Interpolation categories for finite linear groups

Thorsten Heidersdorf

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Newcastle University

Representation stability for S_n: S_∞

Let $S_\infty = \cup_n S_n$.

Let $\mathrm{Rep}(S_\infty)$ be the category of algebraic representations of S_∞ in the sense of Sam and Snowden: All subquotients of direct sums of tensor powers of $\mathbb{C}^\infty$ (the permutation representation of S_∞) .

This is a symmetric monoidal (SM) abelian category generated by $\mathbb{C}^\infty$.

Question. How does this relate to Deligne's $Rep(S_t)$?

To relate these two we have to replace $Rep(S_t)$ by its abelian envelope if $t \in \mathbb{N}$.

Abelian envelopes

Let I be a k-linear SM functor from an additive karoubian k-linear rigid symmetric monoidal category $\mathcal{C}$ to a tensor category $\mathcal{V}$ over k.

Definition: A pair $(\mathcal{V}, I : \mathcal{C} \to \mathcal{V})$ as above is an abelian envelope of $\mathcal{C}$ if for any k-linear tensor category $\mathcal{A}$ the functor

$$- \circ I : \; Fun^{ex}(\mathcal{V}, \mathcal{A}) \; \to \; Fun^{faith}(\mathcal{C}, \mathcal{A})$$

is an equivalence of categories between

- $Fun^{ex}(\mathcal{V}, \mathcal{A})$, the category of exact SM k-linear functors $\mathcal{V} \to \mathcal{A}$,
- $Fun^{faith}(\mathcal{C}, \mathcal{A})$, the category of faithful SM k-linear functors $\mathcal{C} \to \mathcal{A}$.

Example. The category $Rep(S_t)$ has an abelian envelope $Rep(S_t)^{ab}$ (Comes-Ostrik).

Abelian envelopes II

Why we care about abelian envelopes for Deligne categories:

- Satisfy again some universal property.
- Richer structure, e.g. highest weight category for $Rep(S_t)^{ab}$.
- Connection to other settings of stable representation theory, e.g. $Rep(GL_\infty(\mathbb{F}_q))$.
- Extension property very useful

Existence criteria and explicit constructions:

- Some ad-hoc constructions, e.g. by Comes-Ostrik for $Rep(S_t)$.
- Coulembier: General existence criteria for abelian envelopes, but difficult to verify in practice
- Benson-Etingof-Ostrik: More restrictive criteria, but yield (if possible to verify) more explicit description of the hypothetical envelope.

Connection to representation stability for S_t

Theorem. [Barter-Entova-Aizenbud-H.] There is a $\mathbb{C}$-linear symmetric monoidal functor $\mathbf{\Gamma}_t : \mathrm{Rep}(S_\infty) \to \mathrm{Rep}^{ab}(S_t) \hookleftarrow Rep(S_t)$.

- The functor $\mathbf{\Gamma}_t$ is faithful and exact.
- The functor $\mathbf{\Gamma}_t$ takes simple objects in $\mathrm{Rep}(S_\infty)$ to standard objects in the highest-weight category $\mathrm{Rep}^{ab}(S_t)$, and injective objects to tilting objects (these are precisely the objects coming from the Deligne category $\mathrm{Rep}(S_t)$).

Aim. Develop a similar theory for complex representations of $GL_n(\mathbb{F}_q)$ (joint work with Inna Entova-Aizenbud) and possibly $O_n(\mathbb{F}_q)$ and $Sp_{2n}(\mathbb{F}_q)$.

Towards finite linear groups

One recipe to build a Deligne category for some infinite family of groups G_n like $GL_n(\mathbb{F}_q)$:

- Choose your favorite faithful representation V_n for some fixed G_n.
- Need an explicit description of $End(V_n^{\otimes r})$ for all r.
- Should expect that there is some nice algebra $A_r(n)$ such that (i) $A_r(n)$ always surjects onto $End(V_n^{\otimes r})$ and is an iso for r small compared n.
- The composition rule in $A_r(n)$ should depend polynomially on n so that $A_r(t), t \in \mathbb{C}$, also makes sense.
- Start with a skeletal subcategory of objects $[0], [1], [2], \ldots$ and define $End([r], [r]) = A_r(t)$.
- Take the additive idempotent completion of this skeletal subcategory.

