Algorithms and Conjectures for Linear Optimization

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Some Landmarks in LO

1947	simplex method	Dantzig	efficient in practice
1957	Hirsch conjecture	Hirsch	theoretical
1972	exponential example for simplex method	Klee and Minty	theoretical (worst case)
1979	ellipsoid method (polynomial)	Khachiyan	not efficient in practice
1984	projective interior point method	Karmarkar	efficient in practice
1985	analytic center central path	Sonnevend, Megiddo	key setting for modern interior point methods
1989	best complexity for interior point methods	Renegar, Roos/Vial, Gonzaga	$O(n^3L)$ complexity
1989	primal-dual interior point method	Kojima, Mizuno and Yoshise	dominant since then
2004	Klee-Minty Example for Interior Point Methods	Deza, Nematollahi, Peyghami, Terlaky	dominant since then
2010	Hirsch conjecture false	Santos	theoretical (worst case)

Linear Optimization: Primal-Dual Pair

Standard form for linear optimization problem:

$$
\begin{array}{ll}
\text{min} & c^T x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
\qquad\qquad
\begin{array}{ll}
\text{max} & b^T y \\
\text{s.t} & A^T y + s = c \\
 & s \geq 0
\end{array}
$$

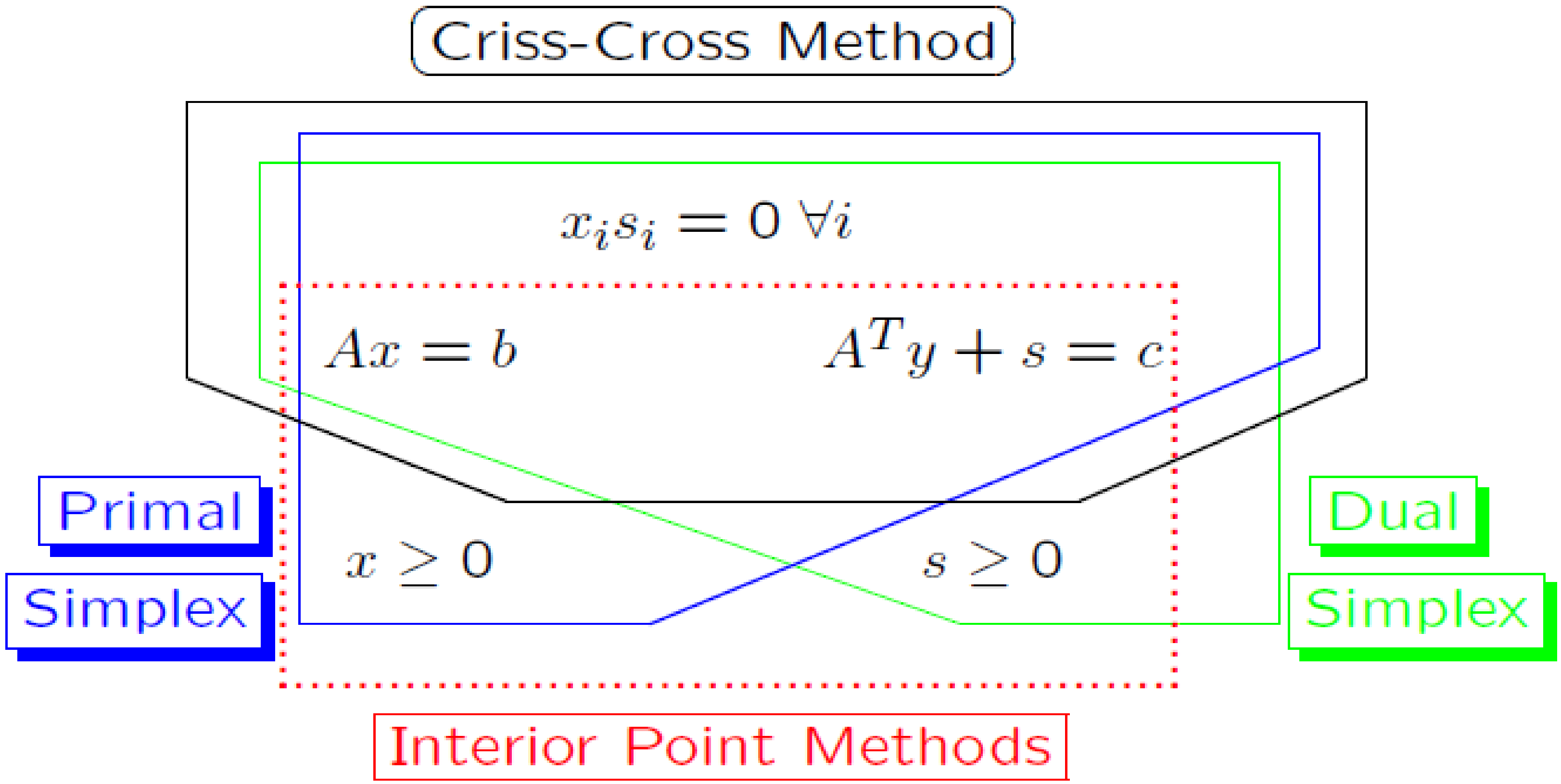
where A:n x m has full row rank. **Optimality:** $x^T s=0$, or $x_i s_i=0$ for all i, or $c^T x = b^T y$.

