Detecting Polygons and Compressing Waves: Some New Twists on Classical Problems in Information Theory

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Outline

- Waves in Space.
- A New Multiterminal Source Coding Problem.
- Some Results.
- Applications.

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Waves in Space

Spatial waves form a most interesting/relevant class of signals to study:

- Pressure waves:
  - Earthquakes/tsunamis, atmospheric variations, noise, ...
  - Speech, underwater tracking, ultrasound images, oil mapping, ...
- Electromagnetic waves:
  - Objects in outer space, light, nuclear radiation, ...
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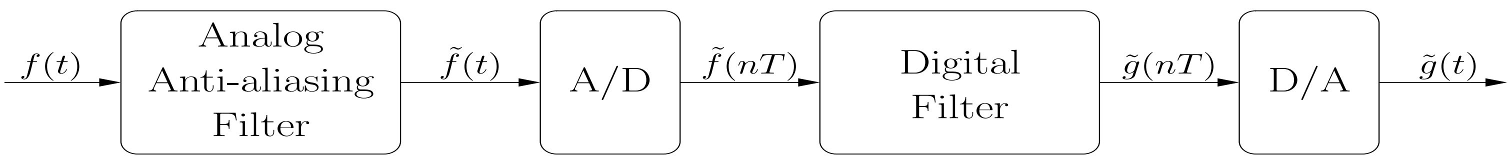
Our goal in this work: **compress waves, subject to a fidelity criterion.**

The Basic Toolbox for Processing Bandlimited Signals

Two fundamental bases: *sincs* and *complex exponentials*.

- Using sincs a suitably chosen discrete set of samples can be interpolated, to recover the original signal.
- Using complex exponentials any LTI operator can be diagonalized, and this greatly simplifies filter design tasks.

The basic signal processing toolbox:



But this is ***not*** *a good way to go about processing spatial waves...*

The Basic Toolbox for Processing Wave Fields?

The model of bandlimited signals and LTI operators is not appropriate here:

- Waves are typically confined to compact sets – *not bandlimited*
- No A/D converter can see an entire wave in space – *no anti-aliasing filter*
- Many typical operations are not LTI ("LSI"?) – *unclear how Fourier basis helps*

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So, what is the right signal model then?

Waves are the solution of a partial differential equation. **(animation)**

The Source Coding Problem for Spatial Waves

So, given:

- a compact set in euclidean space,
- propagation properties and boundary conditions,
- a source of information generating waves,
- a finite set of locations at which waves can be observed,
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we wish to determine the rate/distortion tradeoffs that can be achieved when encoding the solutions of this given PDE.

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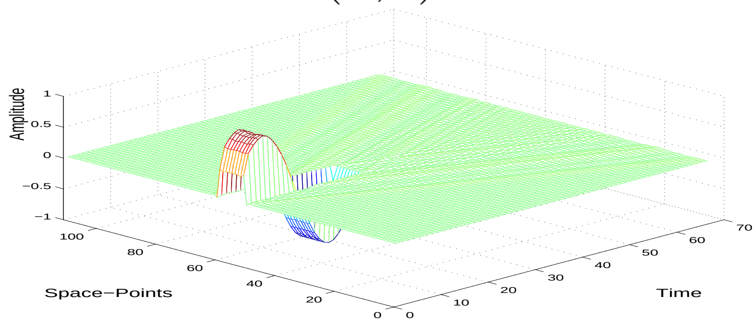
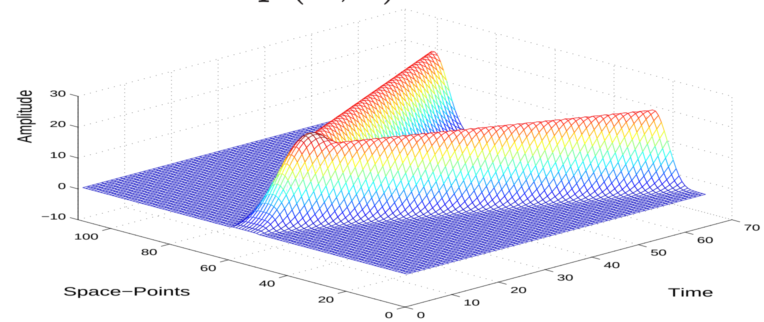
The Wave Equation – Basic Definitions

