# On "real" and on "bad" Linear Programs

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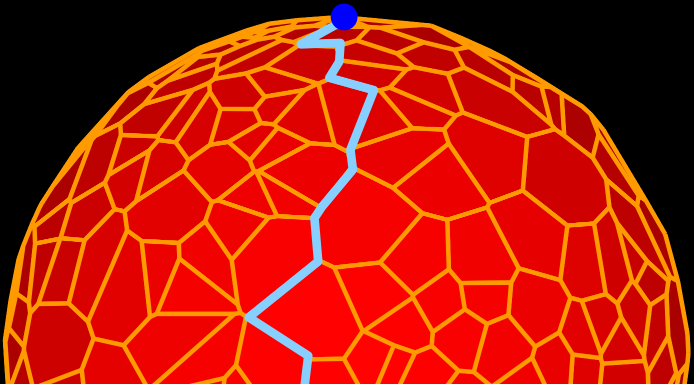

# On “real” and on “bad” LPs

... but when they had taken their seats Her Majesty turned to the president and resumed.

“... was he nevertheless as bad as he was painted? Or more to the point,” and she took up her soup spoon,

# On “real” and on “bad” LPs

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“. . . was he nevertheless as bad as he was painted? Or more to the point,” and she took up her soup spoon, “was he as good?”

Alan Bennett: The Uncommon Reader, 2007

# On "real" and on "bad" LPs

While proving the *existence* of bad instances for the widely used simplex algorithm was a major accomplishment when this work was done originally (around 1969), the period following its discovery was, however, characterized by – what the author calls – *worstcasitis*. The 1970's and 1980's were abundant with articles in the professional journals that reported *negative* "existence" results of this kind for all sorts of problems. Worstcasitis appears to be a very catching sort of phenomenon even today in the 1990's – may be due to the fact that all that it requires for its execution, besides a *brain* of course, is paper, pencil and eraser.

Manfred Padberg, "Linear Optimization and Extensions", 1995

On “real” LPs

On “bad” 3D LPs

On `RandomEdge` on KM-cubes

# On “real” LPs

. . . based on joint work
with Stefan Fischer (1998)
and with Dietmar Weber (1999)

# On “real” LPs

## The NETLIB LP Test Problem Set

The NETLIB Linear Programming test set is a collection of real-life linear programming examples from a variety of sources. The examples are available in MPS format, which is a subset of the SIF format used by CUTEr. Thus, the NETLIB set provide a further collection of interesting examples for those who have CUTEr interfaces to their optimization packages.

# On “real” LPs

`afiro.lp`:

32 variables

8 equations

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19 explicit inequality constraints
32 non-negativities
29 facets

# On "real" LPs

```
\Problem name: afiro.lp

Minimize
 COST: - 0.4 X02 - 0.32 X14 - 0.6 X23 - 0.48 X36 + 10 X39
Subject To
 R09: - X01 + X02 + X03  = 0
 R10: - 1.06 X01 + X04  = 0
 X05: X01 <= 80
 X21: - X02 + 1.4 X14 <= 0
 R12: - X06 - X07 - X08 - X09 + X14 + X15  = 0
 R13: - 1.06 X06 - 1.06 X07 - 0.96 X08 - 0.86 X09 + X16  = 0
 X17: X06 - X10 <= 80
 X18: X07 - X11 <= 0
 X19: X08 - X12 <= 0
 X20: X09 - X13 <= 0
 R19: - X22 + X23 + X24 + X25  = 0
 R20: - 0.43 X22 + X26  = 0
 X27: X22 <= 500
 X44: - X23 + 1.4 X36 <= 0
 R22: - 0.43 X28 - 0.43 X29 - 0.39 X30 - 0.37 X31 + X38  = 0
 R23: X28 + X29 + X30 + X31 - X36 + X37 + X39  = 44
 X40: X28 - X32 <= 500
 X41: X29 - X33 <= 0
 X42: X30 - X34 <= 0
 X43: X31 - X35 <= 0
 X45: 2.364 X10 + 2.386 X11 + 2.408 X12 + 2.429 X13 - X25 + 2.191 X32
      + 2.219 X33 + 2.249 X34 + 2.279 X35 <= 0
 X46: - X03 + 0.109 X22 <= 0
 X47: - X15 + 0.109 X28 + 0.108 X29 + 0.108 X30 + 0.107 X31 <= 0
 X48: 0.301 X01 - X24 <= 0
 X49: 0.301 X06 + 0.313 X07 + 0.313 X08 + 0.326 X09 - X37 <= 0
 X50: X04 + X26 <= 310
 X51: X16 + X38 <= 300
End
```

