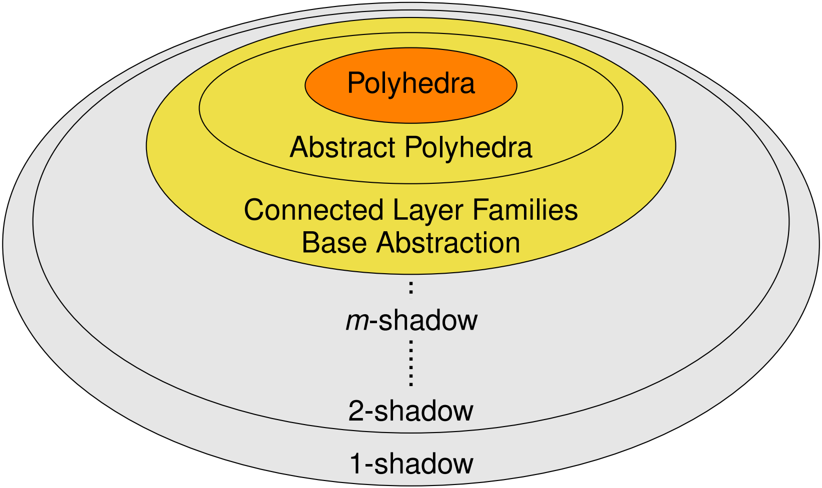
An abstract view on the polynomial Hirsch conjecture

Nicolai Hähnle

Ecole Polytechnique Fédérale de Lausanne, Switzerland

January 2011, “Quo vadis Hirsch conjecture?”

Outline: Hierarchy of Abstractions



The quasi-polynomial upper bound

$\Delta(d, n) =$ max. diameter of a d-dim. polyhedron with n facets

Theorem (Kalai & Kleitman)

$\Delta(d, n) \leq n^{1+\log d}$

The Base Abstraction

- undirected graph $G = (V, E)$
- vertices are d-subsets of $[n]$
- Connectivity: for all $f \subseteq [n]$, the subgraph induced by the vertices that are supersets of f is connected

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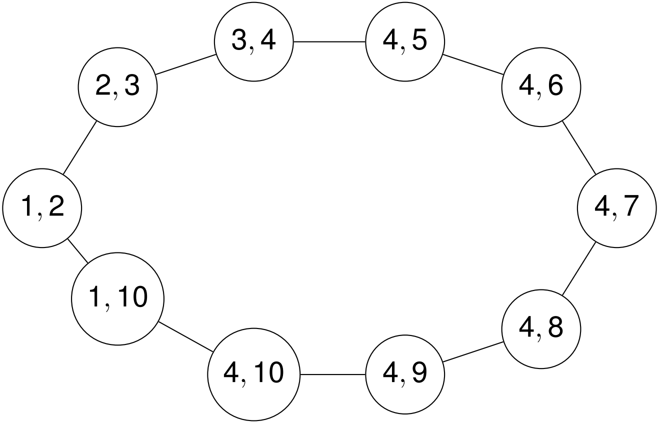

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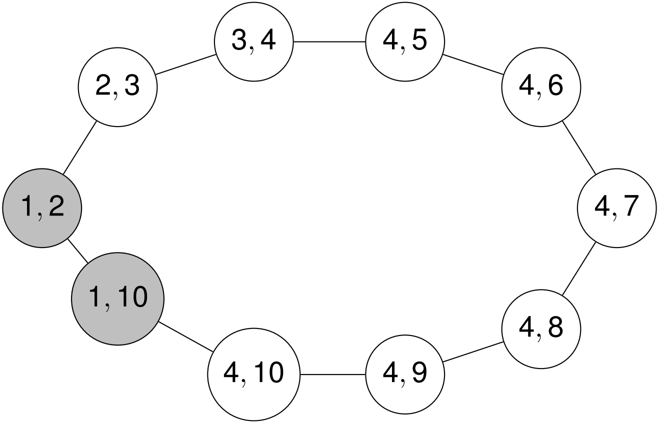