Follow this recipe for $GL_n(\mathbb{F}_q)$

Analog of the permutation representation: $\mathbf{V} = \mathbb{C}\mathbb{F}_q^n$, $V = \mathbb{F}_q^n$ and hence

$$\mathbf{V}_n^{\otimes s} = \mathbb{C}V^{\times s}.$$

Notation: For sums over $\mathbb{F}_q$ we write $\dot{+}$ instead of $+$. For $v_1, \dots, v_s \in V$, we write $(v_1|\dots|v_s) \in V^{\times s}$. Morphisms from $\mathbf{V}_n^{\otimes s}$ to $\mathbf{V}_n^{\otimes k}$:

Let $s, k \in \mathbb{Z}_{\geq 0}$. Let $R \subset \mathbb{F}_q^{s+k} = \mathbb{F}_q^s \times \mathbb{F}_q^k$ be a linear $\mathbb{F}_q$-subspace. We define a G-invariant subspace

$$R^\perp = \{(v_1|\dots|v_{s+k}) \in V^{\times(s+k)} \mid \forall u = (u_1, \dots, u_{s+k}) \in R,\ \dot{\sum_i} u_i v_i = 0\}$$

in $V^{\times(s+k)}$. This allows us to define a map

$$f_R : \mathbf{V}_n^{\otimes s} \to \mathbf{V}_n^{\otimes k},\quad v_1 \otimes \dots \otimes v_s \mapsto \sum_{\substack{(w_1|\dots|w_k) \in V^{\times k} \\ (v_1|\dots|v_s|w_1|\dots|w_k) \in R^\perp}} w_1 \otimes \dots \otimes w_k.$$

Examples of morphisms f_R

Consider the following morphisms ($v, w \in V$):

1. Morphisms

$$\varepsilon := f_{\{\dot{0}\}} : \mathbf{1} \to \mathbf{V}_n, \ , 1 \mapsto \sum_{v \in V} v,$$

$$\varepsilon^* := f_{\{\dot{0}\}} : \mathbf{V}_n \to \mathbf{1}, \ , v \to 1 \ ,$$

$$m := f_{\{(a+b,-a,-b) | a,b \in \mathbb{F}_q\}} : \mathbf{V}_n \otimes \mathbf{V}_n \to \mathbf{V}_n, \ \ v \otimes w \mapsto \delta_{v,w} v \ ,$$

$$m^* := f_{\{(a+b,-a,-b) | a,b \in \mathbb{F}_q\}} : \mathbf{V}_n \to \mathbf{V}_n \otimes \mathbf{V}_n, \ \ v \mapsto v \otimes v \ ,$$

$$\sigma := f_{\{(a,b,-b,-a) | a,b \in \mathbb{F}_q\}} : \mathbf{V}_n \otimes \mathbf{V}_n \to \mathbf{V}_n \otimes \mathbf{V}_n, \ \ v \otimes w \mapsto w \otimes v \ .$$

2. Morphisms

$$z := f_{\mathbb{F}_q^1} : \mathbf{1} \to \mathbf{V}_n, \ \ 1 \mapsto \dot{0},$$

$$\forall a \in \mathbb{F}_q, \ \mu_a := f_{\{(-ab,b) | b \in \mathbb{F}_q\}} : \mathbf{V}_n \to \mathbf{V}_n, \ \ v \mapsto \dot{a} v \quad \text{for } v \in V,$$

$$\dot{+} := f_{\{(b,b,-b) | b \in \mathbb{F}_q\}} : \mathbf{V}_n \otimes \mathbf{V}_n \to \mathbf{V}_n, \ \ v \otimes w \mapsto v \dot{+} w \quad \text{for } v, w \in V.$$

Generating morphism

Interpretation:

- The morphisms $\mu_a, \dot{+}$ give $\mathbf{V}_n$ the structure of vector space over $\mathbb{F}_q$, with $z : \mathbf{1} \to \mathbf{V}_n$ defining the zero vector $\dot{0} \in \mathbf{V}_n$;
- The rest of the morphisms make $\mathbf{V}_n$ into a commutative Frobenius algebra in $\mathrm{Rep}(GL_n(\mathbb{F}_q))$. The algebra is self-dual, via the pairings

$$ev := f_{\{(a,-a) | a \in \mathbb{F}_q\}} : \mathbf{V}_n \otimes \mathbf{V}_n \to \mathbf{1}, \;\; v \otimes w \mapsto \delta_{v,w} \quad \text{for } v, w \in V,$$

$$coev := f_{\{(a,-a) | a \in \mathbb{F}_q\}} : \mathbf{1} \to \mathbf{V}_n \otimes \mathbf{V}_n, \;\; 1 \mapsto \sum_{v \in V} v \otimes v.$$

where

$$ev = \varepsilon^* \circ m, \;\; coev = m^* \circ \varepsilon.$$

Towards a Deligne category

Lemma. [Entova-Aizenbud-H.] The set $\{f_R | R \subset \mathbb{F}_q^{s+k}\}$ spans the space $Hom_G(\mathbf{V}_n^{\otimes s}, \mathbf{V}_n^{\otimes k})$. Furthermore, if $n \geq s + k$ then this set is a basis of $Hom_G(\mathbf{V}_n^{\otimes s}, \mathbf{V}_n^{\otimes k})$. The composition $f_S \circ f_R : \mathbf{V}_n^{\otimes s} \to \mathbf{V}_n^{\otimes l}$ is polynomial in q^n (and equals

$$q^{n \cdot d(R,S)} f_{S \star R}$$

for some explicitely given $d(R, S)$).

Define $Rep(GL_t(\mathbb{F}_q))$

Define the skeletal subcategory $\mathcal{T}(\underline{GL_t})$: Objects $[k]$, $k \in \mathbb{N}$. Define

$$Rel_{s,k} = \{R \subset \mathbb{F}_q^{s+k} \text{ linear subspace } \}, \quad Hom_{\mathcal{T}(\underline{GL_t})}([s],[k]) = \mathbb{C}Rel_{s,k}$$

Composition: for $R \in Rel_{s,k}$, $S \in Rel_{k,l}$ we set

$$S \circ R := t^{d(R,S)} S \star R.$$

Monoidal structure of $\mathcal{T}(\underline{GL_t})$: put $[l] \otimes [k] := [l+k]$.

We define $Rep(GL_t(\mathbb{F}_q))$ as the additive Karoubi envelope of $\mathcal{T}(\underline{GL_t})$.

Results by Knop

Knop: Defined $Rep(GL_t(\mathbb{F}_q))$ (but differently). Proved:

1. $Rep(GL_t(\mathbb{F}_q))$ is abelian and semisimple iff $t \neq q^n$ for some $n \in \mathbb{N}$.
2. For $t = q^n$ there is a specialization functor

$$F_{\mathbf{V}} : Rep(GL_t(\mathbb{F}_q)) \to Rep(GL_n(\mathbb{F}_q))$$

   (the semisimplification functor).
3. For $t \neq q^n$, the simple objects are up to isomorphism in bijection with $\bigcup_n Irr_n$ where Irr_n are the iso classes of irreducible $GL_n(\mathbb{F}_q)$-representations.

$\mathbb{F}_q$-linear Frobenius spaces I

Let $\mathcal{C}$ be a $\mathbb{C}$-linear rigid SM category.

Definition. Let $\mathbf{V} \in \mathcal{C}$ be an object equipped with the following structures:

- $\mathbf{V}$ is equipped with the structure of a Frobenius algebra object in $\mathcal{C}$. That is, $\mathbf{V}$ is equipped with maps $m : \mathbf{V}^{\otimes 2} \to \mathbf{V}$, $\varepsilon : \mathbf{1} \to \mathbf{V}$, $m^* : \mathbf{V} \to \mathbf{V}^{\otimes 2}$, $\varepsilon^* : \mathbf{V} \to \mathbf{1}$ such that:
  - It is a commutative unital algebra object with multiplication m and unit ε, and a cocommutative counital coalgebra object with comultiplication m^* and counit ε^*,
  - Frobenius Relations: $m^* \circ m = (id \otimes m) \circ (m^* \otimes id) = (m \otimes id) \circ (id \otimes m^*)$, and Speciality Relation: $m \circ m^* = id$.