# On "real" LPs: `afiro.lp`

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dimension: 24
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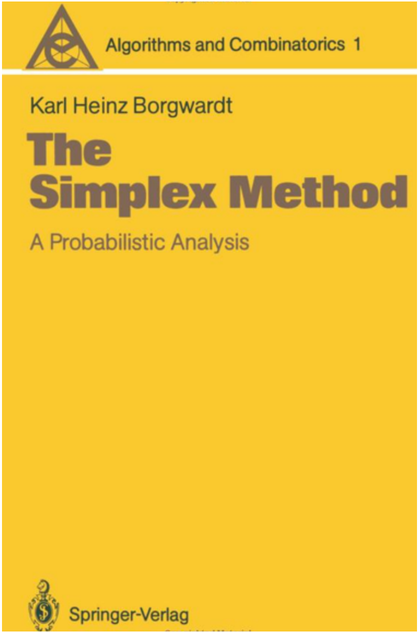
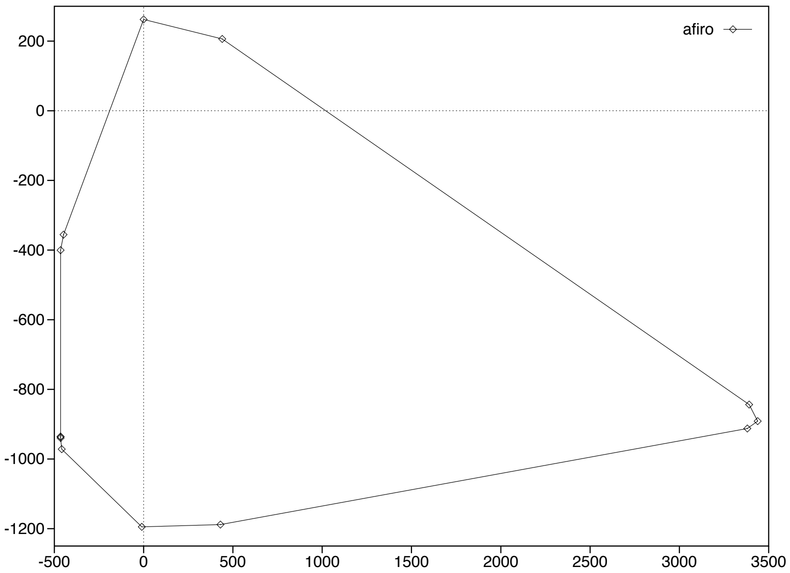
11718 horizontal edges (more than half of the edges horizontal!)
maximal vertex: unique, degree 39
minimal vertex: 4 of them, 2-face
graph distance minimal vertices to maximal vertex: 2
graph diameter: 5 (Hirsch conjecture sharp!)

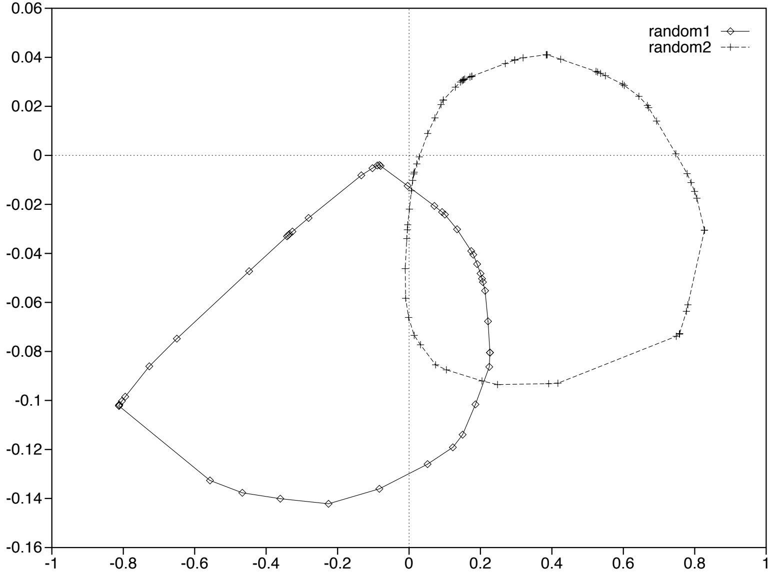
# On "real" LPs: Pictures??



