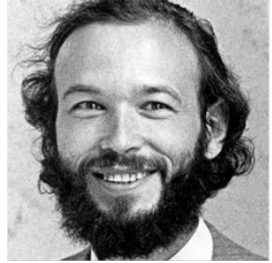
Bregje Pauwels
(joint with K. Coulembier)



TANNAKIAN QUESTIONS

GOAL: WHEN IS A CATEGORY $QC(\mathfrak{X})$ FOR SOME STACK $\mathfrak{X}$?

Let $k = \bar{k}$ be a field,

G an affine group scheme over $\bar{k}$.

$G : Alg/_k \longrightarrow Grp$

$= Hom(k[G], \_)$

$\mathcal{C} = \mathrm{rep}\, G = k[G]\text{-comod}$ (fd G-representations)

TENSOR CATEGORY

- $\mathcal{C}$ is abelian
- $\mathcal{C}$ is symmetric monoidal $(\otimes_k, \mathbb{1} = k)$
- $\mathrm{End}(\mathbb{1})$ is a field k
- Every $X \in \mathcal{C}$ has a dual $X^\vee$

$$\{\text{AFFINE GROUP SCHEMES}\} \hookrightarrow \{\text{TENSOR CATEGORIES}\}$$

$$G \longmapsto \operatorname{rep} G$$

QUESTION: WHAT'S THE IMAGE ?

We have a forgetful functor

$$\operatorname{rep} G \longrightarrow k\text{-VS}$$

FIBER FUNCTOR

- faithful
- exact
- symmetric monoidal

A tensor category with a fiber functor is called

TANNAKIAN

$$\left\{\text{AFFINE GROUP SCHEMES}\right\} \xrightarrow{\simeq} \left\{\text{TANNAKIAN CATEGORIES}\right\} \subset \left\{\text{TENSOR CATEGORIES}\right\}$$

RECOGNITION

QUESTION: WHAT'S THE IMAGE ?

ANSWER: TANNAKIAN CATEGORIES

<Saavedra-Rivano, Deligne>

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TANNAKIAN

$$\left\{\text{AFFINE GROUP SCHEMES}\right\} \xrightarrow{\simeq} \left\{\text{TANNAKIAN CATEGORIES}\right\} \subset \left\{\text{TENSOR CATEGORIES}\right\}$$

DESCRIPTION → USE INTRINSIC PROPERTIES ONLY

QUESTION: WHAT'S THE IMAGE? <Deligne>

(char 0) $\mathcal{C} \simeq \operatorname{rep} G$ for some G $\Leftrightarrow$ $\forall X \in \mathcal{C} : \exists n : \Lambda^n X = 0$

RECONSTRUCTION QUESTION: HOW DO I RECOVER THE GROUP SCHEME?

Given $\omega : \mathcal{C} \longrightarrow k\text{-VS}$ fiber functor

$$k\text{-VS} \xrightarrow{R \otimes -} R\text{-mod}$$

we can build

$$\underline{\mathrm{aut}}^{\otimes}(\omega) : k\text{-alg} \longrightarrow \mathrm{Grp}$$

$$R \longmapsto \mathrm{aut}^{\otimes}((R \otimes -) \circ \omega)$$

$$\underline{\mathrm{aut}}^{\otimes}(\omega) = G$$

this is an affine group scheme and

$$\mathcal{C} \simeq \mathrm{rep}\, G$$

RECONSTRUCTION QUESTION: HOW DO I RECOVER THE GROUP SCHEME?

