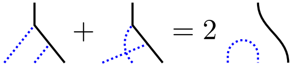
The spin Brauer category



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Outline

Goals:

1. Study the representation theory of the spin and pin groups using diagrammatic techniques.
2. Define interpolating categories for these groups.

Overview:

1. The Clifford algebra
2. The spin and pin groups and their modules
3. The Brauer category
4. The spin Brauer category

The Clifford algebra

Let V be a finite-dimensional vector space of dimension N and let

$$\Phi_V \colon V \times V \to \Bbbk$$

be a nondegenerate symmetric bilinear form.

Define

$$n = \left\lfloor \frac{N}{2} \right\rfloor \in \mathbb{N}, \quad \text{so that} \quad N = \begin{cases} 2n & \text{if } N \text{ is even,} \\ 2n+1 & \text{if } N \text{ is odd.} \end{cases}$$

Let

$$\mathrm{Cl} = \mathrm{Cl}(V) := T(V) / \left(vw + wv - 2\Phi_V(v,w) : v, w \in V \right)$$

denote the Clifford algebra associated to V. Here $T(V)$ is the tensor algebra on V.

The Clifford algebra

Let $e_1, \dots, e_N$ be an orthonormal basis. For $1 \le j \le n$, define

$$\psi_j := \frac{1}{2}\left(e_{2j-1} + \sqrt{-1}e_{2j}\right), \qquad \psi_j^\dagger := \frac{1}{2}\left(e_{2j-1} - \sqrt{-1}e_{2j}\right).$$

Then we have

$$\Phi_V(\psi_i, \psi_j) = 0, \quad \Phi_V(\psi_i^\dagger, \psi_j^\dagger) = 0, \quad \Phi_V(\psi_i, \psi_j^\dagger) = \frac{1}{2}\delta_{ij}.$$

Hence

$$\psi_i\psi_j + \psi_j\psi_i = 0 = \psi_i^\dagger\psi_j^\dagger + \psi_j^\dagger\psi_i^\dagger, \qquad \psi_i\psi_j^\dagger + \psi_j^\dagger\psi_i = \delta_{ij}. \tag{1}$$

When N is even, (1) gives a presentation of Cl. When N is odd, we need to include the additional relations

$$\psi_i e_N + e_N\psi_i = 0 = \psi_i^\dagger e_N + e_N\psi_i^\dagger, \qquad e_N^2 = 1,$$

to obtain a presentation of Cl.

The spin Clifford module

Let

$$S := \Lambda(W) = \bigoplus_{r=0}^{n} \Lambda^r(W), \qquad \text{where} \quad W = \mathrm{Span}_{\Bbbk}\{\psi_i^\dagger : 1 \le i \le n\}.$$

As a $\Bbbk$-module, S has basis

$$x_I := \psi_{i_1}^\dagger \wedge \psi_{i_2}^\dagger \wedge \cdots \wedge \psi_{i_k}^\dagger,$$
$$I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}, \quad i_1 < i_2 < \dots < i_k.$$

In particular,

$$\dim_{\Bbbk}(S) = 2^n.$$

The spin Clifford module

If N is even, we turn S into a Cl-module by defining

$$\begin{aligned} \psi_i^\dagger x_J &= \psi_i^\dagger \wedge x_J, \\ \psi_i x_J &= \begin{cases} (-1)^{|\{j \in J \mid j < i\}|} x_{J \setminus \{i\}} & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases} \end{aligned} \tag{2}$$

If N is odd, then we define two Cl-module structures on S, depending on a choice of $\varepsilon \in \{\pm 1\}$.

We again use the action defined in (2) and additionally define

$$e_N x_I = \varepsilon (-1)^{|I|} x_I.$$

We call S the spin module.

The pin and spin groups

Define the pin group

$$\{v_1 v_2 \cdots v_k : k \in \mathbb{N},\ v_i \in V,\ \Phi_V(v_i, v_i) = 1\ \forall\ i\} \subseteq \mathrm{Cl}(V)^\times$$

and the spin group

$$\{v_1 v_2 \cdots v_k : k \in 2\mathbb{N},\ v_i \in V,\ \Phi_V(v_i, v_i) = 1\ \forall\ i\} \subseteq \mathrm{Pin}(V).$$

The group $\mathrm{Pin}(V)$ acts on V by

$$g \cdot v = (-1)^{\deg g} g v g^{-1}.$$

This yields a short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Pin}(V) \to \mathrm{O}(V) \to 1.$$

Restricting to $\mathrm{Spin}(V)$ yields another short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1.$$

The pin and spin groups

When N is odd, $\mathrm{Pin}(V)$ is generated by $\mathrm{Spin}(V)$ and the central element $e_1 e_2 \cdots e_N$.

