} \\ \to \mathrm{H}^{\bullet}(GL_{m|n(r)}, k).$$

Theorem (Drupieski 2016)

Let $G \subset GL_{m|n(r)}$ be an infinitesimal supergroup scheme, and let ϕ_G be the composition of ϕ with the restriction map in cohomology. Then $\mathrm{H}^{\bullet}(G, k)$ is finite over the image of ϕ_G.

Consequence

Let G be a finite supergroup scheme. Then $\mathrm{H}^{\bullet}(G, k)$ is finitely generated as a k-superalgebra.

Algebra relations in $\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)})$

$$\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)}) = \begin{pmatrix} \mathrm{Ext}^\bullet_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}) \\ \mathrm{Ext}^\bullet_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) & \mathrm{Ext}^\bullet_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}) \end{pmatrix}.$$

$\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)})$ is generated as an algebra by (even) extension classes

$$c_r \in \mathrm{Ext}^{p^r}_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}), \qquad c_r^\Pi \in \mathrm{Ext}^{p^r}_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}),$$

$$e_i \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}), \qquad e_i^\Pi \in \mathrm{Ext}^{2p^{i-1}}_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}),$$

for $1 \leq i \leq r$, subject only to the relations imposed by the matrix ring and

- the e_i and e_i^Π generate commutative subalgebras,
- $(e_i)^p = 0 = (e_i^\Pi)^p \quad$ for $1 \leq i \leq r-1$,
- $e_i \circ c_r = c_r \circ e_i^\Pi$ and $e_i^\Pi \circ c_r^\Pi = c_r^\Pi \circ e_i$,
- $(e_r)^p = c_r \circ c_r^\Pi$ and $(e_r^\Pi)^p = c_r^\Pi \circ c_r$.

For $r > 1$, only proof I know of the last relation is to look at how the classes restrict to certain supergroup schemes, to show $(e_r)^p \neq 0$ and $(e_r^\Pi)^p \neq 0$.

Reinterpreting the algebra relations

Relations in $\mathrm{Ext}^\bullet_{\mathcal{P}}(I^{(r)}, I^{(r)})$ imply that $\phi_{GL_{m|n(r)}}$ factors through a map

$$\overline{\phi}_{GL_{m|n(r)}} : k[C_r(GL_{m|n})] \to \mathrm{H}^\bullet(GL_{m|n(r)}, k),$$

where

$$\begin{aligned} C_r(GL_{m|n}) = \Big\{ &(\alpha_0, \dots, \alpha_{r-1}, \beta) \in (\mathfrak{gl}(m|n)_{\overline{0}})^{\times r} \times \mathfrak{gl}(m|n)_{\overline{1}} : \\ &[\alpha_i, \alpha_j] = [\alpha_i, \beta] = 0 \text{ for all } 0 \leq i, j \leq r-1, \\ &\alpha_i^p = 0 \text{ for all } 0 \leq i \leq r-2, \text{ and } \alpha_{r-1}^p + \tfrac{1}{2}[\beta, \beta] = 0 \Big\}. \end{aligned}$$

Theorem (Drupieski–Kujawa, 2019)

$\overline{\phi}_{GL_{m|n(r)}}$ induces a finite surjective morphism of varieties

$$\mathrm{MaxSpec}\left(\mathrm{H}^\bullet(GL_{m|n(r)}, k)\right) \to C_r(GL_{m|n}).$$

If $r = 1$ and $G \subset GL_{m|n(1)}$, i.e., if G corresponds to a restricted Lie superalgebra, then an analogous result holds for G. More on this subject in Jon Kujawa's talk ...

Other directions

Rational cohomology

A byproduct of the arguments inspecting how the extension class

$$e_1 \in \mathrm{Ext}^2_{\mathcal{P}}(I_0^{(1)}, I_0^{(1)})$$

restricts to the Frobenius kernel $GL(m|n)_1$ is the curious observation:

$$\text{For } m, n \geq 1, \quad \mathrm{Ext}^2_{GL(m|n)}(k, k) \neq 0.$$

Problem

Compute the structure of the rational cohomology ring

$$\mathrm{H}^\bullet(GL_{m|n}, k) = \mathrm{Ext}^\bullet_{GL_{m|n}}(k, k).$$

For $m = n = 1$, appears to be a polynomial ring generated in degree 2.