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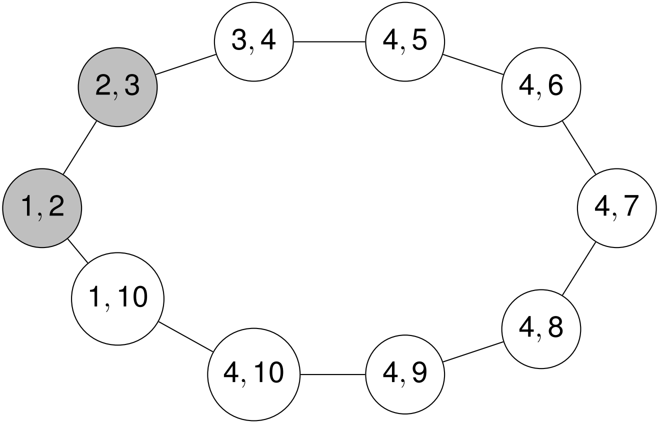

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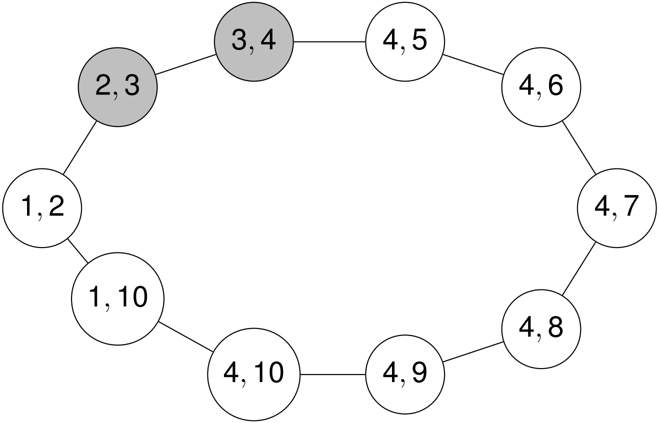

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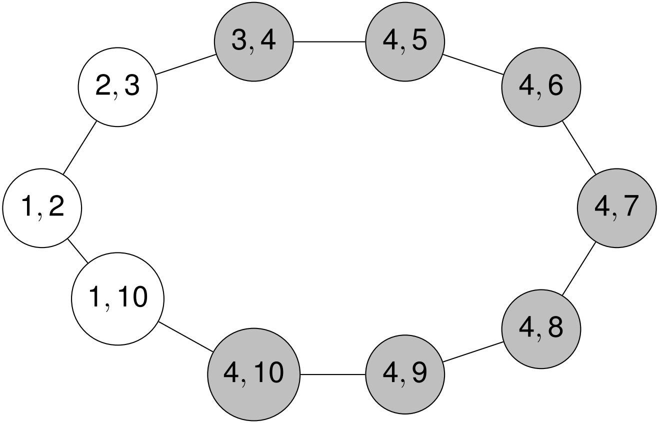
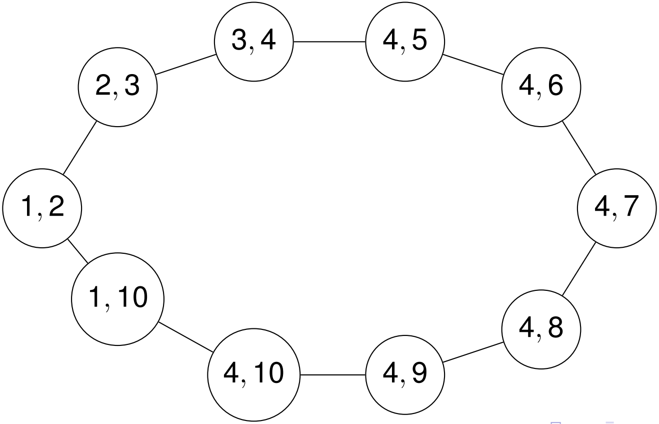


Connected Layer Families

- Sequence $S_1, \ldots, S_t$ of disjoint non-empty families of d-subsets of $[n]$
- Connectivity: for $i < j < k$: if f is covered in S_i and S_k, then f is covered in S_j
  - $f \subseteq [n]$ is *covered in* S_j if S_j contains a superset of f



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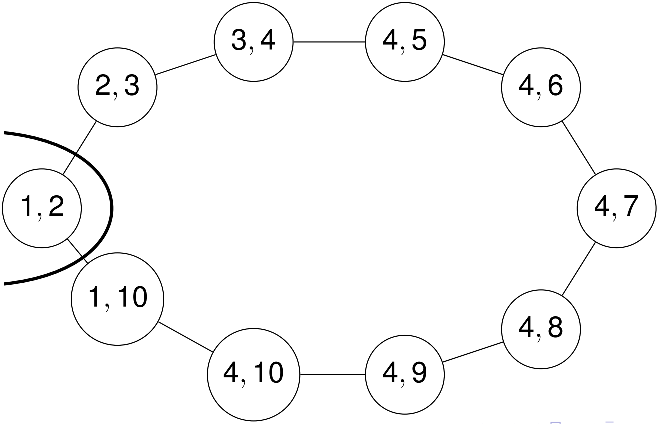