Simplex (criss-cross) Methods

- start from a feasible (any) basis
- use a pivot rule
- find an optimal solution (after finite number of iterations)
- most pivot rules variants are known to be exponential
  nevertheless very efficient implementations exist.

Linear Optimization: Fundamentals

Standard form LO model: Optimality Conditions



Algorithmic concepts

Linear Optimization: Pivot Algorithms

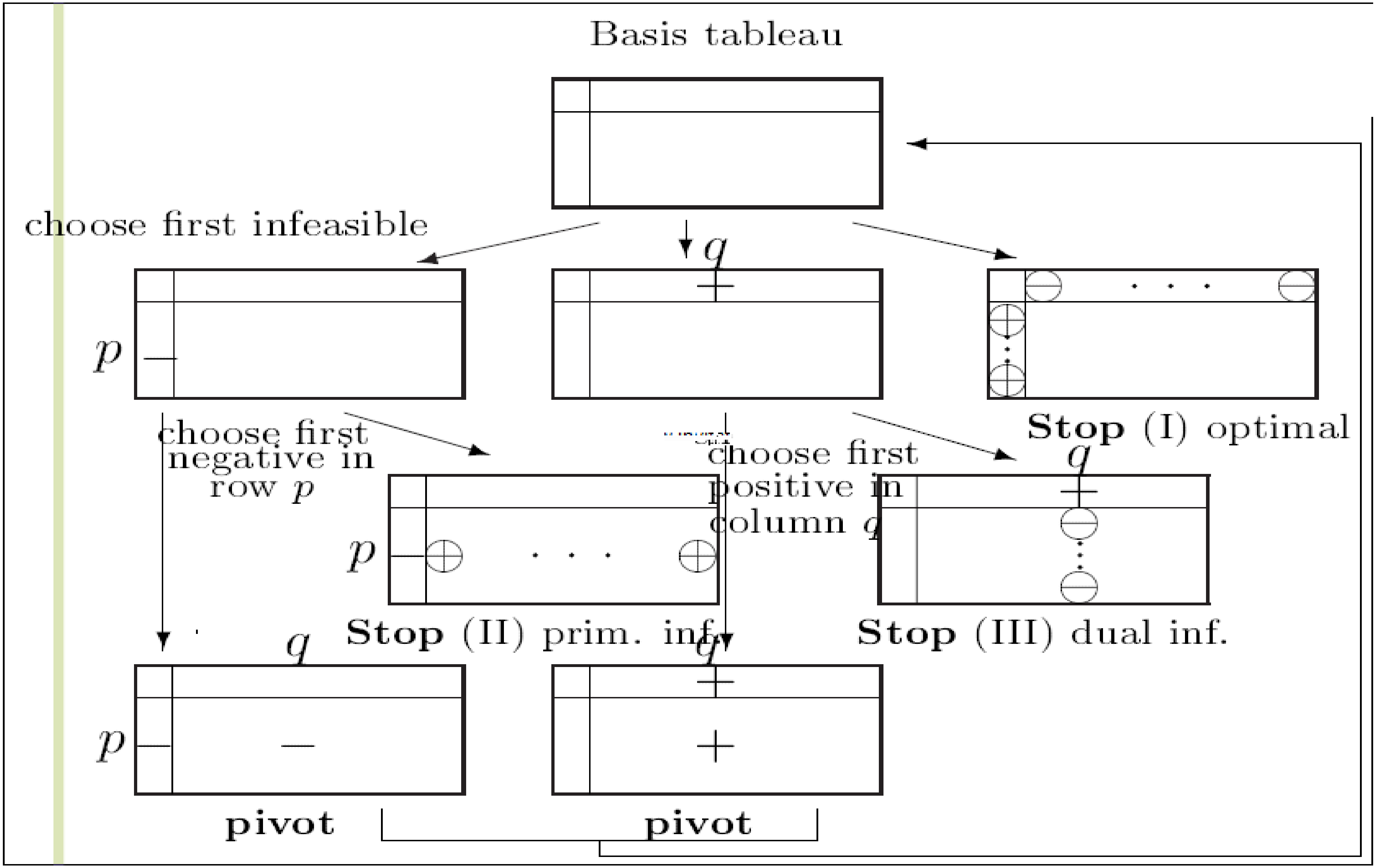
Simplex algorithms

- Primal Simplex
- Dual Simplex (Umbrella)
  == primal simplex for dual LO
- Monotonic-build-up simplex method **(MBUSA)** ⟺
- Rules:
  - **Lexicographic**
  - **Least index**
  - **Shadow vertex**
  - **Steepest edge**
  - **Dantzig's rule**
  - **Largest descent**
  - **Harris**
  - **ZADEH's RULE (cycles)**

Criss-Cross Algorithms

- Dantzig's Parametric Self-dual simplex method
- Zionts' criss-cross
- Least-index C-C
  - Explore flexibilities
- Last-In First-Out (LIFO)