$\mathbb{F}_q$-linear Frobenius spaces II

- $\mathbf{V}$ is a module over the field $\mathbb{F}_q$: $\mathbf{V}$ is equipped with maps

$$\dot{+} : \mathbf{V} \otimes \mathbf{V} \to \mathbf{V},$$
$$\mu : (\mathbb{F}_q, \cdot) \to (End_{\mathcal{C}}(\mathbf{V}), \circ), \quad a \mapsto \mu_a,$$
$$z : \mathbf{1} \to \mathbf{V}$$

  satisfying the following conditions:

  - $\dot{+}$ is associative and commutative: $\dot{+} \circ (\dot{+} \otimes id) = \dot{+} \circ (id \otimes \dot{+})$, $\dot{+} \circ \sigma = \dot{+}$ where $\sigma \in End(\mathbf{V} \otimes \mathbf{V})$ is the symmetry morphism.
  - z serves as "the embedding of the $\dot{0}$ vector": $\dot{+} \circ (z \otimes id) = id, \dot{+} \circ (id \otimes z) = id,$
  - For all $a, b \in \mathbb{F}_q$, $\mu_a \circ \mu_b = \mu_{ab}$ and $\mu_1 = id$, $\mu_0 = z \circ \varepsilon^*$.
  - Linearity of μ_a with respect to $\dot{+}$: for any $a, b \in \mathbb{F}_q$, $\mu_{a+b} = \dot{+} \circ (\mu_a \otimes \mu_b) \circ m^*$.
    Distributivity of μ_a: for any $a \in \mathbb{F}_q$, $\mu_a \circ \dot{+} = \dot{+} \circ (\mu_a \otimes \mu_a)$.

Universal property

Assume furthermore that the above structures satisfy some compatibility relations. Such an object $\mathbf{V}$ is called an $\mathbb{F}_q$-linear Frobenius space in $\mathcal{C}$.

Example. The representation $\mathbf{V}_n$ in $Rep(GL_n(\mathbb{F}_q))$. Dito $\mathbf{V}_n$ seen as a representation of $O_n(\mathbb{F}_q)$, $Sp_{2n}(\mathbb{F}_q)$, ...

Theorem. [Entova-Aizenbud-H.] Let $\mathcal{C}$ be a Karoubi additive rigid SM category, and let $\mathbf{V}$ be an $\mathbb{F}_q$-linear Frobenius space in $\mathcal{C}$. Let $t = \dim(\mathbf{V})$. Then there exists a SM functor

$$F_{\mathbf{V}} : Rep(GL_t(\mathbb{F}_q)) \to \mathcal{C}, \ \ \mathbf{V}_t \longmapsto \mathbf{V}$$

which is unique up to isomorphism.

The infinite case

Let $V := \mathbb{F}_q^\infty = \bigcup_{n\neq 0} \mathbb{F}_q^n$. We denote by $\mathbf{V}_\infty$ the representation $\mathbb{C}\mathbb{F}_q^\infty$ of $GL_\infty(\mathbb{F}_q)$.; this is the countable-dimensional vector space consisting of infinite sequences of elements in $\mathbb{F}_q$ which have finite support.

Let $\mathcal{I}_\infty \subset Rep(GL_\infty(\mathbb{F}_q))$ denote the full subcategory of direct summands in tensor powers of $\mathbb{C}\mathbb{F}_q^\infty$. Let $Rep(GL_\infty(\mathbb{F}_q))$ denote the category of algebraic representations of $GL_\infty(\mathbb{F}_q)$: all subquotients of tensor powers of $\mathbb{C}\mathbb{F}_q^\infty$.

$\mathcal{I}_\infty$ does not have a good notion of duality anymore. In particular $\mathbb{C}\mathbb{F}_q^\infty$ is no longer an $\mathbb{F}_q$-linear Frobenius space.

Theorem. [Entova-Aizenbud-H.] There is a non-full embedding of $\mathcal{I}_\infty \to Rep(GL_t(\mathbb{F}_q))$. The category $\mathcal{I}_\infty$ is universal with respect to a semi-$\mathbb{F}_q$-linear Frobenius space (essentially an $\mathbb{F}_q$-linear Frobenius space without a unit).

Remark. The additive envelope of $\mathcal{I}_\infty$ can be identified with the full subcategory of injective objects in $Rep(GL_\infty(\mathbb{F}_q))$.

Abelian envelopes for $Rep(GL_t(\mathbb{F}_q))$ [careful: work in progress]

Harman-Snowden: Construction of the abelian envelope for $t = q^n$ via oligomorphic groups.