Also an open problem for the algebraic supergroup $Q(n)$, for which Brundan and Kleshchev (2003) computed $\mathrm{Ext}^1_{Q(n)}(k, k) \cong k^{0|1}$. Axtell's "Type II" strict polynomial functors, related to the Schur superalgebra $Q(n, d)$ of Type Q, may be relevant.

Frobenius twists of ordinary strict polynomial functors

Let $F \in \mathcal{P}_d$ be an ordinary (non-super) strict polynomial functor.

In general, there's no obvious way to lift F to the structure for a strict polynomial superfunctor, but for $r \geq 1$ the twists

$$F_0^{(r)} = F \circ \mathbf{I}_0^{(r)}, \qquad F_1^{(r)} = F \circ \mathbf{I}_1^{(r)}$$

do make sense as strict polynomial superfunctors, with action on morphisms defined by

$$\mathbf{\Gamma}^{dp^r} \operatorname{Hom}_k(V, W) \xrightarrow{\mathbf{\Gamma}^d I_0^{(r)}} \mathbf{\Gamma}^d \underbrace{\operatorname{Hom}_k(V_{\bar{0}}^{(r)}, W_{\bar{0}}^{(r)})}_{\text{purely even!}} \xrightarrow{F} \operatorname{Hom}_k(F(V_{\bar{0}}^{(r)}), F(W_{\bar{0}}^{(r)})),$$

$$\mathbf{\Gamma}^{dp^r} \operatorname{Hom}_k(V, W) \xrightarrow{\mathbf{\Gamma}^d I_1^{(r)}} \mathbf{\Gamma}^d \underbrace{\operatorname{Hom}_k(V_{\bar{1}}^{(r)}, W_{\bar{1}}^{(r)})}_{\text{purely even!}} \xrightarrow{F} \operatorname{Hom}_k(F(V_{\bar{1}}^{(r)}), F(W_{\bar{1}}^{(r)})).$$

Giordano's Conjectures

Set $\boldsymbol{E}_r = \mathrm{Ext}^\bullet_{\boldsymbol{\mathcal{P}}}(I_0^{(r)}, I_0^{(r)})$. For $G \in \mathcal{P}$ and $W \in \mathcal{V}$, set $G_W = G(W \otimes -)$.

Conjecture (Giordano)

Let $F, G \in \mathcal{P}$ be ordinary strict polynomial functors. Then

$$\mathrm{Ext}^\bullet_{\boldsymbol{\mathcal{P}}}(F_0^{(r)}, G_0^{(r)}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(F, G_{\boldsymbol{E}_r}).$$

More generally, and let $A, B \in \boldsymbol{\mathcal{P}}$ be additive strict polynomial superfunctors of degree > 1. Then

$$\mathrm{Ext}^\bullet_{\boldsymbol{\mathcal{P}}}(F \circ A, G \circ B) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(F, G_{\mathrm{Ext}^\bullet_{\boldsymbol{\mathcal{P}}}(A,B)}).$$

Additive functors direct sums of $\boldsymbol{I}$, $\boldsymbol{\Pi}$, and (for $r \geq 1$) $I_0^{(r)}$, $I_1^{(r)}$, $I_0^{(r)} \circ \boldsymbol{\Pi}$, $I_1^{(r)} \circ \boldsymbol{\Pi}$.

Touzé established that a result like this holds in the classical (non-super) setting for many F and G, by exploiting some nice injective resolutions constructed by Troesch.

Superized Troesch complexes

Troesch complexes, after Touzé

Goal

For $m, r \geq 1$, describe an injective resolution in $\mathcal{P}_{p^r m}$ of $S^{m(r)}$.

Describe what happens for $r = 1$.

Consequence

Let Ш be the graded k-space with basis $ш_0, \ldots, ш_{p-1}$, $\deg(ш_i) = i$.