Source function $s(x,t)$



Pressure field $p(x,t)$



At all membrane locations $x \in [0, \pi]$, and at all times $t \in \mathbb{R}$, we must have:

$$\frac{\partial^2 p(x,t)}{\partial^2 x} + \frac{1}{c^2}\frac{\partial^2 p(x,t)}{\partial^2 t} + s(x,t) = 0;$$

or equivalently, in frequency (and for a normalized frequency $a = \frac{\omega}{c}$),

$$\frac{\partial^2 \hat{p}(x,a)}{\partial^2 x} + a^2 \hat{p}(x,a) + \hat{s}(x,a) = 0.$$

A. J. Berkhout. *Applied Seisimc Wave Theory*. Elsevier, 1987.

The Wave Equation – Solution

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- a point source $\hat{s}(x,a) = \hat{s}(a)\delta(x - x_o)$; then,

$$\hat{p}(x,a) \;=\; \hat{s}(a)\hat{h}(x,a) \;=\; \hat{s}(a)\left(\frac{2}{\pi}\sum_{n\in\mathbb{Z}} \frac{\cos(nx_o)\cos(nx)}{n^2 - a^2}\right)$$

This is the structure in space and time of the data field we are trying to understand how to compress subject to a fidelity criterion.

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Note: all these codes can do is encode the functions they observe at a fixed location, and deliver these encodings to a decoder that will use them to estimate a solution of the wave equation.

The Distributed Rate/Distortion Function

- Achievability with N encoders:

  $(R_1 ... R_N, D)$ is achievable if and only if there is a fixed finite N and locations $\xi_1 ... \xi_N$, such that for all $\epsilon > 0$ and all n large enough, we can find a $(2^{nR_1} ... 2^{nR_N}, \xi_1 ... \xi_N, n, N, \bar{D})$ code with $\bar{D} < D + \epsilon$.

- Rate region with N encoders:

  $\mathcal{R}_N(D)$: closure of the set of all rates $(R_1 ... R_N)$, such that $(R_1 ... R_N, D)$ is achievable with N encoders.

- Finally: $$\mathcal{R}(D) = \inf_{(R_1 ... R_N) \in \mathcal{R}_N(D),\ N \geq 1} \sum_{k=1}^{N} R_k.$$

End goal: describe $\mathcal{R}_N(D)$ and $\mathcal{R}(D)$ in terms of computable information theoretic quantities.

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A Simple Interpolation Formula

- Temporal sampling – assuming compactly supported $\hat{s}(a)$, for any fixed $\xi \in [0, \pi]$, $p(\xi, a) = s(a)\hat{h}(\xi, a)$ is compactly supported too. *Standard.*

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Good news: $\hat{h}(x, a)$ is analytic in both x and a – so, the number of zeros on any compact set is finite. Better news: it is ***not*** *straightforward to make sure those zeros do not cause problems...*

Example: is there one frequency a_o such that, for all locations $0 \leq \xi \leq \pi$, $\hat{h}(\xi, a_o) = 0$?

A Sampling Theorem for Spatial Waves

There exist functions $f_k(x,a)$, such that for all $0 \le x \le \pi$ and for all a in the support of $\hat{s}$,

$$\hat{p}(x,a) \;=\; \sum_{k=1}^{N} f_k(x,a)\hat{p}\big(\tfrac{k\pi}{N},a\big),$$

provided $N \ge C_o \frac{n_o \log(n_o+1)}{\Delta^2}$, for $a \in I \subset (n_o, n_o+1)$, I compact, $n_o \in \mathbb{Z}$, and Δ is the minimum distance from I to one of the resonant integer frequencies n_o or $n_o + 1$.

- Key idea: for all a, show there is at least one k such that $p\big(\frac{k\pi}{N},a\big) \neq 0$.

- Significance: interpolation possible through linear filtering.

