Detecting Polygons and Compressing Waves: Some New Twists on Classical Problems in Information Theory

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(Presented at the IPAM Workshop on Mathematical Challenges in Sensor Networks, January 10th, 2007.)
Acknowledgements

- My collaborators on parts of this work:
  - Joe Rosenblatt (UIUC/Math).
  - Mingbo Zhao and Georgios N. Lilis (Cornell/ECE).

- Sources of support:
  - *Fundamental Performance Limits of Large-Scale Sensor Networks.*
    NSF CAREER award CCR-0238271.
  - *The Reachback Channel in Wireless Sensor Networks.*
    NSF SENSORS grant CCR-0330059. PI, joint with T. Berger, L. Tong, S. Wicker.
  - *Self-Configuring Sensor Networks for Disaster Prevention, Mitigation and Recovery.*
    NSF ITR grant ANR-0325556. Co-PI, joint with Cornell ECE, CEE and Economics faculty, and staff at NYS Wadsworth Labs.

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• Waves in Space.

• A New Multiterminal Source Coding Problem.

• Some Results.

• Applications.
Outline

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- Some Results.
- Applications.
Waves in Space

Spatial waves form a most interesting/relevant class of signals to study:

- **Pressure waves:**
  - Earthquakes/tsunamis, atmospheric variations, noise, ...
  - Speech, underwater tracking, ultrasound images, oil mapping, ...

- **Electromagnetic waves:**
  - Objects in outer space, light, nuclear radiation, ...
  - Radio communications, radar, (controlled) nuclear energy, ...

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Our goal in this work: **compress waves, subject to a fidelity criterion.**
The Basic Toolbox for Processing Bandlimited Signals

Two fundamental bases: \textit{sincs} and \textit{complex exponentials}.

- Using sincs a suitably chosen discrete set of samples can be interpolated, to recover the original signal.
- Using complex exponentials any LTI operator can be diagonalized, and this greatly simplifies filter design tasks.

The basic signal processing toolbox:

\[ f(t) \xrightarrow{\text{Analog Anti-aliasing Filter}} \tilde{f}(t) \xrightarrow{\text{A/D}} \tilde{f}(nT) \xrightarrow{\text{Digital Filter}} \tilde{g}(nT) \xrightarrow{\text{D/A}} \tilde{g}(t) \]

\textit{But this is not a good way to go about processing spatial waves...}
The Basic Toolbox for Processing Wave Fields?

The model of bandlimited signals and LTI operators is not appropriate here:

- Waves are typically confined to compact sets – *not bandlimited*
- No A/D converter can see an entire wave in space – *no anti-aliasing filter*
- Many typical operations are not LTI ("LSI"?) – *unclear how Fourier basis helps*
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So, what is the right signal model then?

Waves are the solution of a partial differential equation.
The Source Coding Problem for Spatial Waves

So, given:

- a compact set in euclidean space,
- propagation properties and boundary conditions,
- a source of information generating waves,
- a finite set of locations at which waves can be observed,
- and a fidelity criterion;

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we wish to determine the rate/distortion tradeoffs that can be achieved when encoding the solutions of this given PDE.
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The Wave Equation – Basic Definitions

At all membrane locations \( x \in [0, \pi] \), and at all times \( t \in \mathbb{R} \), we must have:

\[
\frac{\partial^2 p(x, t)}{\partial^2 x} + \frac{1}{c^2} \frac{\partial^2 p(x, t)}{\partial^2 t} + s(x, t) = 0;
\]

or equivalently, in frequency (and for a normalized frequency \( a = \frac{\omega}{c} \)),

\[
\frac{\partial^2 \hat{p}(x, a)}{\partial^2 x} + a^2 \hat{p}(x, a) + \hat{s}(x, a) = 0.
\]

The Wave Equation – Solution

Under some assumptions (needlessly restrictive, but very useful to illustrate the points we want to make), we can obtain a solution. Assume:

- a homogeneous, frictionless, one-dimensional membrane, of length $\pi$;
- perfectly reflecting boundaries:
  \[
  \frac{\partial p}{\partial x} \bigg|_{x=0} = \frac{\partial p}{\partial x} \bigg|_{x=0} = 0;
  \]
- a point source $\hat{s}(x, a) = \hat{s}(a) \delta(x - x_o)$; then,
  \[
  \hat{p}(x, a) = \hat{s}(a) \hat{h}(x, a) = \hat{s}(a) \left( \frac{2}{\pi} \sum_{n \in \mathbb{Z}} \cos(nx_o) \cos(nx) \frac{n^2 - a^2}{n^2 - a^2} \right)
  \]

This is the structure in space and time of the data field we are trying to understand how to compress subject to a fidelity criterion.
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Distributed Rate/Distortion Codes

A \((2^{nR_1}...2^{nR_N}, \xi_1...\xi_N, n, N, \bar{D})\) code:

Note: all these codes can do is encode the functions they observe at a fixed location, and deliver these encodings to a decoder that will use them to estimate a solution of the wave equation.
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- A decoding function \(g : \Pi_{i=1}^N\{1\ldots2^{nR_k}\} \rightarrow \mathcal{P}\).
- A distortion measure
  \[
  d(p, q) = \frac{1}{\pi} \int_0^\pi \int_{t \in \mathbb{R}} (p(x, t) - q(x, t))^2 \, dx \, dt,
  \]
  and a resulting distortion
  \[
  \bar{D} = d\left(p, g(f_1(p(\xi_1))\ldots f_N(p(\xi_N)))\right).
  \]
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- A distortion measure \(d(p, q) = \frac{1}{\pi} \int_{x=0}^{\pi} \int_{t \in \mathbb{R}} (p(x, t) - q(x, t))^2 \, dx \, dt\), and a resulting distortion \(\bar{D} = d(p, g(f_1(p(\xi_1)...f_N(p(\xi_N))))\).