Entova-Aizenbud-H: another description of the abelian envelope of the form $C - Comod$, the category of finite-dimensional C-comodules for some coalgebra C (a special case of the Benson-Etingof-Ostrik construction).

Follows loosely the approach by Comes-Ostrik in the $Rep(S_t)$-case.

Splitting objects and a construction of Benson-Etingof-Ostrik

Let $\mathcal{T}$ denote Karoubi rigid SM category. Recall that a morphism $f : X \to Y$ is split if it is the composition $i \circ \pi$ where π is a split epimorphism and i is a split monomorphism.

Definition. An object $X \in \mathcal{T}$ is a splitting object if for any $Q_1, Q_2 \in \mathcal{T}$ and a morphism $f : Q_1 \to Q_2$ the morphisms $1_X \otimes f : X \otimes Q_1 \to X \otimes Q_2$ and $f \otimes 1_X : Q_1 \otimes X \to Q_2 \otimes X$ are split.

Splitting objects form a thick tensor ideal $\mathcal{S}$. Let $\mathbf{I}$ denote the set of isomorphism classes of indecomposable objects $\{P_i\}_{i\in\mathbf{I}}$ in $\mathcal{S}$. Define the coalgebra

$$C := \bigoplus_{i,j\in\mathbf{I}} Hom(P_i, P_j)^*.$$

and denote by $\mathcal{C} = C - Comod$ the category of finite-dimensional C-comodules, i.e. the category of additive functors $\mathcal{S}^{op} \to Vec$.

The abelian envelope

Theorem. [Entova-Aizenbud-H.] For $Rep(GL_t(\mathbb{F}_q))$ we have $\mathcal{S} = \mathcal{N}$, the thick tensor ideal of negligible objects, and $\mathcal{C} = C - Comod$ is the abelian envelope of $Rep(GL_t(\mathbb{F}_q))$.

Corollary. Universal property: Let $\mathcal{C}$ be a pre-tannakian category and let V be an $\mathbb{F}_q$-linear Frobenius space in $\mathcal{C}$.

(i) If V is annihilated by some exterior power and $\dim(V) = q^n$, the functor $F_V : Rep(GL_{q^n}(\mathbb{F}_q)) \to \mathcal{C}$ factors through the specialization functor

$$Rep(GL_{t=q^n}(\mathbb{F}_q)) \to Rep(GL_n(\mathbb{F}_q)).$$

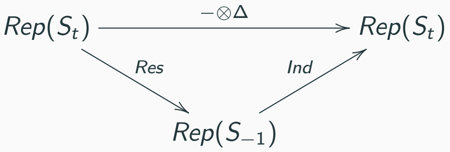
(ii) If V is not annihilated by any exterior power, then F_V factors through the canonical embedding

$$Rep(GL_{q^n}(\mathbb{F}_q)) \to Rep^{ab}(GL_{q^n}(\mathbb{F}_q)).$$

Key idea: the S_t-case

For S_n we can define $\Delta_k^n = \mathbb{C}Inj(\{1,\dots,k\},\{1,\dots,n\}) = Ind_{S_{n-k}}^{S_n}\mathbf{1}$ for $k \leq n$.

These have analogs Δ_k, $k \in \mathbb{N}$ in $Rep(S_t)$. For $Rep(S_{t=n})$ define $\Delta = \Delta_{n+1}$. Then there is a factorization (Deligne, Comes-Ostrik)



Since $Rep(S_{-1})$ is semisimple, Δ is a splitting object.

The $Rep(GL_t(\mathbb{F}_q))$-case

By $P_{k,n}$ we denote the parabolic subgroup of all matrices $\begin{bmatrix} C & A \\ 0 & B \end{bmatrix} \in GL_n(\mathbb{F}_q)$ where $C \in GL_k(\mathbb{F}_q), B \in GL_{n-k}(\mathbb{F}_q)$ and $A \in Mat_{k\times(n-k)}(\mathbb{F}_q)$.

By $H_{k,n} \subset P_{k,n}$ we denote the mirabolic subgroup of all matrices as above for which C is the identity matrix of size $k \times k$.