Scaling Behavior of the Distributed Rate/Distortion Function

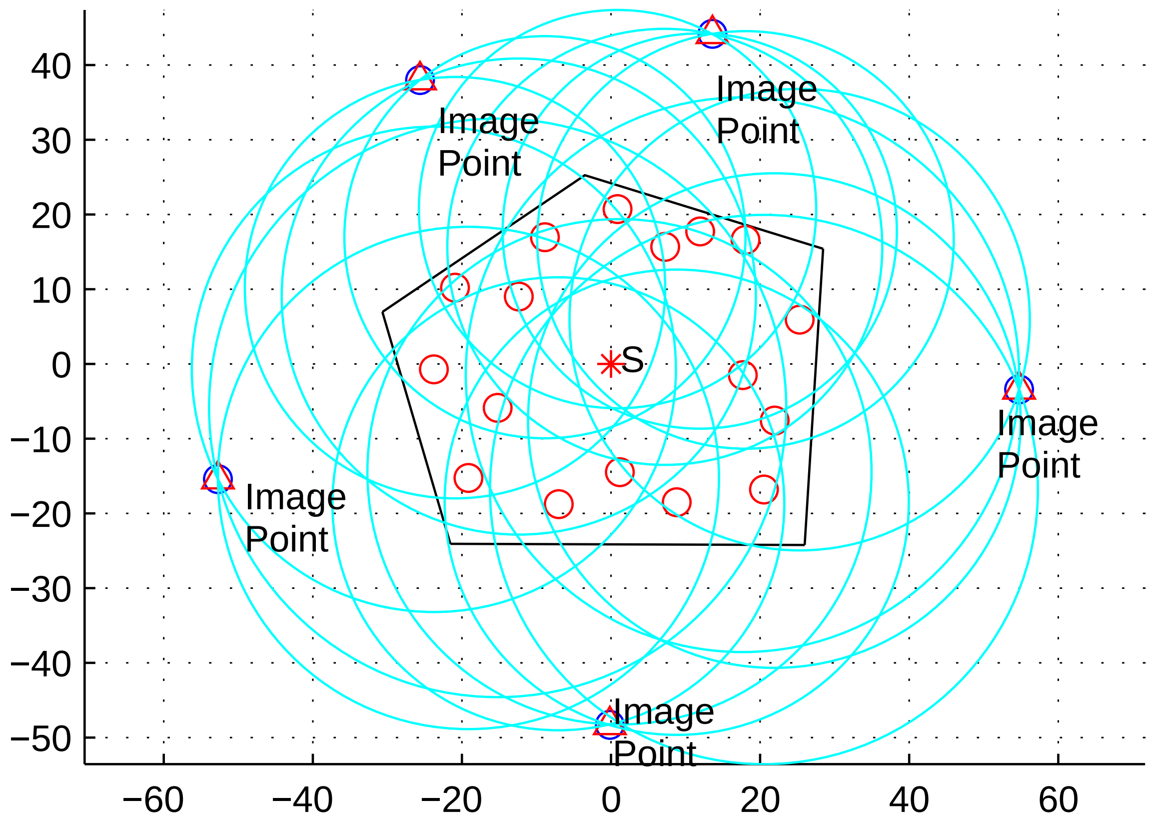
An upper bound under dense sampling:

$$\lim_{N\to\infty} \inf_{(R_1 \ldots R_N)\in\mathcal{R}_N(D)} \sum_{k=1}^{N} R_k \;\leq\; C \;<\; \infty.$$

- Key ideas:
  - At each sampling location, a filtered version of the source is observed.
  - Each sensor encodes only a narrowband segment of the source.
  - Total bitrate $\approx$ bitrate required to encode the source.
- Significance 1: *not* true for processes bandlimited in space.
- Significance 2: Suggests what $\mathcal{R}\big(D\big)$ may look like...

A Sampling Theorem for Polyhedra in $\mathbb{R}^n$

Can reconstruct an arbitrary polyhedron from a set of samples.



So we can hear the shape of a drum after all! (article) Just have to listen carefully...

