Interpolation categories for finite linear groups

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Let $S_\infty = \bigcup_n S_n$.

Let $\text{Rep}(S_\infty)$ be the category of algebraic representations of $S_\infty$ in the sense of Sam and Snowden: All subquotients of direct sums of tensor powers of $C^\infty$ (the permutation representation of $S_\infty$).

This is a symmetric monoidal (SM) abelian category generated by $C^\infty$.

**Question.** How does this relate to Deligne’s $\text{Rep}(S_t)$?

To relate these two we have to replace $\text{Rep}(S_t)$ by its abelian envelope if $t \in \mathbb{N}$.
Abelian envelopes

Let $I$ be a $k$-linear SM functor from an additive karoubian $k$-linear rigid symmetric monoidal category $C$ to a tensor category $V$ over $k$.

**Definition:** A pair $(V, I : C \to V)$ as above is an abelian envelope of $C$ if for any $k$-linear tensor category $A$ the functor

$$- \circ I : \text{Fun}^{\text{ex}}(V, A) \to \text{Fun}^{\text{faith}}(C, A)$$

is an equivalence of categories between

- $\text{Fun}^{\text{ex}}(V, A)$, the category of exact SM $k$-linear functors $V \to A$,
- $\text{Fun}^{\text{faith}}(C, A)$, the category of faithful SM $k$-linear functors $C \to A$.

**Example.** The category $\text{Rep}(S_t)$ has an abelian envelope $\text{Rep}(S_t)^{ab}$ (Comes-Ostrik).
Abelian envelopes II

Why we care about abelian envelopes for Deligne categories:

- Satisfy again some **universal property**.
- Richer structure, e.g. **highest weight category** for \( \text{Rep}(S_t)^{ab} \).
- Connection to other settings of **stable representation theory**, e.g. \( \text{Rep}(GL_\infty(\mathbb{F}_q)) \).
- Extension property very useful

Existence criteria and explicit constructions:

- Some ad-hoc constructions, e.g. by Comes-Ostrik for \( \text{Rep}(S_t) \).
- Coulembier: **General existence criteria** for abelian envelopes, but difficult to verify in practice
- Benson-Etingof-Ostrik: More restrictive criteria, but yield (if possible to verify) **more explicit description** of the hypothetical envelope.
Theorem. [Barter-Entova-Aizenbud-H.] There is a $\mathbb{C}$-linear symmetric monoidal functor $\Gamma_t : \text{Rep}(S_{\infty}) \to \text{Rep}^{ab}(S_t) \leftrightarrow \text{Rep}(S_t)$.

- The functor $\Gamma_t$ is faithful and exact.
- The functor $\Gamma_t$ takes simple objects in $\text{Rep}(S_{\infty})$ to standard objects in the highest-weight category $\text{Rep}^{ab}(S_t)$, and injective objects to tilting objects (these are precisely the objects coming from the Deligne category $\text{Rep}(S_t)$).

Aim. Develop a similar theory for complex representations of $GL_n(\mathbb{F}_q)$ (joint work with Inna Entova-Aizenbud) and possibly $O_n(\mathbb{F}_q)$ and $Sp_{2n}(\mathbb{F}_q)$. 
Towards finite linear groups

One recipe to build a Deligne category for some infinite family of groups $G_n$ like $GL_n(\mathbb{F}_q)$:

- Choose your favorite **faithful representation** $V_n$ for some fixed $G_n$.
- Need an explicit description of $\text{End}(V_n^\otimes r)$ for all $r$.
- Should expect that there is some nice algebra $A_r(n)$ such that (i) $A_r(n)$ always surjects onto $\text{End}(V_n^\otimes r)$ and is an iso for $r$ small compared $n$.
- The composition rule in $A_r(n)$ should **depend polynomially** on $n$ so that $A_r(t)$, $t \in \mathbb{C}$, also makes sense.
- Start with a **skeletal subcategory** of objects $[0], [1], [2], \ldots$ and define $\text{End}([r], [r]) = A_r(t)$.
- Take the **additive idempotent completion** of this skeletal subcategory.
Follow this recipe for $GL_n(\mathbb{F}_q)$