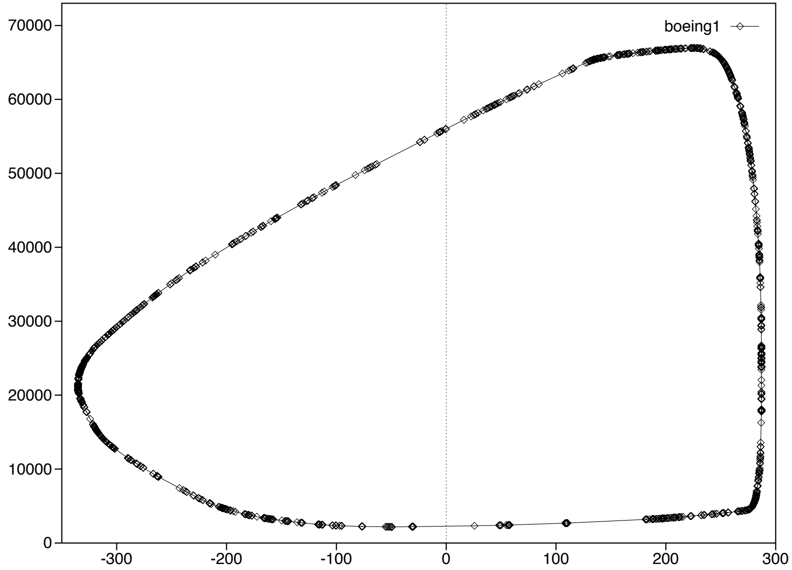
# On “real” LPs: `afiro.lp`



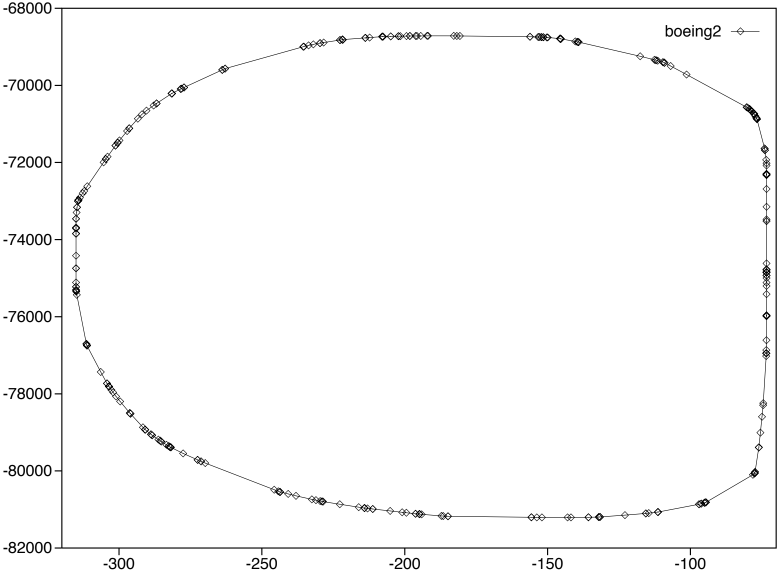
# On “real” LPs: afiro.lp



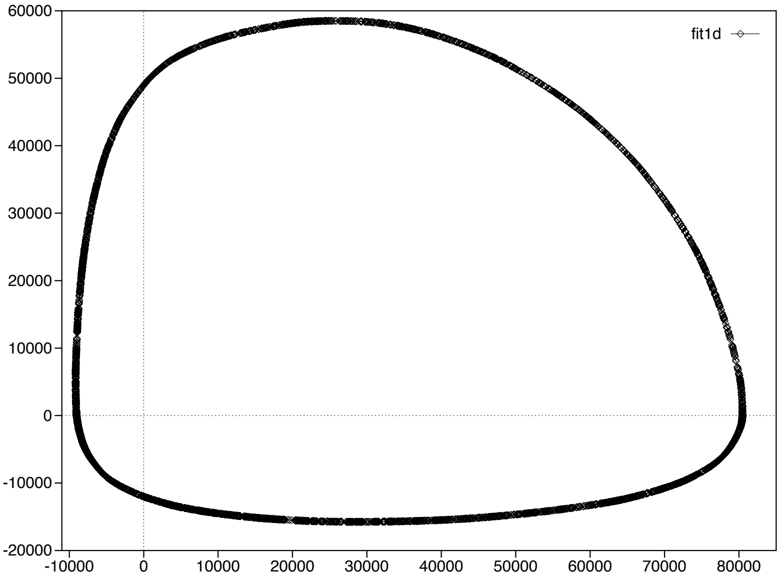
# On “real” LPs: `boeing1.lp`



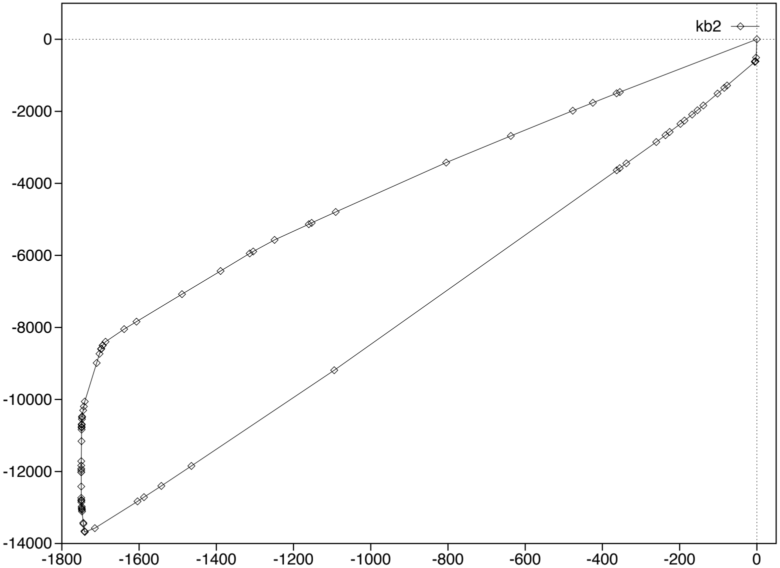
# On “real” LPs: `boeing2.lp`



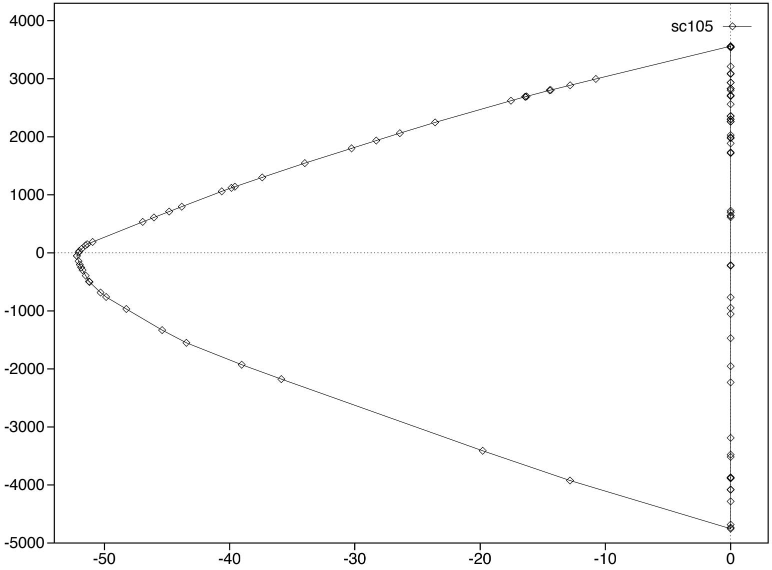
# On “real” LPs: `fit1d.lp`



# On "real" LPs: `kb2.lp`



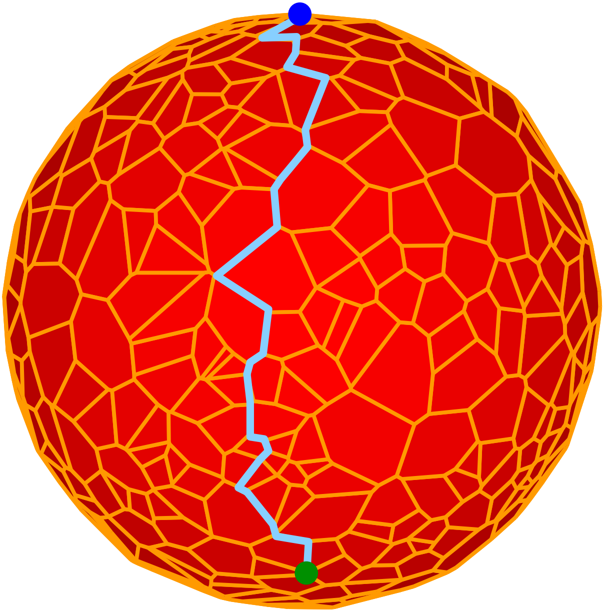
# On "real" LPs: `sc105.lp`



# On “bad” 3D LPs

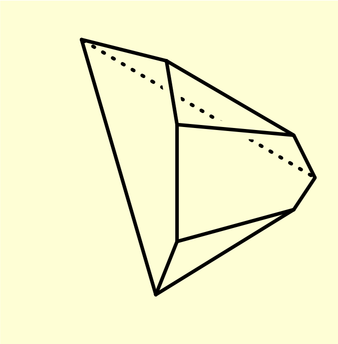
. . . joint work with
Volker Kaibel, Rafael Mechtel
and Micha Sharir
(*SIAM J. Comp.* 2005)

# On “bad” 3D LPs

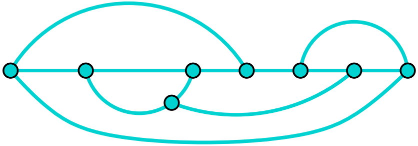