OR

set $k[G] = \int^{X \in \mathcal{C}} X \otimes X^\vee$ (co-end in Ind $\mathcal{C}$)

This is a Hopf algebra and

$\mathcal{C} \simeq k[G]$ - comod

EXAMPLES

① Y topological group

$$\mathcal{C} = \underset{\mathbb{C}}{\mathrm{rep}^{ct}}\, Y \longrightarrow \mathbb{C}\text{-mod} \quad \text{fiber functor}$$

ALGEBRAIC ENVELOPE

↳ $\exists$ affine group scheme G with

$$\mathrm{rep}\, G = \underset{\mathbb{C}}{\mathrm{rep}^{ct}}\, Y$$

f negligent : $\forall g$: $\mathrm{tr}(fg) = 0$

ideal of negligent morphisms

② G affine group scheme

$$\mathcal{C} = \overline{\mathrm{rep}\, G} := \mathrm{rep}\, G \big/ \mathcal{N}$$ semisimplification

$\forall x \in \mathcal{C}$: $\exists n$: $\Lambda^n x = 0$

PRO-REDUCTIVE ENVELOPE

char $k = 0$ $\longrightarrow$ $\exists \hat{G}$ with $\mathrm{rep}\, \hat{G} = \overline{\mathrm{rep}\, G}$

char $k = p$ $\longrightarrow$ $\overline{\mathrm{rep}\, C_p}$ not tannakian :(

↳ need a description criterion for char p.

IDEA: FROBENIUS TWISTS <Coulembier>

$\mathcal{C}$ tensor category, $X \in \mathcal{C}$.

$$\mathrm{Fr}^{(j)} X := \mathrm{im}\Big(\underbrace{\Gamma^{p^j} X}_{\text{divided power}} \hookrightarrow \otimes^{p^j} X \twoheadrightarrow \underbrace{\mathrm{sym}^{p^j} X}_{\text{symmetric power}} \Big)$$

Example: $\mathcal{C} = k\text{-VS}$. $\mathrm{Fr}^{(j)} V = V^{(j)}$ frobenius twist

IDEA: FROBENIUS TWISTS

$\Lambda^2 X = \mathrm{im}(\gamma_{X,X} - 1)$

$\otimes^2 X / \langle \Lambda^2 X \rangle \simeq \mathrm{Sym}\ 1$

$\Gamma^n X = \bigcap \otimes X \, \Gamma^2 X \otimes X$

$\mathcal{C}$ tensor category, $X \in \mathcal{C}$.

$$\mathrm{Fr}^{(j)} X := \mathrm{im}\left(\Gamma^{p^j} X \hookrightarrow \otimes^{p^j} X \twoheadrightarrow \mathrm{sym}^{p^j} X \right)$$