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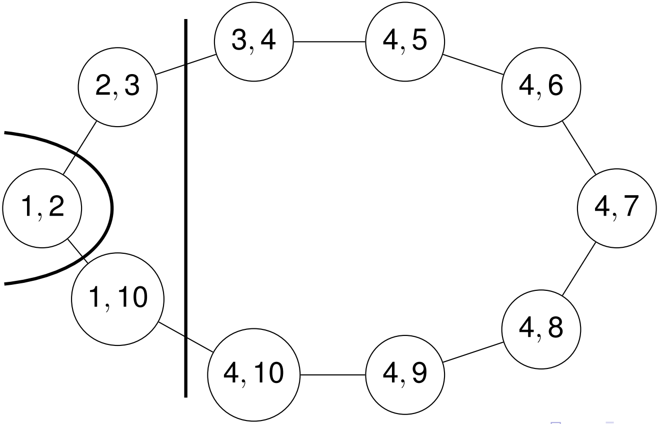

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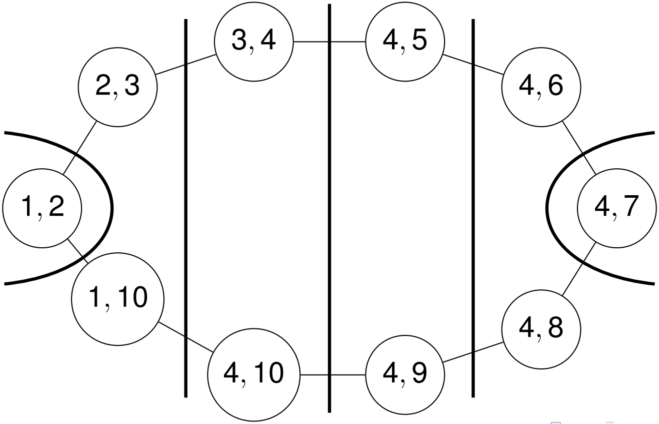

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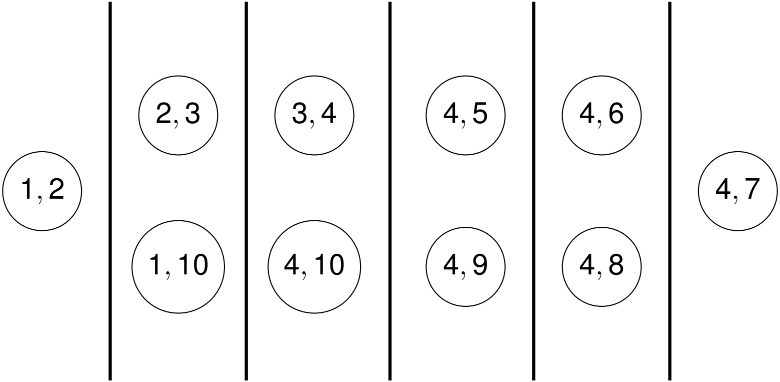

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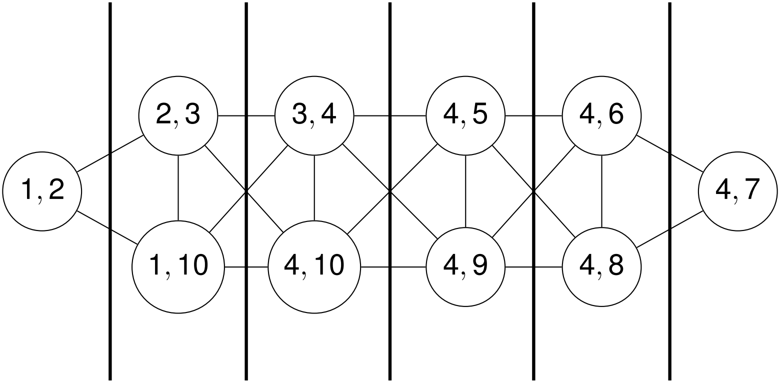