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The Distributed Rate/Distortion Function

- **Achievability with** $N$ **encoders:**

  $(R_1 \ldots R_N, D)$ is achievable if and only if there is a fixed finite $N$ and locations $\xi_1 \ldots \xi_N$, such that for all $\epsilon > 0$ and all $n$ large enough, we can find a $(2^{nR_1} \ldots 2^{nR_N}, \xi_1 \ldots \xi_N, n, N, \bar{D})$ code with $\bar{D} < D + \epsilon$.

- **Rate region with** $N$ **encoders:**

  $\mathcal{R}_N(D)$: closure of the set of all rates $(R_1 \ldots R_N)$, such that $(R_1 \ldots R_N, D)$ is achievable with $N$ encoders.

- **Finally:**

  $\mathcal{R}(D) = \inf_{(R_1 \ldots R_N) \in \mathcal{R}_N(D), \ N \geq 1} \sum_{k=1}^{N} R_k$.

  **End goal:** describe $\mathcal{R}_N(D)$ and $\mathcal{R}(D)$ in terms of computable information theoretic quantities.

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A Simple Interpolation Formula

- Temporal sampling – assuming compactly supported $\hat{s}(a)$, for any fixed $\xi \in [0, \pi]$, $p(\xi, a) = s(a)\hat{h}(\xi, a)$ is compactly supported too. *Standard.*
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• A simple relationship for spatial sampling: $\hat{p}(\xi', a) = \hat{p}(\xi, a) \frac{\hat{h}(\xi', a)}{\hat{h}(\xi, a)}$. But...


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  - $\hat{s}(a)$ has to be supported on compact sets excluding integers.
  (Reminder: $\hat{h}(\xi, a) = \frac{2}{\pi} \sum_{n \in \mathbb{Z}} \cos(n \xi) \frac{\cos(n \xi_0) \cos(n \xi)}{n^2 - a^2}$).

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  – Need to understand when \( \hat{h}(x, a) \) can be zero!
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  - *Need to understand when \( \hat{h}(x, a) \) can be zero!*

*Good news:* \( \hat{h}(x, a) \) is analytic in both \( x \) and \( a \) – so, the number of zeros on any compact set is finite.
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*Good news:* \( \hat{h}(x, a) \) is analytic in both \( x \) and \( a \) – so, the number of zeros on any compact set is finite. *Better news:* it is not straightforward to make sure those zeros do not cause problems...

*Example:* is there one frequency \( a_o \) such that, for all locations \( 0 \leq \xi \leq \pi \), \( \hat{h}(\xi, a_o) = 0 \)?

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A Sampling Theorem for Spatial Waves

There exist functions $f_k(x, a)$, such that for all $0 \leq x \leq \pi$ and for all $a$ in the support of $\hat{s}$,

$$\hat{p}(x, a) = \sum_{k=1}^{N} f_k(x, a) \hat{p}(\frac{k\pi}{N}, a),$$

provided $N \geq C_o \frac{n_o \log(n_o+1)}{\Delta^2}$, for $a \in I \subset (n_o, n_o + 1)$, $I$ compact, $n_o \in \mathbb{Z}$, and $\Delta$ is the minimum distance from $I$ to one of the resonant integer frequencies $n_o$ or $n_o + 1$.

- Key idea: for all $a$, show there is at least one $k$ such that $p(\frac{k\pi}{N}, a) \neq 0$.

- Significance: interpolation possible through linear filtering.
Scaling Behavior of the Distributed Rate/Distortion Function

An upper bound under dense sampling:

$$\lim_{N \to \infty} \inf_{(R_1 \ldots R_N) \in \mathcal{R}_N(D)} \sum_{k=1}^{N} R_k \leq C < \infty.$$ 

• Key ideas:
  – At each sampling location, a filtered version of the source is observed.
  – Each sensor encodes only a narrowband segment of the source.
  – Total bitrate $\approx$ bitrate required to encode the source.

• Significance 1: *not* true for processes bandlimited in space.

• Significance 2: Suggests what $\mathcal{R}(D)$ may look like...

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A Sampling Theorem for Polyhedra in $\mathbb{R}^n$

Can reconstruct an arbitrary polyhedron from a set of samples.

So we can hear the shape of a drum after all! (article) Just have to listen carefully...
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The Class of “Network-in-the-Loop” Problems

Problems where communication between plant and controller happens over a network...

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Real (??) Network-in-the-Loop Problems

Rendering of virtual acoustic sources, noise cancellation, cancer treatment, electronic countermeasures, distributed transmitters/receivers, imaging of solid state objects, ...

In all cases, need a very large number of nodes to achieve good performance.
Main Corollary...

http://cn.ece.cornell.edu/