Consider the functor $S(\text{Ш} \otimes -) : U \mapsto S(\text{Ш} \otimes U)$.

$$S(\text{Ш} \otimes U) \cong S(ш_0 \otimes U) \otimes S(ш_1 \otimes U) \otimes \cdots \otimes S(ш_{p-1} \otimes U)$$

$S(\text{Ш} \otimes U)$ inherits an $\mathbb{N}$-grading from that on Ш:

$$S^n(\text{Ш} \otimes U)^\ell \cong \bigoplus_{\substack{i_0+i_1+\cdots+i_p=n \\ i_0\cdot 0+i_1\cdot 1+\cdots+i_{p-1}\cdot(p-1)=\ell}} S^{i_0}(U) \otimes S^{i_1}(U) \otimes \cdots \otimes S^{i_{p-1}}(U).$$

Troesch complexes, after Touzé

Define $\rho : Ш \to Ш$ by $\quad \rho(ш_i) = \begin{cases} ш_{i+1} & \text{if } 0 \leq i \leq p-2, \\ 0 & \text{if } i = p-1. \end{cases}$

Define $d : S^n(Ш \otimes U)^\ell \to S^n(Ш \otimes U)^{\ell+1}$ to be the composite

$$\begin{aligned} S^n(Ш \otimes U) &\xrightarrow{\Delta} S^{n-1}(Ш \otimes U) \otimes S^1(Ш \otimes U) \\ &\xrightarrow{\mathrm{id} \otimes S(\rho \otimes \mathrm{id}_U)} S^{n-1}(Ш \otimes U) \otimes S^1(Ш \otimes U) \xrightarrow{m} S^n(Ш \otimes U). \end{aligned}$$

Remark

For $r = 1$, the map d is simply the algebra derivation on $S(Ш \otimes U)$ induced by the vector space map $\rho \otimes \mathrm{id}_U : Ш \otimes U \to Ш \otimes U$.

Troesch complexes, after Touzé

Now $d : S^n(\text{Ш} \otimes -)^\ell \to S^n(\text{Ш} \otimes -)^{\ell+1}$ is a p-differential, i.e., $d^p = 0$.

Then the contraction

$$B_n^\bullet : S^n(\text{Ш} \otimes -)^0 \xrightarrow{d} S^n(\text{Ш} \otimes -)^1 \xrightarrow{d^{p-1}} S^n(\text{Ш} \otimes -)^p$$
$$\xrightarrow{d} S^n(\text{Ш} \otimes -)^{p+1} \xrightarrow{d^{p-1}} S^n(\text{Ш} \otimes -)^{2p} \xrightarrow{d} \cdots$$

is an ordinary cochain complex with

$$B_n^{2i} = S^n(\text{Ш} \otimes -)^{pi} \quad \text{and} \quad B_n^{2i+1} = S^n(\text{Ш} \otimes -)^{pi+1}.$$

Theorem (Troesch)

$B_n^\bullet$ is acyclic if $p \nmid n$, and is an injective resolution of $S^{m(1)}$ if $n = pm$.

More generally, he constructs an injective resolution of $S^{m(r)}$, $r \geq 1$.

Note: For fixed n, one has $B_n^i = 0$ for $i \gg 0$.

Why are Troesch complexes so useful?

Yoneda isomorphism, compatible with $\mathbb{Z}$-gradings

Let $F \in \mathcal{P}_m$. Let $F^{(1)} = F \circ I^{(1)}$. Then

$$\mathrm{Hom}_{\mathcal{P}}(F^{(1)}, S^{pm}(\text{Ш} \otimes -)) \cong F^{\#}(\text{Ш}^{(1)})$$

is concentrated in $\mathbb{Z}$-degrees divisible by p.

Then $\mathrm{Hom}_{\mathcal{P}}(F^{(1)}, B^{\bullet}_{pn})$ is concentrated in even degrees.

Corollary

$$\mathrm{Ext}^{\bullet}_{\mathcal{P}}(I^{(1)}, I^{(1)}) \cong \mathrm{Hom}_{\mathcal{P}}(I^{(1)}, B^{\bullet}_{p}) \cong E_1,$$

where E_1 the space Ш regraded so that $\mathrm{deg}(\text{ш}_i) = 2i$ $(0 \leq i < p)$.

Naive generalization of Troesch's construction

Consider Ш as a $\mathbb{Z}$-graded superspace of purely even superdegree.