Analog of the permutation representation: $V = \mathbb{C}\mathbb{F}_q^n$, $V = \mathbb{F}_q^n$ and hence

$$V_n^\otimes s = \mathbb{C}V^{\times s}.$$

Notation: For sums over $\mathbb{F}_q$ we write $\hat{+}$ instead of $+$. For $v_1, \ldots, v_s \in V$, we write $(v_1 \mid \ldots \mid v_s) \in V^{\times s}$. Morphisms from $V_n^\otimes s$ to $V_n^\otimes k$:

Let $s, k \in \mathbb{Z}_{\geq 0}$. Let $R \subset \mathbb{F}_{q}^{s+k} = \mathbb{F}_q^s \times \mathbb{F}_q^k$ be a linear $\mathbb{F}_q$-subspace. We define a $G$-invariant subspace

$$R^\perp = \left\{ (v_1 \mid \ldots \mid v_{s+k}) \in V^{\times (s+k)} \mid \forall u = (u_1, \ldots, u_{s+k}) \in R, \sum_{i} u_i v_i = 0 \right\}$$

in $V^{\times (s+k)}$. This allows us to define a map

$$f_R : V_n^\otimes s \to V_n^\otimes k, \quad v_1 \otimes \ldots \otimes v_s \mapsto \sum_{(w_1 \mid \ldots \mid w_k) \in V^{\times k}} \sum_{(v_1 \mid \ldots \mid v_s \mid w_1 \mid \ldots \mid w_k) \in R^\perp} w_1 \otimes \ldots \otimes w_k.$$
Examples of morphisms $f_R$

Consider the following morphisms $(v, w \in V)$:

1. Morphisms

\[ \varepsilon := f_{\{0\}} : 1 \to V_n, \; 1 \mapsto \sum_{v \in V} v, \]
\[ \varepsilon^* := f_{\{0\}} : V_n \to 1, \; v \mapsto 1, \]
\[ m := f_{\{(a+b,-a,-b)|a,b \in F_q\}} : V_n \otimes V_n \to V_n, \; v \otimes w \mapsto \delta_{v,w} v, \]
\[ m^* := f_{\{(a+b,-a,-b)|a,b \in F_q\}} : V_n \to V_n \otimes V_n, \; v \mapsto v \otimes v, \]
\[ \sigma := f_{\{(a,b,-b,-a)|a,b \in F_q\}} : V_n \otimes V_n \to V_n \otimes V_n, \; v \otimes w \mapsto w \otimes v. \]

2. Morphisms

\[ z := f_{F_1} : 1 \to V_n, \; 1 \mapsto \dot{0}, \]
\[ \forall a \in F_q, \; \mu_a := f_{\{-ab,b\}|b \in F_q} : V_n \to V_n, \; v \mapsto \dot{a}v \quad \text{for } v \in V, \]
\[ + := f_{\{(b,b,-b)|b \in F_q\}} : V_n \otimes V_n \to V_n, \; v \otimes w \mapsto v + w \quad \text{for } v, w \in V. \]
Generating morphism

Interpretation:

• The morphisms $\mu, \hat{+}$ give $V_n$ the structure of vector space over $\mathbb{F}_q$, with $z : 1 \to V_n$ defining the zero vector $\hat{0} \in V_n$;

• The rest of the morphisms make $V_n$ into a commutative Frobenius algebra in $\text{Rep}(GL_n(\mathbb{F}_q))$. The algebra is self-dual, via the pairings

$$ev := f_{\{(a,-a)|a \in \mathbb{F}_q\}} : V_n \otimes V_n \to 1, \quad v \otimes w \mapsto \delta_{v,w} \quad \text{for } v, w \in V,$$

$$coev := f_{\{(a,-a)|a \in \mathbb{F}_q\}} : 1 \to V_n \otimes V_n, \quad 1 \mapsto \sum_{v \in V} v \otimes v.$$