So the difference between the representation theory of $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ is not significant when N is odd.

Define

$$\mathrm{G}(V) := \begin{cases} \mathrm{Pin}(V) & \text{if } N \text{ is even,} \\ \mathrm{Spin}(V) & \text{if } N \text{ is odd.} \end{cases}$$

Goals

1. Study the representation theory of $\mathrm{G}(V)$ using diagrammatic techniques.
2. Define interpolating categories.

The spin and vector modules

The spin module

Restriction of the $\mathrm{Cl}(V)$-action on S gives natural actions of $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ on S.

We call this the spin module.

The vector module

We view V as a $\mathrm{Pin}(V)$-module with action

$$g \cdot v = gvg^{-1}.$$

We call this the vector module.

Building blocks

All simple finite-dimensional $\mathrm{G}(V)$-modules are summands of tensor products of S.

Bilinear form

Define a bilinear form on S by

$$\Phi_S(x_I, x_J) = \begin{cases} (-1)^{\binom{|I|}{2} + nN|I| + |\{(i,j) \in I \times I^\complement : i > j\}|} & \text{if } J = I^\complement, \\ 0 & \text{otherwise.} \end{cases}$$

This form is $\mathrm{G}(V)$-invariant,

$$\Phi_S(gx, gy) = \Phi_S(x, y), \qquad g \in \mathrm{G}(V),\ x, y \in S,$$

and hence yields a homomorphism of $\mathrm{G}(V)$-modules

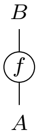
$$S \otimes S \to \text{trivial module}.$$

It is also (skew-)symmetric,

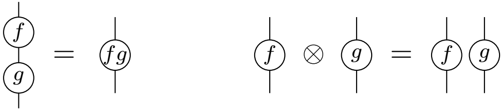
$$\Phi_S(x, y) = (-1)^{\binom{n}{2} + nN} \Phi_S(y, x).$$

String diagrams

In a strict monoidal category, we will denote a morphism $f\colon A \to B$ by:



Composition is vertical stacking and tensor product is horizontal juxtaposition:



The Brauer category

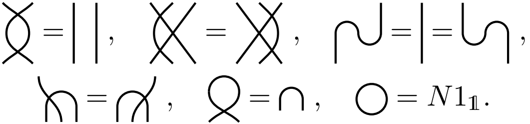
Fix $N \in \mathbb{C}$. The Brauer category $\mathcal{B}(N)$ is the strict linear monoidal category defined as follows.

One generating object: B

Three generating morphisms:

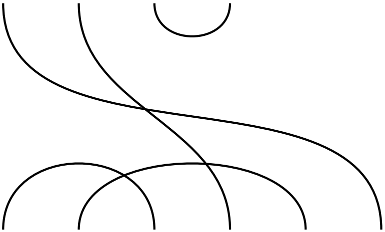
 $\colon \mathbb{1} \to \mathsf{B}^{\otimes 2}$,  $\colon \mathsf{B}^{\otimes 2} \to \mathbb{1}$,  $\colon \mathsf{B}^{\otimes 2} \to \mathsf{B}^{\otimes 2}$.

Relations:



The Brauer category

An arbitrary morphism in $\mathcal{B}(N)$ is a linear combination of Brauer diagrams. E.g.



$: \mathsf{B}^{\otimes 6} \to \mathsf{B}^{\otimes 4}$.

Composition: vertical "gluing", replace closed components by a factor of N.

Tensor product: horizontal juxtaposition.

The incarnation functor

Universal property

Any linear symmetric monoidal category $\mathcal{C}$ with a (skew-)symmetrically self-dual object V of dimension N admits a linear monoidal functor

$$\mathcal{B}(N) \to \mathcal{C}, \quad \mathsf{B} \mapsto V.$$

Corollary

We have a linear monoidal incarnation functor

$$\mathcal{B}(N) \to \mathrm{O}(N)\text{-mod}, \qquad \mathsf{B} \mapsto V = \text{natural module}.$$

$$\times \mapsto \left(V^{\otimes 2} \to V^{\otimes 2}, \quad v \otimes w \mapsto w \otimes v\right),$$
$$\cap \mapsto \left(V^{\otimes 2} \to \mathbb{C}, \quad v \otimes w \mapsto \langle v, w \rangle\right),$$
$$\cup \mapsto \left(\mathbb{C} \to V^{\otimes 2}, \quad 1 \mapsto \textstyle\sum_{v \in \mathbf{B}_V} v \otimes v\right),$$

where $\langle\ ,\ \rangle$ is the bilinear form, and $\mathbf{B}_V$ is an orthonormal basis.