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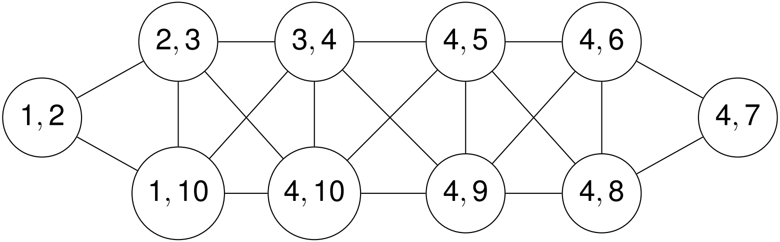

Connected Layer Families: Useful Properties

Lemma (Equivalence of Base Abstraction and CLF)

For every (d, n)*-base abstraction with diameter* δ*, there exists a* (d, n)*-connected layer family of length* $\delta + 1$*, and vice versa.*

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For every (d, n)-base abstraction with diameter δ, there exists a (d, n)-connected layer family of length $\delta + 1$, and vice versa.

Lemma

Every subsequence of a (d, n)-CLF is a (d, n)-CLF.

Lemma (Dimension reduction)

Let $f \subseteq [n]$ be covered in a (d, n)-CLF $S_1, \ldots, S_t$.
Let $S_a, \ldots, S_b$ be the sequence of families that cover f.
Then $S'_a, \ldots, S'_b$ is a $(d-1, n-1)$-CLF, where

$$S'_j := \{a \setminus f \mid f \subset a \in S_j\}.$$

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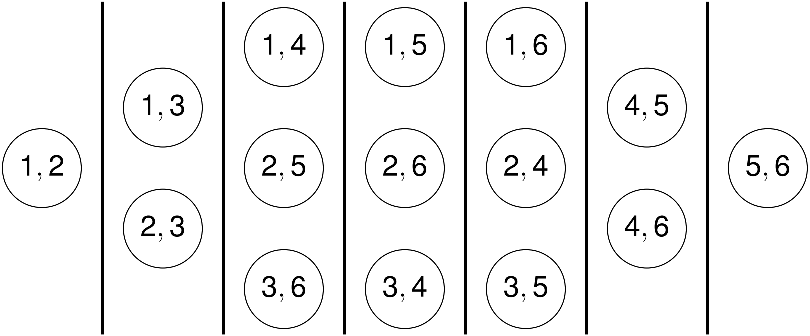
Theorem

The length of a (d, n)-CLF is bounded by $n^{1+\log d}$.

Another Example

- sequence of disjoint non-empty families of d-subsets of $[n]$
- Connectivity: for all $f \subseteq [n]$, the families that cover f form an interval



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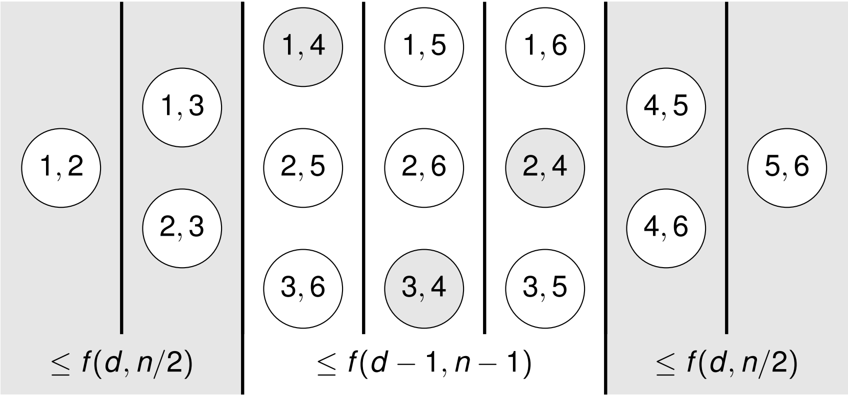


An Almost Quadratic Lower Bound

Theorem (Eisenbrand, H., Razborov, Rothvoß)

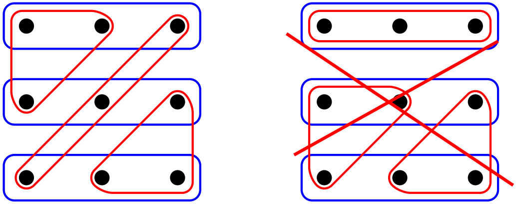
There exist $(n/4, n)$-connected layer families of length $\Omega(n^2/\log n)$.

- Problem: How to keep subsets “alive” for long intervals
- Solution: Use covering designs!