where

$$ev = \varepsilon^* \circ m, \quad coev = m^* \circ \varepsilon.$$
Lemma. [Entova-Aizenbud-H.] The set \( \{ f_R \mid R \subset \mathbb{F}_q^{s+k} \} \) spans the space \( \text{Hom}_G(V_n \otimes^s, V_n \otimes^k) \). Furthermore, if \( n \geq s + k \) then this set is a basis of \( \text{Hom}_G(V_n \otimes^s, V_n \otimes^k) \). The composition \( f_S \circ f_R : V_n \otimes^s \rightarrow V_n \otimes^l \) is polynomial in \( q^n \) (and equals
\[
q^n \cdot d(R,S) f_{S \star R}
\]
for some explicitly given \( d(R, S) \)).
Define $\text{Rep}(\mathbb{G}L_t(\mathbb{F}_q))$

Define the skeletal subcategory $\mathcal{T}(\mathbb{G}L_t)$: Objects $[k]$, $k \in \mathbb{N}$. Define

$$\text{Rel}_{s,k} = \{ R \subset \mathbb{F}^{s+k} \text{ linear subspace } \}, \quad \text{Hom}_{\mathcal{T}(\mathbb{G}L_t)}([s],[k]) = \mathbb{C}\text{Rel}_{s,k}$$

**Composition:** for $R \in \text{Rel}_{s,k}$, $S \in \text{Rel}_{k,l}$ we set

$$S \circ R := t^{d(R,S)} S \star R.$$ 

**Monoidal structure of $\mathcal{T}(\mathbb{G}L_t)$:** put $[l] \otimes [k] := [l + k]$.

We define $\text{Rep}(\mathbb{G}L_t(\mathbb{F}_q))$ as the additive Karoubi envelope of $\mathcal{T}(\mathbb{G}L_t)$. 
Knop: Defined $\text{Rep}(GL_t(\mathbb{F}_q))$ (but differently). Proved:

1. $\text{Rep}(GL_t(\mathbb{F}_q))$ is abelian and semisimple iff $t \neq q^n$ for some $n \in \mathbb{N}$.
2. For $t = q^n$ there is a specialization functor

$$F_V : \text{Rep}(GL_t(\mathbb{F}_q)) \to \text{Rep}(GL_n(\mathbb{F}_q))$$

(the semisimplification functor).

3. For $t \neq q^n$, the simple objects are up to isomorphism in bijection with $\bigcup_n \text{Irr}_n$ where $\text{Irr}_n$ are the iso classes of irreducible $GL_n(\mathbb{F}_q)$-representations.
Let $C$ be a $\mathbb{C}$-linear rigid SM category.

**Definition.** Let $V \in C$ be an object equipped with the following structures:

- $V$ is equipped with the structure of a Frobenius algebra object in $C$. That is, $V$ is equipped with maps $m : V \otimes 2 \to V$, $\varepsilon : 1 \to V$, $m^* : V \to V \otimes 2$, $\varepsilon^* : V \to 1$ such that:
  - It is a commutative unital algebra object with multiplication $m$ and unit $\varepsilon$, and a cocommutative counital coalgebra object with comultiplication $m^*$ and counit $\varepsilon^*$,
  - Frobenius Relations: $m^* \circ m = (id \otimes m) \circ (m^* \otimes id) = (m \otimes id) \circ (id \otimes m^*)$, and Speciality Relation: $m \circ m^* = id$. 
• $V$ is a module over the field $\mathbb{F}_q$: $V$ is equipped with maps

$$\bar{+} : V \otimes V \to V,$$

$$\mu : (\mathbb{F}_q, \cdot) \to (\text{End}_C(V), \circ), \ a \mapsto \mu_a,$n

$$z : 1 \to V$$

satisfying the following conditions:

• $\bar{+}$ is associative and commutative: $\bar{+} \circ (\bar{+} \otimes \text{id}) = \bar{+} \circ (\text{id} \otimes \bar{+}),$

$\bar{+} \circ \sigma = \bar{+}$ where $\sigma \in \text{End}(V \otimes V)$ is the symmetry morphism.