divided power

symmetric power

THEOREM <Coulembier>

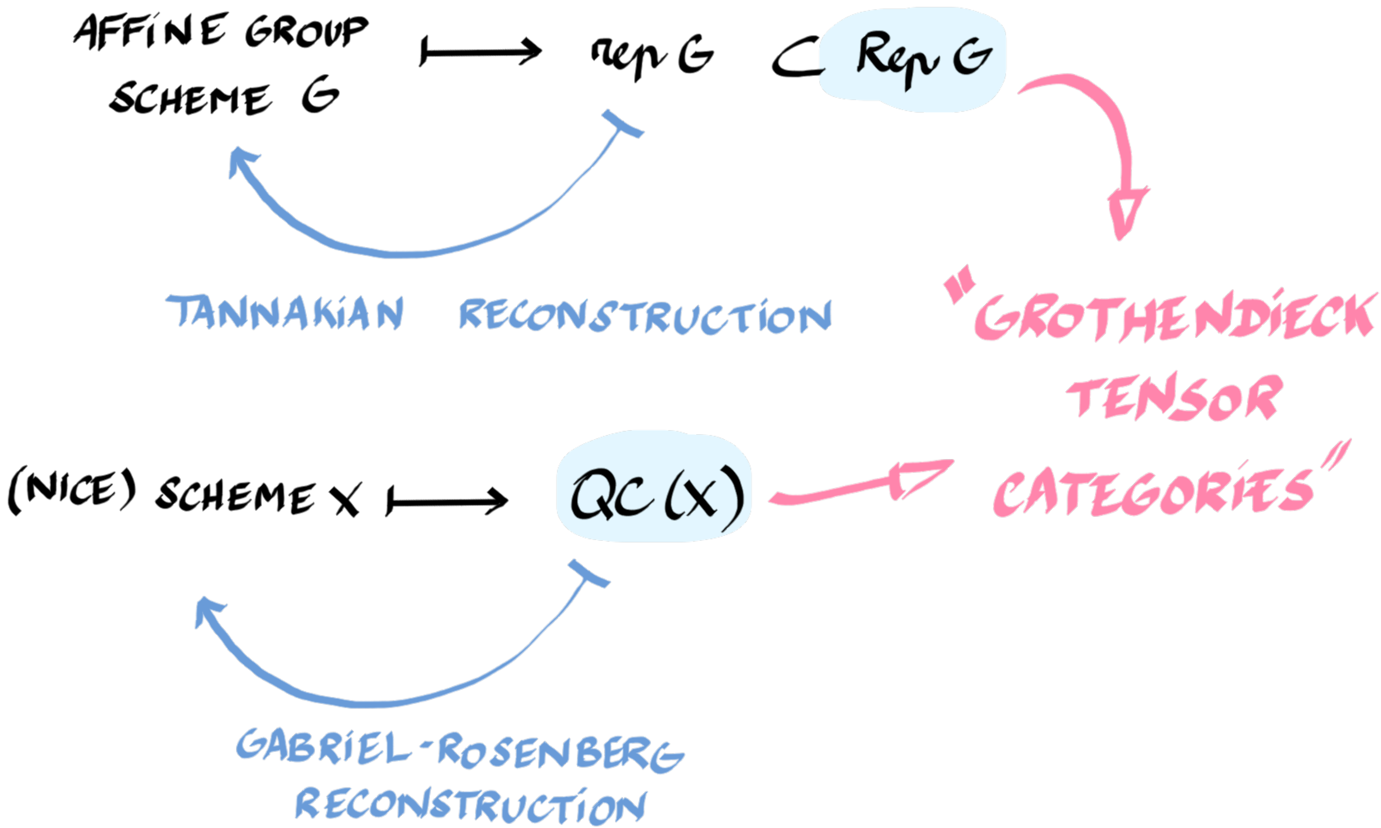
Let $\mathrm{char}\, k = p$. Then

$\mathcal{C} \simeq \mathrm{rep}\, G$ for some G $\iff$

$$\begin{cases} \text{(I)} \ \forall X \in \mathcal{C} : \exists n : \Lambda^n X = 0 \\ \text{(II)} \ \exists j : \Lambda^n \mathrm{Fr}^{(j)} X = 0 \Rightarrow \Lambda^n X = 0 \end{cases}$$

$$\begin{tikzcd} \text{AFFINE GROUP SCHEME } G \arrow[r, mapsto] & \mathrm{rep}\,G \arrow[l, bend left, mapsto, "\text{TANNAKIAN RECONSTRUCTION}"] & \subset \mathrm{Rep}\,G \end{tikzcd}$$

$$\begin{tikzcd} \text{(NICE) SCHEME } X \arrow[r, mapsto] & \mathrm{QC}(X) \arrow[l, bend left, mapsto, "\text{GABRIEL-ROSENBERG RECONSTRUCTION}"] \end{tikzcd}$$


AFFINE GROUP SCHEME G
rep G
Rep G
TANNAKIAN RECONSTRUCTION
"GROTHENDIECK TENSOR CATEGORIES"
(NICE) SCHEME X
QC(X)
GABRIEL-ROSENBERG RECONSTRUCTION

$$\left\{\begin{array}{c}\text{ALGEBRO-GEOMETRIC}\\ \text{OBJECTS}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{GROTHENDIECK}\\ \text{TENSOR CATEGORIES}\end{array}\right\}$$

(homs: left adjoints)

$$\begin{tikzcd}
X \arrow[d, "f"] & \longmapsto & QC(X) \\
Y & & QC(Y) \arrow[u, "f^*"']
\end{tikzcd}$$