Properties of the incarnation functor

The incarnation functor is full. In particular, we have a surjection

$$\mathrm{B}_r(N) \cong \mathrm{End}_{\mathcal{B}(N)}(\mathsf{B}^{\otimes r}) \twoheadrightarrow \mathrm{End}_{\mathrm{O}(N)}(V^{\otimes r}),$$

where $\mathrm{B}_r(N)$ is the Brauer algebra.

Fact

Every f.d. $\mathrm{O}(N)$-module is a summand of $V^{\otimes r}$ for some r.

The induced incarnation functor

$$\mathrm{Kar}(\mathcal{B}(N)) \to \mathrm{O}(N)\text{-mod}$$

is full and essentially surjective. Its kernel is the tensor ideal of negligible morphisms.

The category $\mathrm{Kar}(\mathcal{B}(N))$ is Deligne's interpolating category, defined for any value of N, even $N \notin \mathbb{Z}$.

What's missing?

Pulling back representations along the homomorphism

$$\mathrm{Pin}(V) \twoheadrightarrow \mathrm{O}(V)$$

yields a functor

$$\mathrm{O}(V)\text{-mod} \to \mathrm{Pin}(V)\text{-mod}.$$

Thus, we have a functor

$$\mathcal{B}(N) \to \mathrm{Pin}(V)\text{-mod}.$$

Problem: The above functor is no longer essentially surjective.

What's missing? The spin module!

Goal

Enlarge the Brauer category so that the corresponding incarnation functor to $\mathrm{Pin}(V)$-mod is essentially surjective. Add the spin module!

The spin Brauer category

Fix $d, D \in \Bbbk$ and $\kappa \in \{\pm 1\}$.

The spin Brauer category $\mathcal{SB}(d, D; \kappa)$ is the strict $\Bbbk$-linear monoidal category presented as follows.

The generating objects are S and V, with identity morphisms

 $:= 1_{\mathsf{S}}$, $:= 1_{\mathsf{V}}$.

The generating morphisms are

$\colon \mathsf{S} \otimes \mathsf{S} \to \mathbb{1}$, $\colon \mathbb{1} \to \mathsf{S} \otimes \mathsf{S}$,

$\colon \mathsf{V} \otimes \mathsf{V} \to \mathbb{1}$, $\colon \mathbb{1} \to \mathsf{V} \otimes \mathsf{V}$,

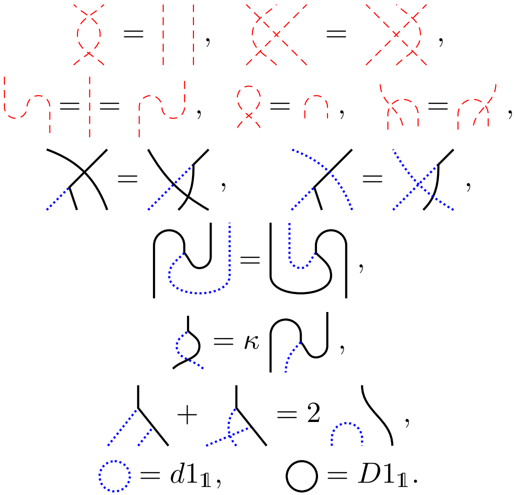
$\colon \mathsf{S} \otimes \mathsf{S} \to \mathsf{S} \otimes \mathsf{S}$, $\colon \mathsf{V} \otimes \mathsf{V} \to \mathsf{V} \otimes \mathsf{V}$,

$\colon \mathsf{V} \otimes \mathsf{S} \to \mathsf{S} \otimes \mathsf{V}$, $\colon \mathsf{S} \otimes \mathsf{V} \to \mathsf{V} \otimes \mathsf{S}$,

$\colon \mathsf{V} \otimes \mathsf{S} \to \mathsf{S}$.

The spin Brauer category

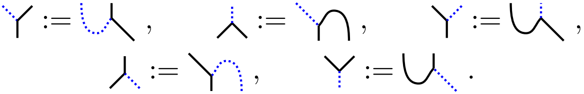
The defining relations on morphisms are:



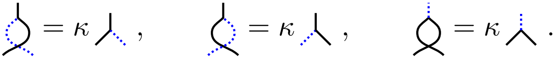
Here, the dashed red strands denote either $|$ or $\vdots$.