• $z$ serves as “the embedding of the $\bar{0}$ vector”:

$$\bar{+} \circ (z \otimes \text{id}) = \text{id}, \ \bar{+} \circ (\text{id} \otimes z) = \text{id},$$

• For all $a, b \in \mathbb{F}_q$, $\mu_a \circ \mu_b = \mu_{ab}$ and $\mu_1 = \text{id}$, $\mu_0 = z \circ \varepsilon^*.$

• Linearity of $\mu_a$ with respect to $\bar{+}$: for any $a, b \in \mathbb{F}_q$,

$$\mu_{a+b} = \bar{+} \circ (\mu_a \otimes \mu_b) \circ m^*.$$ 

Distributivity of $\mu_a$: for any $a \in \mathbb{F}_q$, $\mu_a \circ \bar{+} = \bar{+} \circ (\mu_a \otimes \mu_a).$
Assume furthermore that the above structures satisfy some compatibility relations. Such an object $V$ is called an $\mathbb{F}_q$-linear Frobenius space in $\mathcal{C}$.

**Example.** The representation $V_n$ in $\text{Rep}(GL_n(\mathbb{F}_q))$. Dito $V_n$ seen as a representation of $O_n(\mathbb{F}_q)$, $Sp_{2n}(\mathbb{F}_q)$, ... 

**Theorem.** [Entova-Aizenbud-H.] Let $\mathcal{C}$ be a Karoubi additive rigid SM category, and let $V$ be an $\mathbb{F}_q$-linear Frobenius space in $\mathcal{C}$. Let $t = \dim(V)$. Then there exists a SM functor

$$F_V : \text{Rep}(GL_t(\mathbb{F}_q)) \to \mathcal{C}, \quad V_t \mapsto V$$

which is unique up to isomorphism.
The infinite case

Let $V := \mathbb{F}_q^\infty = \bigcup_{n \neq 0} \mathbb{F}_q^n$. We denote by $V_\infty$ the representation $\mathbb{C} \mathbb{F}_q^\infty$ of $GL_\infty(\mathbb{F}_q)$; this is the countable-dimensional vector space consisting of infinite sequences of elements in $\mathbb{F}_q$ which have finite support.

Let $\mathcal{I}_\infty \subset \text{Rep}(GL_\infty(\mathbb{F}_q))$ denote the full subcategory of direct summands in tensor powers of $\mathbb{C} \mathbb{F}_q^\infty$. Let $\text{Rep}(GL_\infty(\mathbb{F}_q))$ denote the category of algebraic representations of $GL_\infty(\mathbb{F}_q)$: all subquotients of tensor powers of $\mathbb{C} \mathbb{F}_q^\infty$.

$\mathcal{I}_\infty$ does not have a good notion of duality anymore. In particular $\mathbb{C} \mathbb{F}_q^\infty$ is no longer an $\mathbb{F}_q$-linear Frobenius space.

**Theorem.** [Entova-Aizenbud-H.] There is a non-full embedding of $\mathcal{I}_\infty \rightarrow \text{Rep}(GL_t(\mathbb{F}_q))$. The category $\mathcal{I}_\infty$ is universal with respect to a semi-$\mathbb{F}_q$-linear Frobenius space (essentially an $\mathbb{F}_q$-linear Frobenius space without a unit).

**Remark.** The additive envelope of $\mathcal{I}_\infty$ can be identified with the full subcategory of injective objects in $\text{Rep}(GL_\infty(\mathbb{F}_q))$. 
Harman-Snowden: Construction of the abelian envelope for $t = q^n$ via oligomorphic groups.

Entova-Aizenbud-H: another description of the abelian envelope of the form $C - Comod$, the category of finite-dimensional $C$-comodules for some coalgebra $C$ (a special case of the Benson-Etingof-Ostrik construction).

Follows loosely the approach by Comes-Ostrik in the $Rep(S_t)$-case.
Let $\mathcal{T}$ denote Karoubi rigid SM category. Recall that a morphism $f : X \to Y$ is **split** if it is the composition $i \circ \pi$ where $\pi$ is a split epimorphism and $i$ is a split monomorphism.