# On “bad” 3D LPs

Geometry vs. Combinatorial Model [Steinitz 1922]:



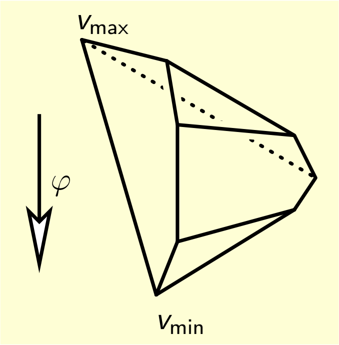
3-polytope $P$

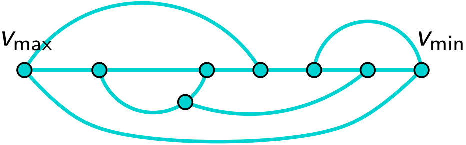


planar, 3-connected graph $G$
(no loops, no parallel edges)

# On "bad" 3D LPs

Geometry vs. Combinatorial Model [Mihalisin & Klee 2000]:




3-polytope $P$
generic linear function
$\varphi : \mathbb{R}^3 \to \mathbb{R}$

- 3-polytopal graph $G$
- acyclic orientation with unique sink in every face (AOF)
- three disjoint monotone paths from $v_{max}$ to $v_{min}$

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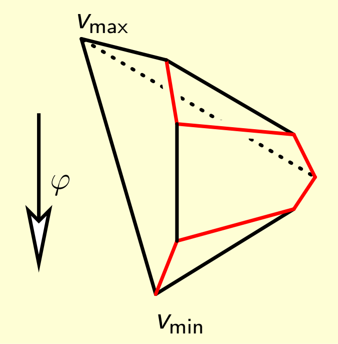


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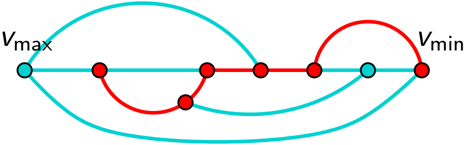