The spin Brauer category

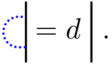
We introduce other trivalent morphisms by successive rotation:



Then one can show that



We can also show that



The incarnation functor: ingredients

Fix an inner product space (V, Φ_V) of finite dimension N, and let

$$n = \left\lfloor \frac{N}{2} \right\rfloor, \qquad \text{so that} \quad N = 2n \quad \text{or} \quad N = 2n+1.$$

Recall

$$\mathrm{G}(V) := \begin{cases} \mathrm{Pin}(V) & \text{if } N \text{ is even,} \\ \mathrm{Spin}(V) & \text{if } N \text{ is odd.} \end{cases}$$

Let

$$\sigma_N := (-1)^{\binom{n}{2} + nN} \qquad \text{and} \qquad \kappa_N := (-1)^{nN},$$
$$\mathcal{SB}(V) := \mathcal{SB}(\underbrace{N}_{d}, \underbrace{\sigma_N 2^n}_{D}; \kappa_N).$$

(Recall that σ_N is the sign describing the symmetry of the form Φ_S.)

The incarnation functor: ingredients

Fix a basis $\mathbf{B}_S$ of S, and let $\mathbf{B}_S^\vee = \{x^\vee : x \in \mathbf{B}_S\}$ denote the left dual basis with respect to Φ_S, defined by

$$\Phi_S(x^\vee, y) = \delta_{x,y}, \qquad x, y \in \mathbf{B}_S.$$

We fix a basis $\mathbf{B}_V$ of V and define the left dual basis $\mathbf{B}_V^\vee = \{v^\vee : v \in V\}$ similarly.

Then we have $\mathrm{G}(V)$-module homomorphisms

$$\Phi_S^\vee \colon \mathbb{C} \to S \otimes S, \qquad \lambda \mapsto \lambda \sum_{x \in \mathbf{B}_S} x \otimes x^\vee, \quad \lambda \in \mathbb{C},$$

$$\Phi_V^\vee \colon \mathbb{C} \to V \otimes V, \qquad \lambda \mapsto \lambda \sum_{v \in \mathbf{B}_V} v \otimes v^\vee, \quad \lambda \in \mathbb{C}.$$

Let

$$\tau \colon V \otimes S \to S, \quad v \otimes x \mapsto vx,$$

denote the homomorphism of $\mathrm{G}(V)$-modules induced by multiplication in the Clifford algebra $\mathrm{Cl}(V)$.

The incarnation functor

Theorem (McNamara–S.)

There is a unique monoidal functor

$$\mathbf{F}\colon \mathcal{SB}(V) \to \mathrm{G}(V)\text{-mod}$$

given on objects by S $\mapsto S$, V $\mapsto V$, and on morphisms by

 $\mapsto \Phi_S,$  $\mapsto \Phi_V,$  $\mapsto \tau,$

 $\mapsto \sigma_N \,\mathrm{flip}_{S,S},$  $\mapsto \mathrm{flip}_{S,V},$  $\mapsto \mathrm{flip}_{V,S},$  $\mapsto \mathrm{flip}_{V,V}\,.$

Furthermore, we have

 $\mapsto \Phi_S^\vee,$  $\mapsto \Phi_V^\vee.$

We call $\mathbf{F}$ the incarnation functor.

Properties of the incarnation functor

Theorem (McNamara–S.)

1. The functor $\mathbf{F}$ is full.
2. After passing to the Karoubi envelope, $\mathbf{F}$ is essentially surjective.
3. When N is even, the kernel of $\mathbf{F}$ is the tensor ideal of negligible morphisms. Thus, $\mathrm{G}(V)$-mod is equivalent to the semisimplification of $\mathrm{Kar}(\mathcal{SB}(V))$.
4. When N is odd, the same is true if we impose one additional relation in $\mathcal{SB}(V)$.

More incarnation!

There exist other possible incarnation functors:

$$\mathcal{SB}(N, \sigma_N(m-2k)2^n; \kappa_N) \to (\mathrm{G}(V) \times \mathrm{OSp}(m|2k))\text{-mod},$$
$$\mathsf{S} \mapsto S \otimes W, \qquad \mathsf{V} \mapsto V,$$

where W is the natural $\mathrm{OSp}(m|2k)$-supermodule.