Grothendieck category,
symmetric monoidal,
generated by a set of rigid objects
with $\mathbb{1}$ compact

"GROTHENDIECK TENSOR CATEGORIES"

Examples

$\mathrm{Ind}\, T$, where T is a tensor category

$R\text{-}\mathrm{Mod}$, where R is a ring

k commutative ring

$$\left\{\begin{array}{c}\text{ALGEBRO-GEOMETRIC}\\ \text{OBJECTS}\end{array}\right\} \xrightarrow{QC(-)} \left\{\begin{array}{c}\text{GROTHENDIECK}\\ \text{TENSOR CATEGORIES}/k\end{array}\right\}$$

$G \longmapsto \operatorname{Rep} G$ $\qquad$ G group scheme

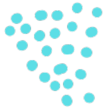
$X \dashrightarrow QC(X)$ $\qquad$ X scheme

$\mathfrak{X} \dashrightarrow QC(\mathfrak{X})$ $\qquad$ $\mathfrak{X}$ stack

FUNCTOR OF POINTS

(k ring)

SCHEME X/k

$X : \mathrm{Alg}/k \longrightarrow \mathrm{Sets}$ 

$\quad \|$

$\mathrm{Hom}(\mathrm{Spec}(\_), X)$

+ extra conditions

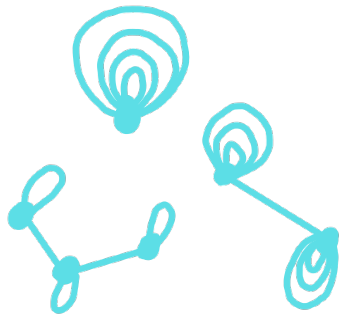
AFFINE GROUP SCHEME G/k

$G : \mathrm{Alg}/k \longrightarrow \mathrm{Grp}$ 

$\quad \|$

$\mathrm{Hom}(k[G], \_)$

STACK $\mathfrak{X}$



$\mathfrak{X} : \mathrm{Alg}/k \longrightarrow \mathrm{Groupoids}$

+ extra conditions

ALGEBRAIC STACK $\mathfrak{X}$

$\mathfrak{X} : \mathrm{Alg}/_k \longrightarrow \mathrm{Groupoids}$ represented by (A, Γ)

$B \longmapsto$ groupoid with objects $\mathrm{Hom}(A, B)$

& morphisms $\mathrm{Hom}(\Gamma, B)$

Hopf algebroid (A, Γ) has structure

$s : A \to \Gamma$ source

$t : A \to \Gamma$ target

$\iota : \Gamma \to A$ identity maps

$\Gamma \to \Gamma \otimes \Gamma$ composition

$c : \Gamma \to \Gamma$ inverse

An algebraic stack is a stack $\mathcal{X}$ with affine diagonal and such that $\exists\, \mathcal{U}$ affine with

$$\mathcal{U} \longrightarrow \mathcal{X} \quad \text{faithfully flat.}$$

$$\left\{ \begin{array}{c} \text{algebraic stacks} \\ \left(\begin{array}{c} \text{with choice of} \\ \text{presentation} \end{array} \right) \end{array} \right\} \overset{\simeq}{\longleftrightarrow} \{\text{flat Hopf algebroids}\}$$

(s is a flat map)

$$\mathcal{U} \longrightarrow \mathcal{X} \qquad\qquad \left(\mathcal{U},\ \mathcal{U} \underset{\mathcal{X}}{\times} \mathcal{U}\right)$$

An Adams stack is an algebraic stack which satisfies the $\underbrace{\text{strong resolution property}}_{\text{dualisable sheaves generate } QC(\mathcal{X})}$.