**Definition.** An object $X \in \mathcal{T}$ is a **splitting object** if for any $Q_1, Q_2 \in \mathcal{T}$ and a morphism $f : Q_1 \to Q_2$ the morphisms $1_X \otimes f : X \otimes Q_1 \to X \otimes Q_2$ and $f \otimes 1_X : Q_1 \otimes X \to Q_2 \otimes X$ are split.

Splitting objects form a **thick tensor ideal** $S$. Let $I$ denote the set of isomorphism classes of indecomposable objects $\{P_i\}_{i \in I}$ in $S$. Define the coalgebra

$$C := \bigoplus_{i,j \in I} \text{Hom}(P_i, P_j)^*.$$ 

and denote by $\mathcal{C} = C - \text{Comod}$ the category of finite-dimensional $C$-comodules, i.e. the category of additive functors $S^{\text{op}} \to \text{Vec}$. 
The abelian envelope

**Theorem.** [Entova-Aizenbud-H.] For $\text{Rep}(GL_t(\mathbb{F}_q))$ we have $S = \mathcal{N}$, the thick tensor ideal of negligible objects, and $\mathcal{C} = \mathcal{C} - \text{Comod}$ is the abelian envelope of $\text{Rep}(GL_t(\mathbb{F}_q))$.

**Corollary.** Universal property: Let $\mathcal{C}$ be a pre-tannakian category and let $V$ be an $\mathbb{F}_q$-linear Frobenius space in $\mathcal{C}$.

(i) If $V$ is annihilated by some exterior power and $\dim(V) = q^n$, the functor $F_V : \text{Rep}(GL_{q^n}(\mathbb{F}_q)) \to \mathcal{C}$ factors through the specialization functor

$$\text{Rep}(GL_{t=q^n}(\mathbb{F}_q)) \to \text{Rep}(GL_{t=q}(\mathbb{F}_q)).$$

(ii) If $V$ is not annihilated by any exterior power, then $F_V$ factors through the canonical embedding

$$\text{Rep}(GL_{q^n}(\mathbb{F}_q)) \to \text{Rep}^{ab}(GL_{q^n}(\mathbb{F}_q)).$$
Key idea: the $S_t$-case

For $S_n$ we can define $\Delta_k^n = \mathbb{C}Inj(\{1, \ldots, k\}, \{1, \ldots, n\}) = Ind_{S_{n-k}}^{S_n} 1$ for $k \leq n$.

These have analogs $\Delta_k, k \in \mathbb{N}$ in $Rep(S_t)$. For $Rep(S_{t-n})$ define $\Delta = \Delta_{n+1}$. Then there is a factorization (Deligne, Comes-Ostrik)

$$Rep(S_t) \xrightarrow{-\otimes\Delta} Rep(S_t) \xrightarrow{Res} \xrightarrow{Ind} Rep(S_{t-1})$$

Since $Rep(S_{t-1})$ is semisimple, $\Delta$ is a splitting object.
The $Rep(\mathbb{GL}_t(\mathbb{F}_q))$-case

By $P_{k,n}$ we denote the parabolic subgroup of all matrices
\[
\begin{bmatrix}
C & A \\
0 & B
\end{bmatrix} \in GL_n(\mathbb{F}_q) \text{ where } C \in GL_k(\mathbb{F}_q), B \in GL_{n-k}(\mathbb{F}_q) \text{ and } A \in \text{Mat}_{k \times (n-k)}(\mathbb{F}_q).
\]

By $H_{k,n} \subset P_{k,n}$ we denote the mirabolic subgroup of all matrices as above for which $C$ is the identity matrix of size $k \times k$.