Consider the left regular representation $\mathbb{C}[GL_k(\mathbb{F}_q)]$ of $GL_k(\mathbb{F}_q)$. We inflate it to a representation of $P_{k,n}$ by requiring that $H_{k,n}$ acts trivially. We denote:

$$\Delta_k^n := \mathbb{C}Inj_{\mathbb{F}_q}(\mathbb{F}_q^k, \mathbb{F}_q^n) = Ind_{P_{k,n}}^{GL_k(\mathbb{F}_q)\times GL_n(\mathbb{F}_q)} \mathbb{C}[GL_k(\mathbb{F}_q)].$$

This is a representation of $GL_k(\mathbb{F}_q) \times GL_n(\mathbb{F}_q)$. If we restrict the action just to $GL_n(\mathbb{F}_q)$, this becomes

$$\Delta_k^n|_{GL_n(\mathbb{F}_q)} \cong Ind_{H_{k,n}}^{GL_n(\mathbb{F}_q)} \mathbf{1}.$$

Knop's machinery

Can construct analogs Δ_k, $k \in \mathbb{N}$, in $Rep(GL_t(\mathbb{F}_q))$.

Suggests that Δ_k for $Rep(GL_t(\mathbb{F}_q)$ should be induced from a parabolic version of the Deligne category.

This can be done via a general machinery developed by Knop to construct interpolating categories.

Fix $k \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{A}_k$ denote the category whose objects are pairs (V, p_V) where $V \in \mathrm{Vect}_{\mathbb{F}_q}$ and $p_V : V \twoheadrightarrow \mathbb{F}_q^k$ be a surjective $\mathbb{F}_q$-linear map. The morphisms in this category are given by

$$Mor_{\mathcal{A}_k}((V, p_V), (W, p_W)) = \{f \in Hom_{\mathbb{F}_q}(V, W) \,:\, f \circ p_W = p_V\}.$$

Knop's construction gives a family of karoubian SM categories $\mathcal{T}(\mathcal{A}_k, t)$ for $t \in \mathbb{C}$.

The category $\mathcal{T}(\mathcal{A}_k, t)$ is semisimple whenever $t \notin q^{\mathbb{Z}_{\geq k}}$.

The analog of $Rep(S_{-1})$

For $t = q^N$ where $N \in \mathbb{Z}_{\geq k}$, we have a full, essentially surjective SM functor

$$F_N : \mathcal{T}(\mathcal{A}_k, t) \to \mathrm{Rep}(H_{k,N}^{tr})$$

where

$$H_{k,N}^{tr} = \left\{ \begin{bmatrix} 1_k & 0 \\ A & B \end{bmatrix} \in GL_n(\mathbb{F}_q) \right\}$$

Since $H_{k,N}^{tr}$ is obtained from $H_{k,N}$ by the automorphism $X \mapsto (X^{-1})^{tr}$ of $GL_n(\mathbb{F}_q)$, we have $\mathrm{Rep}(H_{k,N}) \cong \mathrm{Rep}(H_{k,N}^{tr})$, so the family $\mathcal{T}(\mathcal{A}_k, t)$, $t \in \mathcal{C}$ interpolates" the categories $\mathrm{Rep}(H_{k,N})$ for $t = q^N$.

Splitting property

Fix $t \in \mathbb{C}$. We have a functor

$$Res_{t,k} : \mathrm{Rep}(GL_t(\mathbb{F}_q)) \longrightarrow \mathcal{T}(\mathcal{A}_k, t)$$

given by the universal property of $\mathrm{Rep}(GL_t(\mathbb{F}_q))$.

Key lemma. Δ_{n+1} is a projective object in $Prshv(Rep(GL_t(\mathbb{F}_q))$ for $t = q^n$.

Proof. Show that there is an induction functor from the parabolic Deligne category to $Prshv(Rep(GL_t))$ which sends $\mathbf{1}$ to Δ_{n+1}, and this functor is left adjoint to an exact restriction functor, hence sends projectives to projectives. For $k = n + 1$, $\mathcal{T}(\mathcal{A}_k, t)$ is semisimple.

Open questions

- Develop the representation theory of $Rep(GL_\infty(\mathbb{F}_q))$ similarly to Sam-Snowden's work for S_∞. Some partial results by Nagpal.
- Understand the abelian envelope better (highest weight structure etc).
- We suspect that there are two different Deligne categories for the finite linear group, ours and a bigger one which should be generated by $\tilde{\Delta}$-objects induced from a Levi subgroup. We don't even know how to construct the latter diagrammatically.