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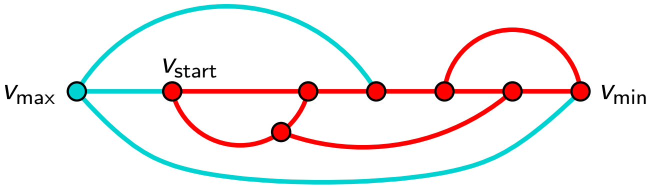
# On "bad" 3D LPs: RandomEdge

RandomEdge takes a step to an improving neighbor chosen uniformly at random:



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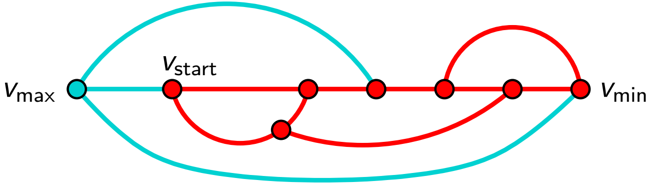



**Theorem:**
*The worst-case running time of* RandomEdge *on a 3D polytope with $n$ facets is bounded by*

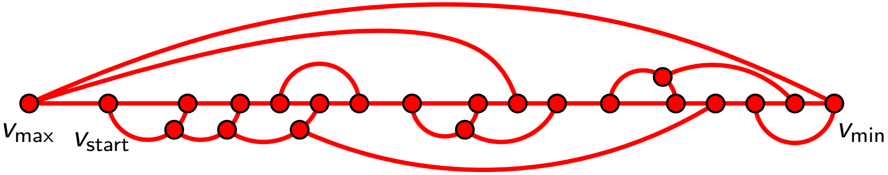
$$1.3473\, n \;\leq\; \texttt{RandomEdge}(n) \;\leq\; 1.4943\, n.$$

# On "bad" 3D LPs: `RandomEdge`

For $n \leq 12$, we enumerated 3-connected cubic graphs with $n$ faces using `plantri` by [Brinkmann & McKay]

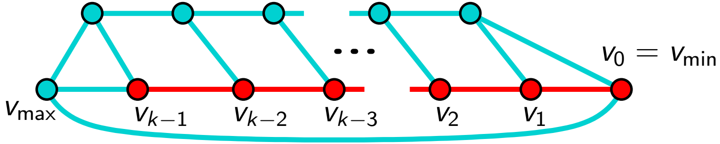
... and all the "abstract objective functions" on each of these ...

... to see what worst-case examples *look like*:



# On "bad" 3D LPs: `RandomEdge`

the "backbone":



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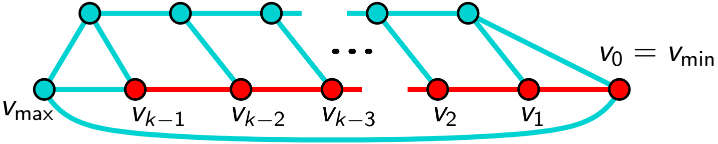
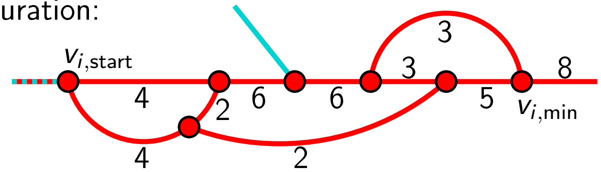



a configuration:



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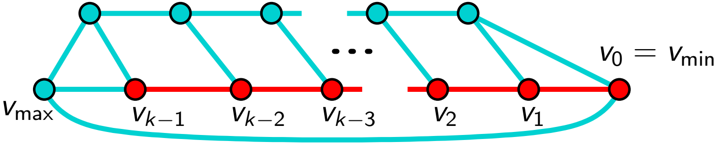


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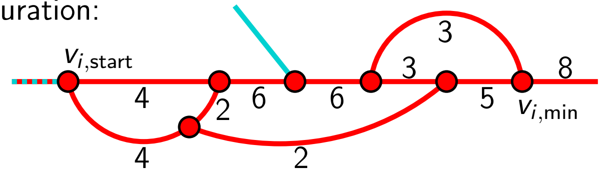


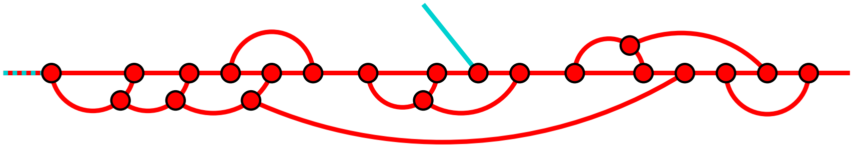

flow costs per configuration: $\frac{43}{8}$

facets per configuration: $3 + 1$

$\implies$ $\texttt{RandomEdge}(n)/n \geq \frac{43}{8 \cdot 4} \approx 1.3437$

# On “bad” 3D LPs: RandomEdge

a worse configuration:



flow costs per configuration: $\frac{1897}{128}$

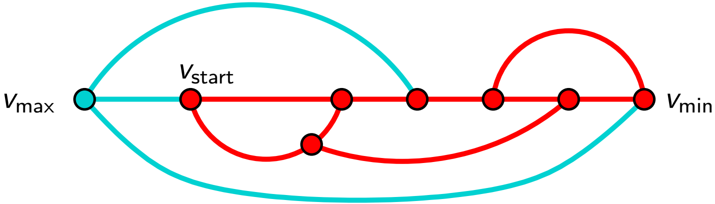
facets per configuration: $10 + 1$

$\Longrightarrow \quad \texttt{RandomEdge}(n)/n \geq \frac{1897}{1408} \approx 1.3473.$