The extra relation: motivation

Case D_n: $N = 2n$

$$\Lambda^0(V), \quad \Lambda^1(V), \; \dots \;, \quad \Lambda^N(V)$$

are pairwise nonisomorphic $\mathrm{Pin}(V)$-modules. In particular,

$$\Lambda^N(V) \text{ is not the trivial module.}$$

Case B_n: $N = 2n + 1$

$$\Lambda^k(V) \cong \Lambda^{N-k}(V) \quad \text{as } \mathrm{Pin}(V)\text{-modules}, \quad 0 \le k \le N.$$

In particular,

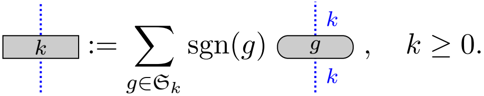
$$\Lambda^N(V) \text{ is the trivial module.}$$

Problem: We don't seem to have an diagrammatic isomorphism corresponding to the isomorphism

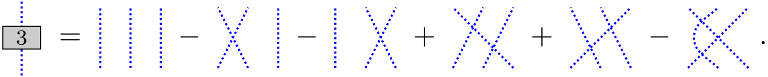
$$\Lambda^N(V) \cong \text{trivial module.}$$

The extra relation: antisymmetrizer

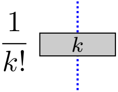
Define the antisymmetrizer



For example,



Under the incarnation functor,



corresponds to the projection

$$V^{\otimes k} \twoheadrightarrow \Lambda^k(V).$$

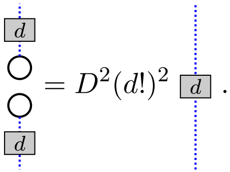
The extra odd relation

Suppose $d = N \in 2\mathbb{N} + 1$.

Corresponding to the isomorphism

$$\Lambda^N(V) \cong \text{trivial module},$$

we add in the relation



We seem to need this relation in order to evaluate closed diagrams.

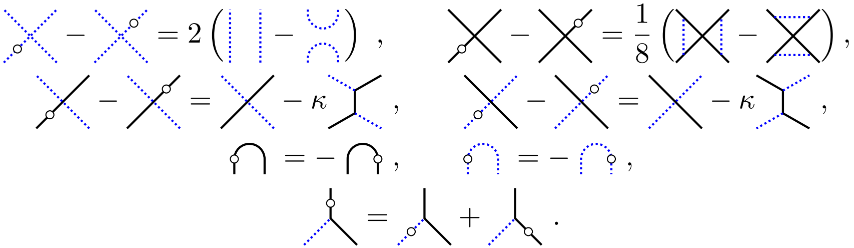
This relation is respected by the incarnation functor.

The affine spin Brauer category

There exists an affine spin Brauer category $\mathcal{ASB}(d, D; \kappa)$, obtained from the spin Brauer category by adding morphisms

: $\mathsf{S} \to \mathsf{S}$, : $\mathsf{V} \to \mathsf{V}$,

subject to the relations.



The affine incarnation functor

Then we have an affine incarnation functor

$$\mathcal{ASB}(V) := \mathcal{ASB}(N, \sigma_N 2^n; \kappa_N) \to \mathcal{End}(\mathrm{G}(V)\text{-mod}),$$
$$\mathsf{S} \mapsto S \otimes -, \qquad \mathsf{V} \mapsto V \otimes -.$$

The dots are sent to natural transformations given by multiplication by

$$\Delta(C) - 1 \otimes C,$$

where $C \in U(\mathfrak{so}(V))$ is the Casimir element and Δ is the usual comultiplication on $U(\mathfrak{so}(V))$.

This induces an algebra homomorphism

$$\mathrm{End}_{\mathcal{ASB}(V)}(\mathbb{1}) \twoheadrightarrow \mathcal{End}(\mathrm{id}) \cong Z(U(\mathfrak{so}(V)))$$

whose image is $Z(U(\mathfrak{so}(V)))^{\mathrm{G}(V)}$.

Further directions

Basis theorem

It would be nice to describe an explicit basis for the morphism spaces of the spin Brauer category and the affine spin Brauer category.

Description of the kernel

We know the kernel of the incarnation functor is the tensor ideal of negligible morphisms.

It would be nice to find explicit generators for this tensor ideal.

Such a description is known for the Brauer category.

Quantum version

Kauffman skein category

The Kauffman skein category is a quantum version of the Brauer category.

Its endomorphism algebras are BMW algebras.

There is a natural functor to $U_q(\mathfrak{so}(N))$-mod.

But this functor is not essentially surjective; it misses the quantum spin module.

There should exist a quantum version of the spin Brauer category—a spin Kauffman skein category.