Families of Disjoint Coverings

- An (n, k, r)-covering of a set X of n elements is a collection of k-subsets of X that covers each r-subset of X at least once.
- $DC(n, k, r)$ is the size of a largest family of pairwise disjoint (n, k, r)-coverings.

Example of Disjoint $(9, 3, 1)$-Coverings



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Theorem (Eisenbrand, H., Razborov, Rothvoß)
$DC(n, r+1, r) \geq (n-r)/(3 \ln n)$

Note: $DC(n, r+1, r) \leq n - r$

Large families of disjoint coverings

Theorem (Eisenbrand, H., Razborov, Rothvoß)
$DC(n, r+1, r) \geq (n-r)/(3 \ln n)$

Proof sketch.

- Color $(r+1)$-subsets randomly using $(n-r)/(3 \ln n)$ colors
- Each color class will be one of the coverings
  - coverings are disjoint
- Bad events: an r-subset not covered in one color class
- Use Lovász Local Lemma

□

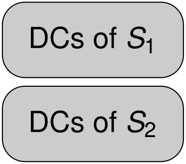
First Attempt: Disjoint Coverings

- Recall: $DC(n, r+1, r) \geq (n-r)/(3 \ln n)$
- Take a family of disjoint $(n, d, d-1)$-coverings $\mathcal{L}_1, \ldots, \mathcal{L}_{(n-r)/(3 \ln n)}$.
- This is a connected layer family of length $(n-d)/(3 \ln n)$.
- No improved lower bound yet.

DCs of $[n]$

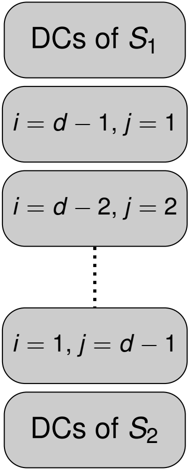
Second Attempt with Split Set of Symbol

- Instead of $[n]$, use two disjoint sets of symbols S_1 and S_2, $|S_1| = |S_2| = m$.
- Take separate families of disjoint $(m, d, d-1)$-coverings and concatenate them.
- Get a connected layer family of length $2(m-d)/(3\ln m)$.
- Length is still sublinear, but now there are many unused potential vertices.

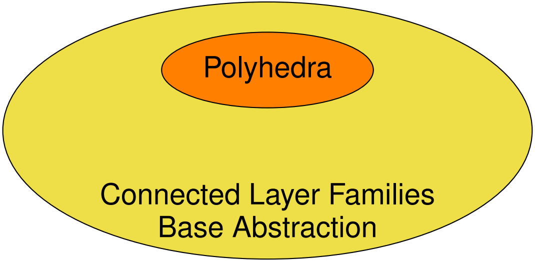


Mixing Sets of Symbols

- Add intermediate blocks for all $i, j > 0$ with $i + j = d$ as follows:
  - Disjoint $(m, i, i-1)$-coverings $A_0, \dots A_{k-1}$ of S_1
  - Disjoint $(m, j, j-1)$-coverings $B_0, \dots B_{k-1}$ of S_2
  - Form the q-th layer by combining sets from A_a with sets from B_b whenever $a + b = q \mod k$.
- Length is now $(d+1) \cdot (m-d)/(3 \ln m)$.
- Yields lower bound $\Omega(n^2 / \ln n)$ for $d = n/4$.



Outline: Hierarchy of Abstractions



Abstract Polyhedra

- Stronger properties can be added onto the Base Abstraction:
  - uv an edge iff $|u \cap v| = d - 1$ [Adler & Dantzig, Kalai]
  - Every existing $(d-1)$-set appears in exactly two vertices [Adler & Dantzig]
- Best lower bounds are linear
- Open problem: Find a separation from Connected Layer Families
  - e.g. every abstract polyhedron yields a strictly larger CLF

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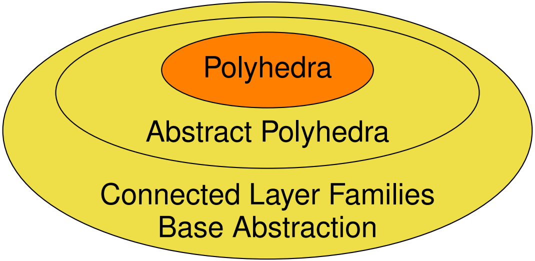

1-shadows cast by the Polymath Project

Definition (Volvovskiy)