# On “bad” 3D LPs: `RandomEdge`

Recursion formula for the expected number of pivot steps $\mathbf{E}(v)$ “starting from $v$”:

$$\mathbf{E}(v) \;=\; 1 + \frac{1}{|\,\delta^+(v)|} \sum_{u:(v,u)\in\delta^+(v)} \mathbf{E}(u),$$



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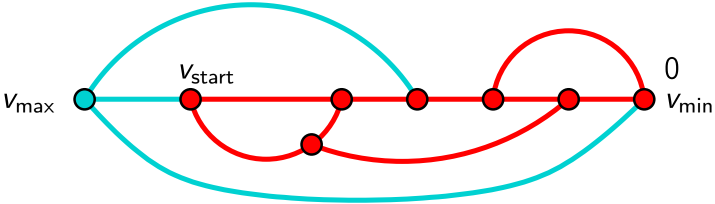

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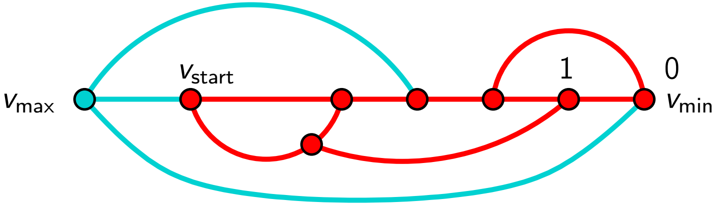

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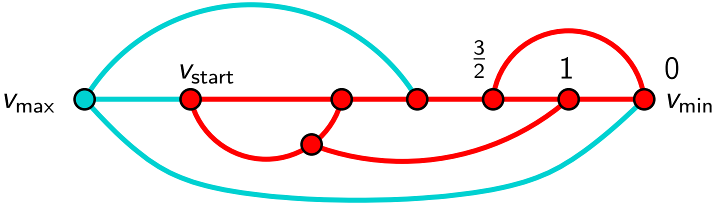

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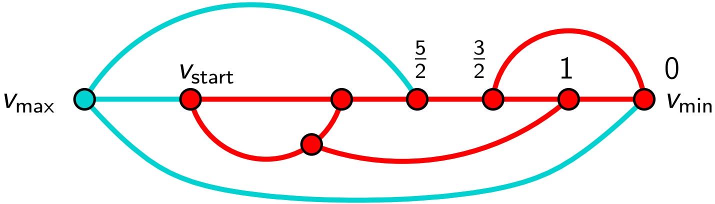

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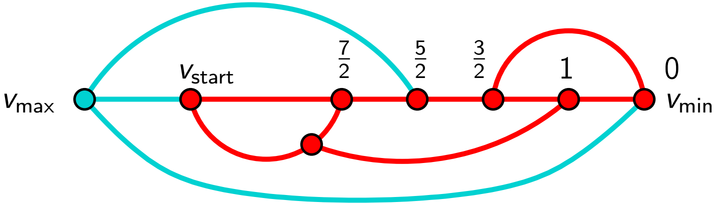

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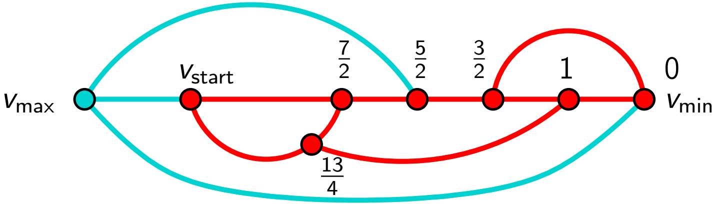

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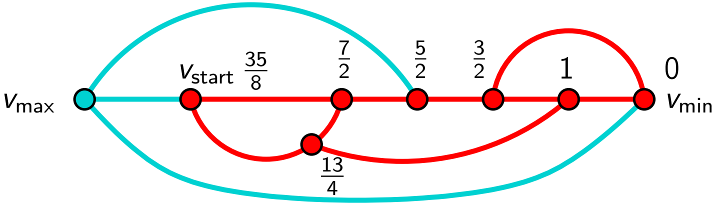

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*The worst-case running time of* `RandomEdge` *on a 3D polytope with n facets is bounded by*

$$1.3473\, n \;\le\; \texttt{RandomEdge}(n) \;\le\; 1.4943\, n.$$

**Proof of Upper bound:**
Complicated, LP-optimized induction
based on the numbers of 1- and 2-vertices "ahead"
by case analysis of possible local structure.