In other words,

$QC(\mathcal{X})$ is a Grothendieck tensor category examples:

G affine group scheme $\quad (k, k[G])$

X qcqs scheme with the strong resolution property

eg X regular noetherian separated scheme

$(\sqcup \operatorname{spec} A_i = \mathcal{U}, \quad \mathcal{U} \times_X \mathcal{U})$

$\mathfrak{X}$ moduli stack of formal groups

Hopf algebroid $(R_*, R_* R)$ for many ring spectra

eg $R_* = MU, MSp, K, KO, H\mathbb{F}_p$

k commutative ring

$$\left\{\text{ADAMS STACKS}\right\} \xrightarrow{QC(-)} \left\{\begin{array}{l}\text{GROTHENDIECK}\\ \text{TENSOR CATEGORIES}/k\end{array}\right\}$$

THEOREM <Lurie, Brandenburg - Chirvasitu, Schäppi>

$QC(-)$ is full & faithful.

QUESTION: WHAT'S THE IMAGE?

RECOGNITION <Schappi>

$\mathcal{C} = QC(\mathfrak{X}) \iff \exists\, \mathcal{C} \longrightarrow R\text{-mod}$ fiber functor

$\mathfrak{X}$ adams stack

for some k-algebra R

<Schappi, Coulembier-P>

DESCRIPTION If $\mathrm{End}(1)$ is a $\mathbb{Q}$-algebra,

$\mathcal{C} = QC(\mathfrak{X}) \iff \forall x \in \mathcal{C}^{d}\ \exists n : \Lambda^n X = 0$ & X is locally faithful

DESCRIPTION (char $k = p$) < Coulembier-P >

$\mathcal{C} = QC(\mathcal{X}) \iff \forall X \in \mathcal{C}^d$:

1. X is locally faithful
2. $\mathrm{Sym}^i X$ is flat & $\Lambda^n X$ is rigid
3. $\forall j$: $Fr^{(j)}(X^\vee)$ is dual to $Fr^{(j)}(X)$
4. $\exists m : \Lambda^m X = 0$
5. $\forall Y \in \mathcal{C}^d$: $\Lambda^n Fr^{(j)} X \otimes Y = 0 \Rightarrow \Lambda^n X \otimes Y = 0$

RECONSTRUCTION QUESTION:

Given $\omega : \mathcal{C} \longrightarrow R\text{-mod}$ fiber functor,

consider

$$\Gamma := \int^{x \in \mathcal{C}^d} \omega(x) \otimes \omega(x^\vee)$$

Then (R, Γ) is a Hopf-algebroid and

$$\mathcal{C} = (R, \Gamma)\text{-Comod}.$$

RECONSTRUCTION, MORE GENERALLY

suppose $\mathcal{D} \boxtimes \mathcal{D} \xrightarrow{(id, id)} \mathcal{D}$ is exact & faithful

Given $\omega : \mathcal{C} \longrightarrow \mathcal{D}_A$ fiber functor,

A comm. ring in $\mathcal{D}$

consider

$$\Gamma := \int^{x \in \mathcal{C}^d} \omega(x) \otimes \omega(x^\vee)$$

then (A, Γ) is a Hopf algebroid in $\mathcal{D}$ and

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\simeq"] \arrow[d] & (A,\Gamma)\text{-}\mathrm{Comod}^{\heartsuit}_{\mathcal{D}} \arrow[d, hook] \\
\mathcal{C} \boxtimes \mathcal{D} \arrow[r, "\simeq"] \arrow[dr, "{(\omega,\, A\otimes -)}"'] \arrow[r, phantom, shift right=6ex, "\circlearrowleft"] & (A,\Gamma)\text{-}\mathrm{Comod}_{\mathcal{D}} \arrow[d, "\text{forget}"] \\
 & \mathcal{D}_A
\end{tikzcd}
$$

THEOREM

The 2-category of Grothendieck-tensor categories has all finite coproducts and

$$QC(\mathcal{X} \times_k \mathcal{Y}) \simeq QC(\mathcal{X}) \boxtimes_k QC(\mathcal{Y})$$

for all Adams stacks $\mathcal{X}$ and $\mathcal{Y}$.