A sequence $S_1, \dots, S_t$ of subsets of X is a valid sequence of 1-shadows if

- $\emptyset$ appears at most once
- Convexity: $S_i \cap S_k \subseteq S_j$ for all $i < j < k$
- Restriction: For any $x \in X$, let $S_a, \dots, S_b$ be the subinterval on which x appears. Then there must exist a valid sequence $T_a, \dots, T_b \subseteq X \setminus \{x\}$ with $T_j \subseteq S_j$ for all j

Some valid sequences:

- $\emptyset$
- $\emptyset, \{1\}$
- $\{1,2\}, \{1,2\}, \emptyset$

Not valid:

- $\{1\}, \{1\}$
- $\{1,2\}, \emptyset, \{1,2\}$

1-shadows: Useful properties

Lemma

Every subsequence of a valid 1*-shadow is a valid* 1*-shadow.*

1-shadows: Useful properties

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Every subsequence of a valid 1-shadow is a valid 1-shadow.

Theorem
The length of a 1-shadow on n elements is bounded by $n^{1+\log n}$.

Proof.
$y(n) \leq 2y(n/2) + y(n-1)$ □

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The length of a 1*-shadow on* n *elements is bounded by* $n^{1+\log n}$*.*

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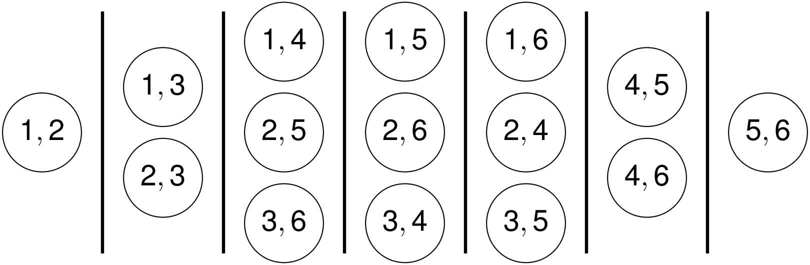
Lemma
The sequence of 1*-shadows (supports) of families of a CLF is a valid sequence of* 1*-shadows.*

1-shadow of a Connected Layer Family



$$\{1,2\}, \{1,2,3\},$$
$$\{1,2,3,4,5,6\}, \{1,2,3,4,5,6\}, \{1,2,3,4,5,6\},$$
$$\{4,5,6\}, \{5,6\}$$

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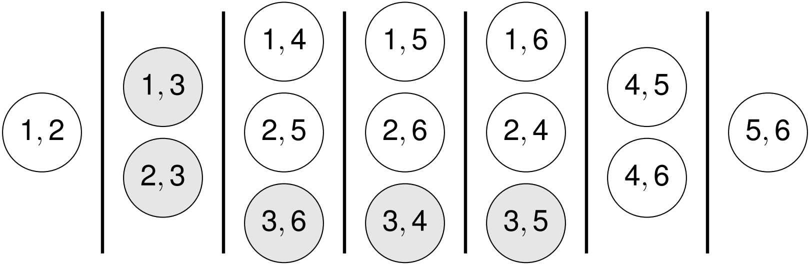

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Restriction to 3: $\{1,2\},\{6\},\{4\},\{5\}$

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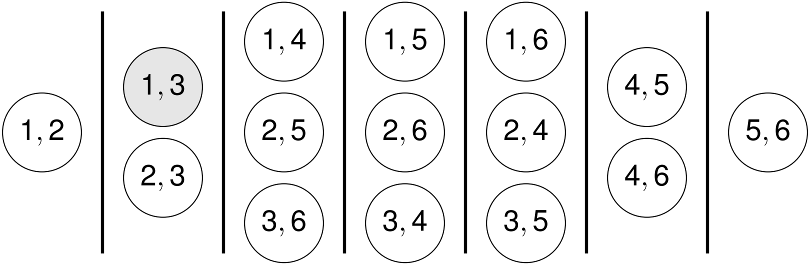

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Restriction to 1: $\emptyset$