# On `RandomEdge` on KM-cubes

. . . joint work with Bernd Gärtner

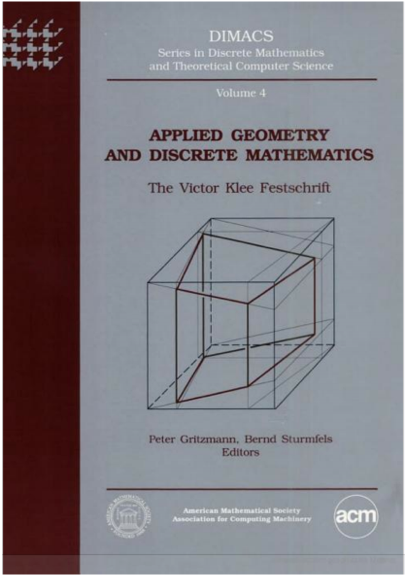
(*Combinatorica* 1998)

# RandomEdge on KM-cubes

[Klee & Minty 1972]:

$$\min x_n : \quad \begin{aligned} 0 \le \quad & x_1 \quad \le 1, \\ \varepsilon x_{i-1} \le \quad & x_i \quad \le 1 - \varepsilon x_{i-1} \qquad \text{for } 2 \le i \le n. \end{aligned}$$

# RandomEdge on KM-cubes



# RandomEdge on KM-cubes

[Klee & Minty 1972] – Geometry:

"deformed product" [Amenta & Z. 1998] [Sanyal & Z. 2010]

# `RandomEdge` on KM-cubes

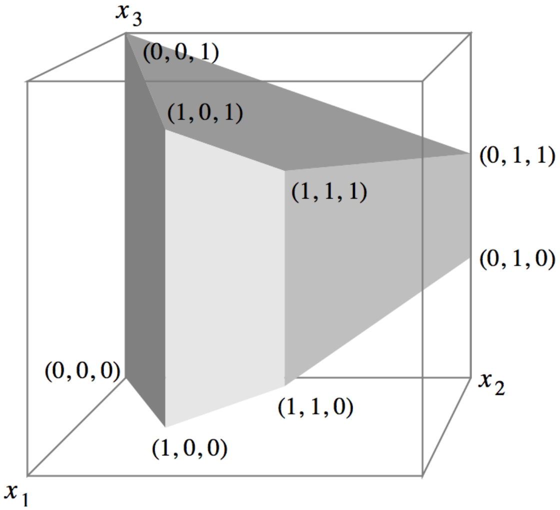
[Klee & Minty 1972] – Geometry:

"deformed product" [Amenta & Z. 1998] [Sanyal & Z. 2010]

… *not* a projective cube, cf. [Gritzmann & Klee 1993]

# RandomEdge on KM-cubes

[Klee & Minty 1972] – Combinatorial model:



# RandomEdge on KM-cubes

[Klee & Minty 1972] – Combinatorial model:

110101000**1**0101010111001

↓

# RandomEdge on KM-cubes

[Klee & Minty 1972] – Combinatorial model:

$$\begin{array}{c} 1101010001010101011 1001 \\ \downarrow \\ 1101010000101010 1000110 \\ \downarrow \end{array}$$

# RandomEdge on KM-cubes

[Klee & Minty 1972] – Combinatorial model:

$$1101010001010101011 1001$$

$$\downarrow$$

$$11010100001010101000110$$

$$\downarrow$$

$$11010100001010010111001$$

$$\downarrow$$

# RandomEdge on KM-cubes

[Klee & Minty 1972] – Combinatorial model:

1101010001010101010111001

↓

1101010000101010101000110

↓

1101010000101001010111001

↓

1010101111010110101000110

↓

# RandomEdge on KM-cubes

[Klee & Minty 1972] – Combinatorial model:

110101000**1**0101010111001

↓

1101010000101**0**1**1**01000110

↓

11010**1**0000101001011 1001

↓

10101011**1**10101101000110

↓

1010101**1**00101001011 1001

↓

# RandomEdge on KM-cubes

[Klee & Minty 1972] – Combinatorial model:

11010100010101010111001

↓

11010100001010101000110

↓

11010100001010010111001

↓

10101011110101101000110

↓

10101011001010010111001

↓

10101010110101101000110

↓

...

# `RandomEdge` on KM-cubes

Recursion for running time of `RandomEdge`:

$$\mathsf{E}(v) \;=\; 1 + \frac{1}{|\,\delta^{+}(v)|} \sum_{u:(v,u)\in\delta^{+}(v)} \mathsf{E}(u),$$

# RandomEdge on KM-cubes

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Conjectures/claims to deal with:

$\mathbf{E}(\mathrm{KM}_n) = O(n \log^2 n)$ [Kelly 1981]

$\mathbf{E}(\mathrm{KM}_n) = O(n)$ [Papadimitriou & Steiglitz 1982]

Worst starting vertex is "111111111" [Kelly 1981]