Consider the left regular representation $\mathbb{C}[GL_k(\mathbb{F}_q)]$ of $GL_k(\mathbb{F}_q)$. We inflate it to a representation of $P_{k,n}$ by requiring that $H_{k,n}$ acts trivially. We denote:

\[
\Delta_n^k := \mathbb{C}\text{Inj}_{\mathbb{F}_q}(\mathbb{F}_q^k, \mathbb{F}_q^n) = \text{Ind}_{P_{k,n}}^{GL_k(\mathbb{F}_q) \times GL_n(\mathbb{F}_q)} \mathbb{C}[GL_k(\mathbb{F}_q)].
\]

This is a representation of $GL_k(\mathbb{F}_q) \times GL_n(\mathbb{F}_q)$. If we restrict the action just to $GL_n(\mathbb{F}_q)$, this becomes

\[
\Delta_n^k|_{GL_n(\mathbb{F}_q)} \cong \text{Ind}^{GL_n(\mathbb{F}_q)}_{H_{k,n}} 1.
\]
Knop’s machinery

Can construct analogs $\Delta_k$, $k \in \mathbb{N}$, in $\text{Rep}(GL_t(\mathbb{F}_q))$.

Suggests that $\Delta_k$ for $\text{Rep}(GL_t(\mathbb{F}_q))$ should be induced from a parabolic version of the Deligne category.

This can be done via a general machinery developed by Knop to construct interpolating categories.

Fix $k \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{A}_k$ denote the category whose objects are pairs $(V, p_V)$ where $V \in \text{Vect}_{\mathbb{F}_q}$ and $p_V : V \to \mathbb{F}_q^k$ be a surjective $\mathbb{F}_q$-linear map. The morphisms in this category are given by

$$\text{Mor}_{\mathcal{A}_k}((V, p_V), (W, p_W)) = \{f \in \text{Hom}_{\mathbb{F}_q}(V, W) : f \circ p_W = p_V\}.$$

Knop’s construction gives a family of karoubian SM categories $\mathcal{T}(\mathcal{A}_k, t)$ for $t \in \mathbb{C}$.

The category $\mathcal{T}(\mathcal{A}_k, t)$ is semisimple whenever $t \not\in q^{\mathbb{Z}_{\geq k}}$. 
The analog of $\text{Rep}(S_{-1})$

For $t = q^N$ where $N \in \mathbb{Z}_{\geq k}$, we have a full, essentially surjective SM functor

$$F_N : \mathcal{T}(A_k, t) \to \text{Rep}(H_{k,N}^{tr})$$

where

$$H_{k,N}^{tr} = \left\{ \begin{bmatrix} 1_k & 0 \\ A & B \end{bmatrix} \in GL_n(\mathbb{F}_q) \right\}$$

Since $H_{k,N}^{tr}$ is obtained from $H_{k,N}$ by the automorphism $X \mapsto (X^{-1})^\text{tr}$ of $GL_n(\mathbb{F}_q)$, we have $\text{Rep}(H_{k,N}) \cong \text{Rep}(H_{k,N}^{tr})$, so the family $\mathcal{T}(A_k, t)$, $t \in \mathcal{C}$ interpolates" the categories $\text{Rep}(H_{k,N})$ for $t = q^N$. 
Splitting property

Fix $t \in \mathbb{C}$. We have a functor

$$Res_{t,k} : \text{Rep}(GL_t(\mathbb{F}_q)) \rightarrow \mathcal{T}(\mathcal{A}_k, t)$$

given by the universal property of $\text{Rep}(GL_t(\mathbb{F}_q))$.

**Key lemma.** $\Delta_{n+1}$ is a projective object in $\text{Prshv}(\text{Rep}(GL_t(\mathbb{F}_q)))$ for $t = q^n$.

**Proof.** Show that there is an induction functor from the parabolic Deligne category to $\text{Prshv}(\text{Rep}(GL_t))$ which sends $1$ to $\Delta_{n+1}$, and this functor is left adjoint to an exact restriction functor, hence sends projectives to projectives. For $k = n + 1$, $\mathcal{T}(\mathcal{A}_k, t)$ is semisimple.
Open questions

- Develop the representation theory of $\text{Rep}(GL_\infty(\mathbb{F}_q))$ similarly to Sam-Snowden's work for $S_\infty$. Some partial results by Nagpal.
- Understand the abelian envelope better (highest weight structure etc).
- We suspect that there are two different Deligne categories for the finite linear group, ours and a bigger one which should be generated by $\tilde{\Delta}$-objects induced from a Levi subgroup. We don’t even know how to construct the latter diagrammatically.