- Edmonds-Fukuda Rule
- Todd's primal-dual lexicography
- General recursions

The Least-Index Criss-Cross Scheme



Some open and solved problems

Fact:* *Polynomial pivot alg. is strongly polynomial.

Solved

- Consider admissible pivots **(respecting primal or dual simplex sign restrictions, but no ration test, no feasibility requirement)**

  ***From any basis* there exists an *admissible pivot sequence* to an optimal basis with at most *m* pivots**

Open problems

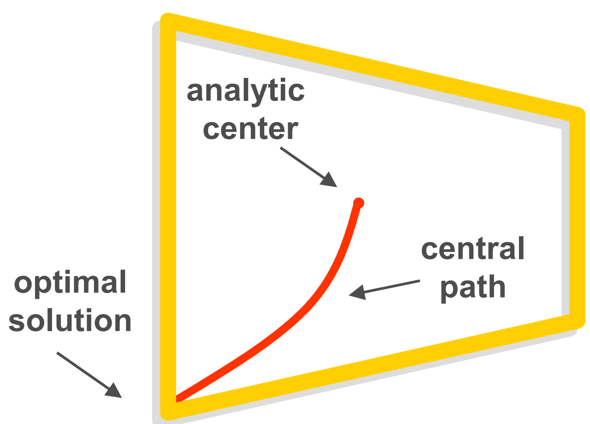
- **Is there a strongly polynomial time algorithm to solve LO?**
- **Is there a s-polynomial pivot algorithm to solve LO?**
- **Is there a s-polynomial admissible pivot alg. to solve LO?**
- **From any given feasible basis is there a feasible pivot sequence of polynomial (linear) length to an optimal basis?**
- **Is there a s-polynomial simplex algorithm to solve LO?**
- **Polynomial (linear) Hirsch conjecture?**