# RandomEdge on KM-cubes

**Theorem:** [Gärtner, Henk & Z. 1998]

$$\frac{n^2}{4 \ln n} \;\le\; \mathbf{E}(KM_n) \le .32\, n^2$$

# RandomEdge on KM-cubes

**Theorem:** [Gärtner, Henk & Z. 1998]

$$\frac{n^2}{4 \ln n} \leq \mathbf{E}(KM_n) \leq .32\, n^2$$

**Theorem:** [Balogh & Pemantle 2007]

$$\mathbf{E}(KM_n) = \Theta(n^2)$$

# RandomEdge on KM-cubes

**Theorem:** [Gärtner, Henk & Z. 1998]
*For $n = 18$, the "worst" starting vertex is not "111111111",*

```
1361663177777584468336302012312709410281356053324920159268403046401503659774916417793789249705463620
0014861829248286017175089542461228218802989390186162377748591120627796205623721274772410697146678060
1096675034342888422786286326399224062694418947409232169057846585014034955997834680378890167612182220
7951451033736676735785044550350731373607453530013707590565861401280889111865631640837448167004916720
7166072497354996638189878102467117451912337103346033095758655541503342507772700446077240059270064950
6305746549090037936487607501761933101578404903655821528503848926053516823767931083443898021126182840
1614272810610376487170928250259503884429560878889631798414010547998386203959837598713021428200303300
2052105152771736370751467816130883622125864123214684927809076675535445933100062020394631057459368050
9164712870765625784398603492731015948824318559801105435316612287356531723641848939276240045046659110
0116943212840182943096449606773730907217985181511018585460692333801910443297850720355254772152895690
85400094903678817126979242634733
                                    /
2496291446958319156636789992396003596640419584806186767408425732535144354163570710815596122625324870
8789077906419399357235040754088248831346248044200918832296530817579383490835071177547072306360021580
0765762337436212646695721708327780432254302122068497800620269764500611292572756268849995101900835220
7320299869843308292367041046047108762061616574160647672349410603907269497899539762736245788609695030
0996350444990026613028146681290461119909853980542193260108618871426465727837929625521663877284022410
0357286421933258150684537608721292656864711442853225504758275720083398147051800881880568830663414790
1333838085809804789755619177701459783321339749058258330528564296148410322864550687944381013225922510
3863515233531187923343044085698617343029677874335003360472318789969971282626154255624606754882428890
8819919126696798095680976420617693531925698516674643609719123572292420464412866621328622978307150680
3243621987053704093779359678915870448936997500297408069568430319667687549224857916309386295913835970
80655177475508923698668685694367

*** ROUNDBF turned on to increase accuracy

54.5474439468

2#101010101010101010
```

# RandomEdge on KM-cubes

**Theorem:** [Gärtner, Henk & Z. 1998]

*For $n = 18$, the "worst" starting vertex is not "111111111", but*

```
1361668442565599386880993034282968141651374597211754205969128255878926535938140559202773823506716853
2649094474502003282183145834841470837035995972128447951947130506979667382332374946828505170577127314
9743146612077498828676102187714788642448518642967746683023065650514011256729973419973774731691404711
4945652566890599084339529257062797350550415380335478368936623772478862118743831624530759469457884484
2966129076787888242615886240011646458028300528705172152355076183820248053796028631320621537010448652
3317918515595118742547864414827236101520955286219420603918447294229618054364363953678303134049147181
8264425308985016735882996127064468965001181902100688921358761869506807941689559009436142330396623491
9793924008985871969359359437485822557491942167677047580529125681383103332274589436402198639643389044
6750266844939068101800489157270326481138465335046284528886276366194391151087107687034539327118273296
6016324493571748936964960847791892891535138576704490731009570051858209663082317241306932139811940038
82068757071877644544929382637646
                               /
2496291446958319156636789992396003596640419584806186767408425732535144354163570710815596122625324878
8789077906419399357235040754088248831346248044200918832296530817579383490835071177547072306360021580
0765762337436212646695721708327780432254302122068497800620269764500611292572756268849995101900835227
7320299869843308292367041046047108762061616574160647672349410603907269497899539762736245788609695030
0996350444990026613028146681290461119909853980542193260108618871426465727837929625521663877284022410
0357286421933258150684537608721292656864711442853225504758275720083398147051800881880568830663414791
1333838085809804789755619177701459783321339749058258330528564296148410322864550687944381013225922513
3863515233531187923343044085698617343029677874335003360472318789969971282626154255624606754882428898
8819919126696798095680976420617693531925698516674643609719123572292420464412866621328622978307150683
2436219870537040937793596789158704489369975002974080695684303196676875492248579163093862959138359780
6551774755089236986686856943 67
```

54.5476548512

2#101010101010101011

# `RandomEdge` on KM-cubes

**Conclusions:**

`RandomEdge` is charming and simple, but awful to analyze.

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Experiments help.

# On "real" and on "bad" LPs

While proving the *existence* of bad instances for the widely used simplex algorithm was a major accomplishment when this work was done originally (around 1969), the period following its discovery was, however, characterized by – what the author calls – *worstcasitis*. The 1970's and 1980's were abundant with articles in the professional journals that reported *negative* "existence" results of this kind for all sorts of problems. Worstcasitis appears to be a very catching sort of phenomenon even today in the 1990's – may be due to the fact that all that it requires for its execution, besides a *brain* of course, is paper, pencil and eraser.

# On "real" and on "bad" LPs

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In any case, a good cure against worstcasitis is known: it consists of a *heavy dose* of numerical computing using a *real* digital computer and not a *hypothetical* one. Just set $n = 100$ in the above example and pick $a = b = 2$ and $c = 5$. You will never see a problem like that if you truly *compute* because even the most powerful computers existing today or tomorrow are and will be just "too small" to process the numbers involved, yet we have a linear program with only 100 variables. But then a hypothetical computer

Manfred Padberg, "Linear Optimization and Extensions", 1995

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Mathematical
School

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