For $U = U_{\bar{0}} \oplus U_{\bar{1}}$, consider

$$S(\text{Ш} \otimes U) \cong S(ш_0 \otimes U) \otimes S(ш_1 \otimes U) \otimes \cdots \otimes S(ш_{p-1} \otimes U).$$

Define $d : S(\text{Ш} \otimes U)^{\ell} \to S(\text{Ш} \otimes U)^{\ell+1}$ exactly as before.

Cocycles (by virtue of d being a derivation when $r = 1$)

For $u \in U_{\bar{0}}$, get

$$(ш_0 \otimes u)^p \in S^p(\text{Ш} \otimes U)^0.$$

New for super: If $u \in U_{\bar{1}}$, get

$$u^{(1)} := (ш_0 \otimes u) \otimes (ш_1 \otimes u) \otimes \cdots \otimes (ш_{p-1} \otimes u) \in S^p(\text{Ш} \otimes U)^{p(p-1)/2}$$

in the exterior algebra part of $S(\text{Ш} \otimes U) \cong S(\text{Ш} \otimes U_{\bar{0}}) \otimes \Lambda(\text{Ш} \otimes U_{\bar{1}})$

Naive generalization of Troesch's construction

Let $B_n^\bullet$ be the contracted complex of strict polynomial superfunctors

$$B_n^\bullet : S^n(\text{Ш} \otimes -)^0 \xrightarrow{d} S^n(\text{Ш} \otimes -)^1 \xrightarrow{d^{p-1}} S^n(\text{Ш} \otimes -)^p \xrightarrow{d} S^n(\text{Ш} \otimes -)^{p+1} \xrightarrow{d^{p-1}} S^n(\text{Ш} \otimes -)^{2p} \xrightarrow{d} \cdots$$

Theorem (Drupieski–Kujawa)

$$\mathrm{H}^\bullet(B_n) \cong \begin{cases} 0 & \text{if } p \nmid n, \\ S^{m(1)} & \text{if } n = pm. \end{cases}$$

In the latter case, for $0 \leq \ell \leq m$, the summand

$$(S^{m-\ell} \circ I_0^{(1)}) \otimes (\Lambda^\ell \circ I_1^{(1)})$$

of $S^{m(1)}$ is in cohomological degree $\ell \cdot (p-1)$.

Resolutions of injectives

In the case $n = p$, get a complex of injective objects

$$B_p^0 \to B_p^1 \to \cdots \to B_p^{p-1} \to \cdots \to B_p^{2(p-1)}$$

with $\quad \mathrm{H}^0(B_p) \cong I_0^{(1)}, \quad \mathrm{H}^{p-1}(B_p) \cong I_1^{(1)}, \quad$ and $\quad \mathrm{H}^i(B_p) = 0$ otherwise.

These complexes can be spliced together:

Splice at beginning:

$$\begin{tikzcd}
0 & & p^r-1 & p^r & p^r+1 & & 2p^r-2 & 2p^r-1 & & 3p^r-2 \\
T^0 \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & T^{p^r-1} \arrow[r, "\partial"] \arrow[ddr, "\varepsilon"] & T^{p^r} \arrow[r, "\partial"] \arrow[ddr, "\varepsilon"] & T^{p^r+1} \arrow[r, "\partial"] \arrow[ddr, "\varepsilon"] & \cdots \arrow[r, "\partial"] \arrow[ddr, "\varepsilon"] & T^{2p^r-2} \arrow[ddr, "\varepsilon"] & & & \\
 & & & \oplus & \oplus & & \oplus & & & \\
 & & & \overline{T}^0 \arrow[r, "\bar\partial"] & \overline{T}^1 \arrow[r, "\bar\partial"] & \cdots \arrow[r, "\bar\partial"] & \overline{T}^{p^r-2} \arrow[r, "\bar\partial"] & \overline{T}^{p^r-1} \arrow[r, "\bar\partial"] & \cdots \arrow[r, "\bar\partial"] & \overline{T}^{2p^r-2}
\end{tikzcd}$$