Analytic center, central path and complexity

- The central path start from the analytic center
- Central Path-Following IPMs follow the central path
- It converges to a strictly complementary optimal solution
- IMs are polynomial time algorithms for linear optimization

$O(\sqrt{m}L)$: number of iterations
m : number of inequalities
L : input-data bit-length

μ : central path parameter



$$\min \quad c^{\mathrm{T}}x - \mu \sum_i \ln(Ax-b)_i$$

$$Ax \geq b$$

Notes on Interior Point Methods

Best Complexity Result (1988): Renegar, Gonzaga, Roos-Vial

For central path-following interior point methods is

$O(\sqrt{m}L)$ iterations with total $O(m^3L)$ arithmetic operations

Complexity depends on the number of inequalities

Note: **Iteration complexity of the Ellipsoid method depends on "*n*" the dimension, with separation oracle it is independent of "*m*" the number of inequalities, that might be exponential in the dimension.**

Complexity of Volumetric Center IPMs (1993): Atkinson, Vaidya

$O(\sqrt[4]{mn}L)$ iterations

Complexity of Volumetric Center Cutting Plane IPMs (1993-99):

Anstreicher (improving on Vaidya)

$O(\sqrt{n}L)$ iterations → IPMs Fully Outperform Ellipsoid Methods

Linear Programming: Complexity

Linear Feasibility Problem: Polytopes & Arrangements

COMPLEXITY	Worst Case	Average Case	Practice
Simplex	No polynomial time variant known	$O(m)$	$\sim 2(n+m)$
IPMs	$O(\sqrt{m}L)m^3$	High-Prob: $O(m)$	$20 \sim 60$
Ellipsoid	$O(n^2L)\ m^2$	$O(n^2L)$??	$O(n^2L)$

Average Case: Probabilistic models **v/s** averaging over arrangements

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- ***Simplex methods follow an edge-path on the polytope of the feasible set***
- ***Interior Point Methods follow the central-path***

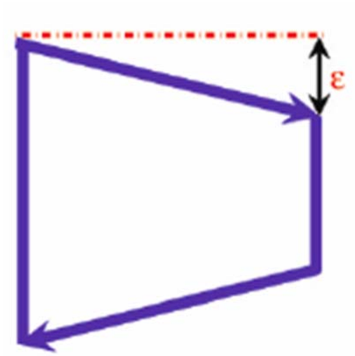
Klee-Minty worst-case example for simplex methods (1972)

Simplex methods may take $2^n - 1$ pivots to reach the optimum on Klee-Minty cubes

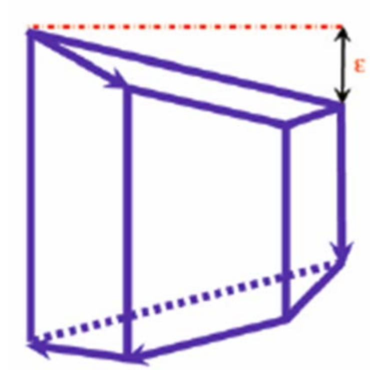
(the edge-path followed by the simplex method visits all the 2^n vertices)

Note: The exponential example is not proved for e.g. Zadeh's rule.

$$\begin{aligned} \textbf{min} \quad & x_n \\ \textbf{subject to} \quad & 0 \le x_1 \le 1 \\ & \varepsilon\, x_{k-1} \le x_k \le 1 - \varepsilon\, x_{k-1} \quad \text{for} \quad k = 2, \ldots, n \end{aligned}$$



Klee-Minty 2-cube



Klee-Minty 3-cube

n **variables**

m=2n **constrains**

How curly can the central path be?

Note: The central path depends on the representation of the feasible set;
It is an analytic, not a geometric object.