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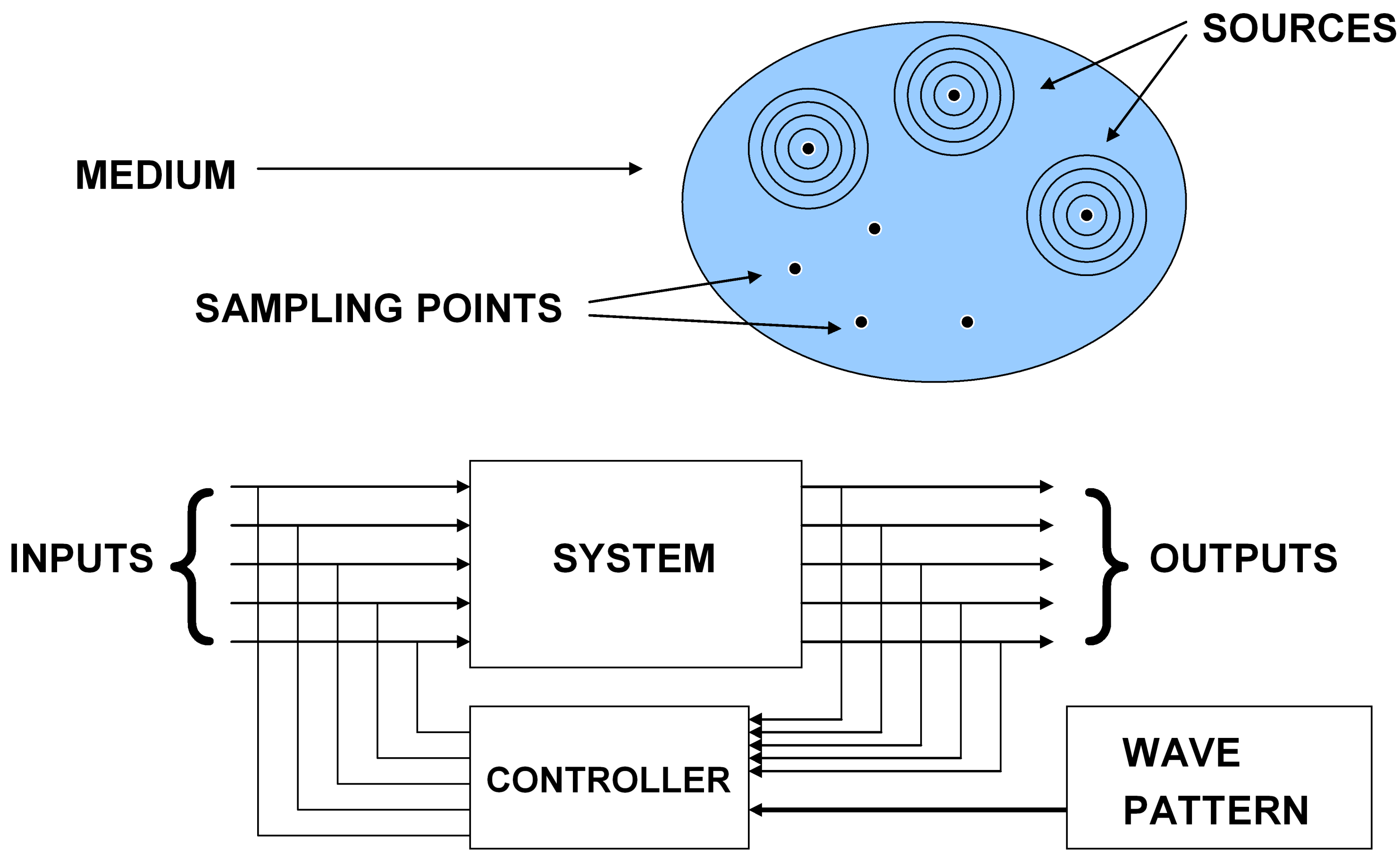
The Class of “Network-in-the-Loop” Problems



Problems where communication between plant and controller happens over a network...

Real (??) Network-in-the-Loop Problems

Rendering of virtual acoustic sources, noise cancellation, cancer treatment, electronic countermeasures, distributed transmitters/receivers, imaging of solid state objects, ...

In all cases, need a *very large* number of nodes to achieve good performance.

Main Corollary...

http://cn.ece.cornell.edu/