Continue splicing:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] \arrow[ddr, "\varepsilon"] & T^{2p^r-2} \arrow[ddr, "\varepsilon"] & & T^0 \arrow[r, "\partial"] & T^1 \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & T^{p^r-2} \arrow[r, "\partial"] & T^{p^r-1} \arrow[r, "\partial"] \arrow[ddr, "\varepsilon"] & T^{p^r} \arrow[r, "\partial"] \arrow[ddr, "\varepsilon"] & \cdots \\
 & \oplus & & \oplus & \oplus & & \oplus & & \oplus & \\
\cdots \arrow[r, "\bar\partial"] & \overline{T}^{p^r-2} \arrow[r, "\bar\partial"] & \overline{T}^{p^r-1} \arrow[r, "\bar\partial"] \arrow[uur, "\bar\varepsilon"] & \overline{T}^{p^r} \arrow[r, "\bar\partial"] \arrow[uur, "\bar\varepsilon"] & \overline{T}^{p^r+1} \arrow[r, "\bar\partial"] \arrow[uur, "\bar\varepsilon"] & \cdots \arrow[r, "\bar\partial"] \arrow[uur, "\bar\varepsilon"] & \overline{T}^{2p^r-2} \arrow[uur, "\bar\varepsilon"] & & \overline{T}^0 \arrow[r, "\bar\partial"] & \cdots
\end{tikzcd}$$

Calculations

End result of splicing

For all $r \geq 1$, construct periodic injective resolutions

$$I_0^{(r)} \to J(r) \quad \text{and} \quad I_1^{(r)} \to \bar{J}(r).$$

Corollary ("quick" recalculation)

$$\mathrm{Ext}^{\bullet}_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) \cong \mathrm{Hom}_{\mathcal{P}}(I_0^{(r)}, J(r)) \cong \bigoplus_{n \geq 0} E_r\langle 2np^r \rangle$$

$$\mathrm{Ext}^{\bullet}_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}) \cong \mathrm{Hom}_{\mathcal{P}}(I_1^{(r)}, J(r)) \cong \bigoplus_{n \geq 0} E_r\langle (2n+1)p^r \rangle$$

where $E_r = \bigoplus_{0 \leq i < p^r} k\langle 2i \rangle$.

More calculations (after Franjou, Friedlander, Scorichenko, and Suslin)

For $1 \leq j \leq r$ and $\ell \in \{0,1\}$, set

$$V_{j,\ell} = \mathrm{Ext}^\bullet_{\mathcal{P}}(I_\ell^{(r)}, S_0^{p^{r-j}(j)}), \qquad W_{j,\ell} = \mathrm{Ext}^\bullet_{\mathcal{P}}(I_\ell^{(r)}, \Lambda_0^{p^{r-j}(j)}),$$

$$\overline{V}_{j,\ell} = \mathrm{Ext}^\bullet_{\mathcal{P}}(I_\ell^{(r)}, S_1^{p^{r-j}(j)}), \qquad \overline{W}_{j,\ell} = \mathrm{Ext}^\bullet_{\mathcal{P}}(I_\ell^{(r)}, \Lambda_1^{p^{r-j}(j)}).$$

Using the superized Troesch complexes in lieu of the de Rham and Koszul complexes:

Theorem

Let $\ell \in \{0,1\}$. For all $d \geq 1$ and all $1 \leq j \leq r$, the cup product maps

$$(V_{j,\ell})^{\otimes d} \to \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, S_0^{dp^{r-j}(j)}), \qquad (W_{j,\ell})^{\otimes d} \to \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, \Lambda_0^{dp^{r-j}(j)}),$$

$$(\overline{V}_{j,\ell})^{\otimes d} \to \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, S_1^{dp^{r-j}(j)}), \qquad (\overline{W}_{j,\ell})^{\otimes d} \to \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, \Lambda_1^{dp^{r-j}(j)})$$

factor to induce isomorphisms of graded vector spaces

$$S^d(V_{j,\ell}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, S_0^{dp^{r-j}(j)}), \qquad \Lambda^d(W_{j,\ell}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, \Lambda_0^{dp^{r-j}(j)}),$$

$$S^d(\overline{V}_{j,\ell}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, S_1^{dp^{r-j}(j)}), \qquad \Lambda^d(\overline{W}_{j,\ell}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}}(\Gamma_\ell^{d(r)}, \Lambda_1^{dp^{r-j}(j)}).$$