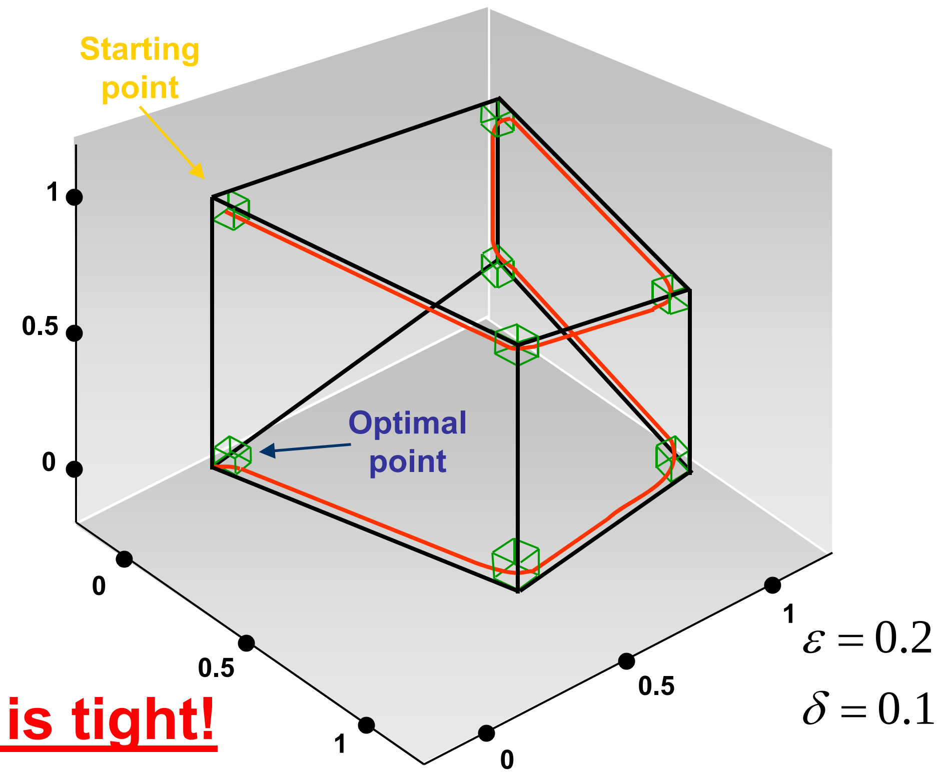
Q: Can the central path be bent along the edge-path followed by the simplex method on the Klee-Minty cube?
(can the central path visit an arbitrary small neighborhood of all 2^n vertices?)

Yes! - if

we carefully add
an exponential number
of redundant constrains

Four different Constructions!

IPMs iteration complexity bound is tight!



Polytopes: Diameter & Curvature

(Motivations and algorithmic issues)