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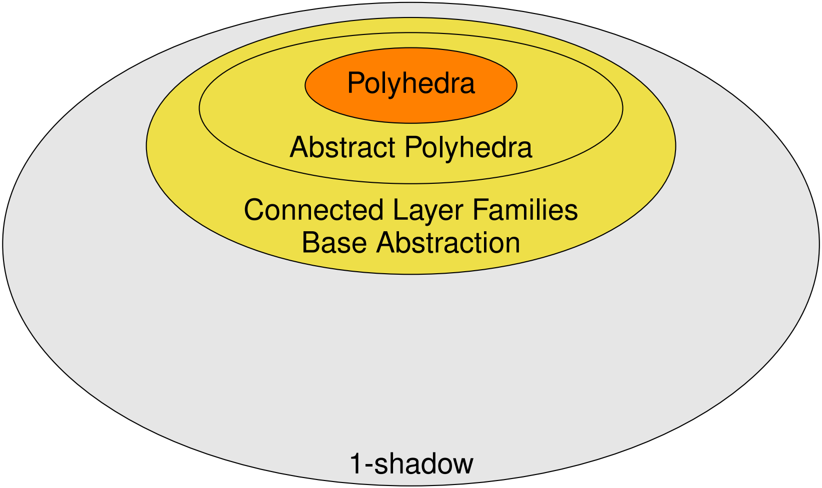

Quasi-polynomial Lower Bound

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Lemma

The sequence of $y(n)$ copies of $[n+1]$ is valid.

Proof.

- $\emptyset$ does not appear
- Convexity
- Restriction on $x \in [n+1]$:
  Let $T_1, \ldots, T_{y(n)}$ be a max. length sequence on n elements
  Map its elements to $[n+1] \setminus \{x\}$ arbitrarily

$\square$

Quasi-polynomial Lower Bound (cont'd)

Definition (Sequence $\mathcal{S}_{n,k}$)

Let A and B be disjoint sets of $n + k$ elements each.
The sequence $\mathcal{S}_{n,k}$ is defined as

- one block of $y(n)$ copies of A,
- followed by $(k - 2)$ blocks of $y(n)$ copies of $A \cup B$,
- followed by one block of $y(n)$ copies of B

Total length of $ky(n)$ on $2(n + k)$ elements.

$$\underbrace{(\mathbf{A}, \mathbf{A} \cup \mathbf{B}, \ldots, \mathbf{A} \cup \mathbf{B}, \mathbf{B})}_{k \text{ blocks}}$$

Quasi-polynomial Lower Bound (cont'd)

Lemma

$\mathcal{S}_{n,k}$ is valid for all $n \geq 1$, $k \geq 2$.

Proof.

$k = 2$: $\mathcal{S}_{n,2} = (\mathbf{A}, \mathbf{B})$

- Restriction to $a \in A$: $(\mathbf{A}, \mathbf{B}) \to (\mathbf{A}')$, $A' = A \setminus \{a\}$
- $y(n)$ copies of A' are valid by previous Lemma
- Restriction to $b \in B$ analogous

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$k \geq 3$: $\mathcal{S}_{n,k} = (\mathbf{A}, \mathbf{A} \cup \mathbf{B}, \dots, \mathbf{A} \cup \mathbf{B}, \mathbf{A} \cup \mathbf{B}, \mathbf{B})$

- Restriction to $a \in A$:

$$\begin{aligned} &(\mathbf{A}, \mathbf{A} \cup \mathbf{B}, \dots, \mathbf{A} \cup \mathbf{B}, \mathbf{A} \cup \mathbf{B}, \mathbf{B}) \\ \to &(\mathbf{A}', \mathbf{A}' \cup \mathbf{B}', \dots, \mathbf{A}' \cup \mathbf{B}', \mathbf{B}') \cong \mathcal{S}_{n,k-1}, \end{aligned}$$

where $A' = A \setminus \{a\}$, $B' = B \setminus \{b\}$, $b \in B$ arbitrary

$\square$

Quasi-polynomial Lower Bound (cont'd)

Lemma

$y(4n) \geq ny(n)$

Proof.

$\mathcal{S}_{n,n}$ is a sequence of length $ny(n)$ on $2(n+n) = 4n$ elements. $\square$

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Theorem

$y(n) \geq n^{\Omega(\log n)}$

The Magic Trick

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- Is there a corresponding quasi-polynomial CLF?

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- Stylize $\mathcal{S}_{n,k}$ as $\{1\}, \{1,2\}, \{1,2\}, \{2\}$
- For the recursion, we construct the uniform sequence $\{1,2,3\}, \{1,2,3\}, \{1,2,3\}, \{1,2,3\}$
  - Restriction to 3: $\{1\}, \{1,2\}, \{1,2\}, \{2\}$
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  - Restriction to 2:

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  - Restriction to 2: $\{1\}, \{1,3\}, \{1,3\}, \{3\}$
- Inconsistency in the first set: we get 1 when restricting on 2, but we *do not* get 2 when restricting on 1

- This kind of inconsistency does not occur with 1-shadows that are derived from CLFs.

Commutativity

Definition (Commutativity)

A valid 1-shadow sequence is commutative if restrictions can be done in any order without changing the result.

Lemma

A commutative valid 1*-shadow sequence is the* 1*-shadow of a CLF consisting of arbitrary-size subsets.*

m-shadows

- Generalize 1-shadows to m-shadows
- Quasi-polynomial lower bound also for 2-shadows [H.]
  - Slightly worse constant in the exponent
- Open problems:
  - Understand possible constructions for 2-, 3-, m-shadows
  - Probably quasi-polynomial for all constant m
  - Better upper bound for 2-shadows, leading to separation statements?

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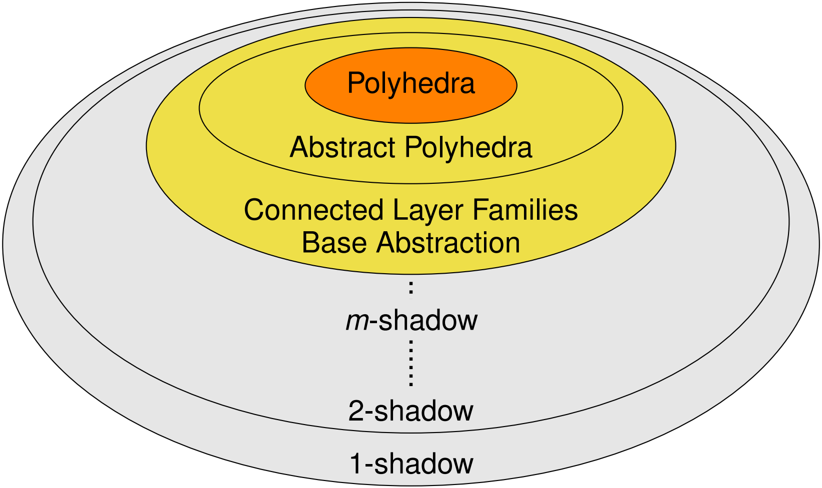

And now for something different

- Consider CLF, but with d-multisets instead of d-sets
- In a sense, the max. diameter of set-CLF and multiset-CLF are almost equal (generalization of our construction)
- Two very simple constructions give length $d(n-1)+1$

$1\ldots111$

$1\ldots112$

$1\ldots122$

$\vdots$

$2\ldots222$

$2\ldots223$

$\vdots$

$n\ldots nnn$

- $X = \{0, 1, \ldots, n-1\}$
- $\phi(a) = \sum_{j \in a} j$ for a a d-multiset of X
- $S_j = \phi^{-1}(j)$, $j = 0 \ldots d(n-1)$
- $S_0, \ldots, S_{d(n-1)}$ is a CLF

And now for something different

- Consider CLF, but with d-multisets instead of d-sets
- In a sense, the max. diameter of set-CLF and multiset-CLF are almost equal (generalization of our construction)
- Two very simple constructions give length $d(n-1)+1$

Conjecture

These constructions are best possible, i.e. the max. length of multiset-CLFs is $d(n-1)+1$.

"Evidence"

- The conjecture holds when each family is a singleton (by a potential function proof)
- The conjecture holds when the multiset-CLF contains *all* possible d-multisets (by induction on d)
- Computational checks for small cases

$d =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$n = 1$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
2	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
3	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
4	$\surd$	$\surd$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	
5	$\surd$	$\surd$	$+$	$+$	$+$	$+$	$+$							
6	$\surd$	$\surd$	$+$	$+$	$+$									
7	$\surd$	$\surd$	$+$	$+$										
8	$\surd$	$\surd$	$+$											
9	$\surd$	$\surd$												

Summary

- Main results:
  - Use abstractions to understand the gap between linear constructions of polytopes and quasi-polynomial upper bound
  - Quadratic lower bound for Connected Layer Families using Lovász Local Lemma
  - Quasi-polynomial lower bound for 1-shadows
- Open problems:
  - Close the gap between $3n$ and $4n$ for CLF with $d = 3$
  - Constructions of long m-shadow sequences
    - see the Polymath 3 threads on Gil Kalai's blog
  - Separation between Abstract Polyhedra and CLF
  - Separation between 1-shadow and m-shadow
  - Resolve the multiset-CLF conjecture