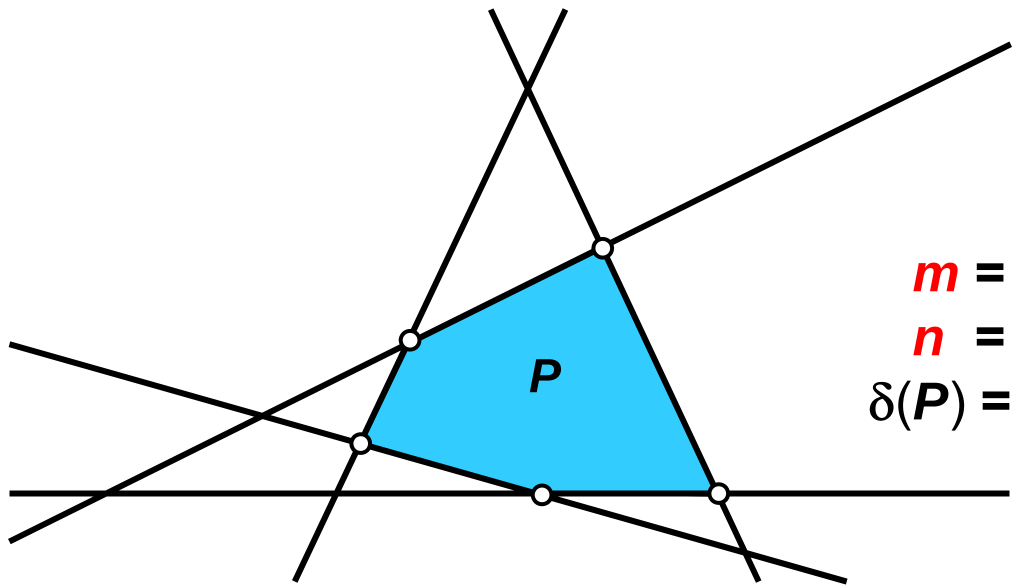
Diameter (***of a polytope***)**:**

lower bound for the number of iterations
for the **simplex method** (feasible *pivot methods*)
(not for criss-cross~admissible pivot methods
short admissible pivot paths do exist!)

Curvature **(*of the central path associated to a polytope*):**

large curvature indicates large number of iterations
for (*central path following*) **interior point methods**

Polytopes & Arrangements : Diameter



$m = 5$: *inequalities*
$n = 2$: *dimension*
$\delta(P) = 2$: *diameter*

Diameter $\delta(P)$: smallest number such that any two vertices can be connected by a path with at most $\delta(P)$ edges

Hirsch Conjecture (1957): $\delta(P) \leq m - n$

Counterexample Santos (2010) maybe: $\delta(P) \leq 2m$

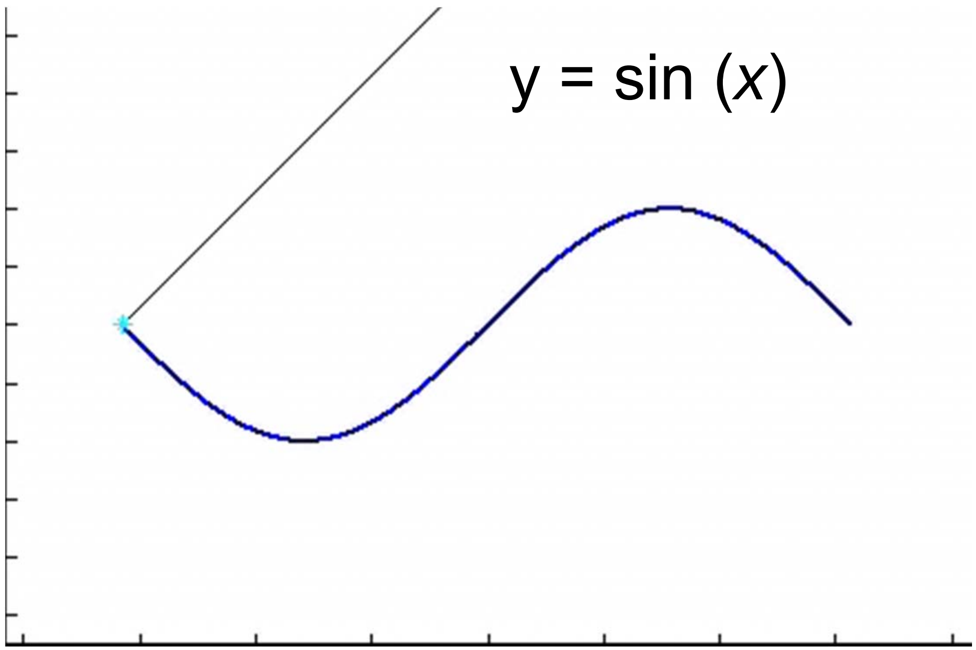
Polytopes & Arrangements : Curvature

C^2 curve $\Psi : [a, b] \to \mathbf{R}^n$ parameterized by its *arc length t*

(note: $\|\Psi'(t)\| = 1$)

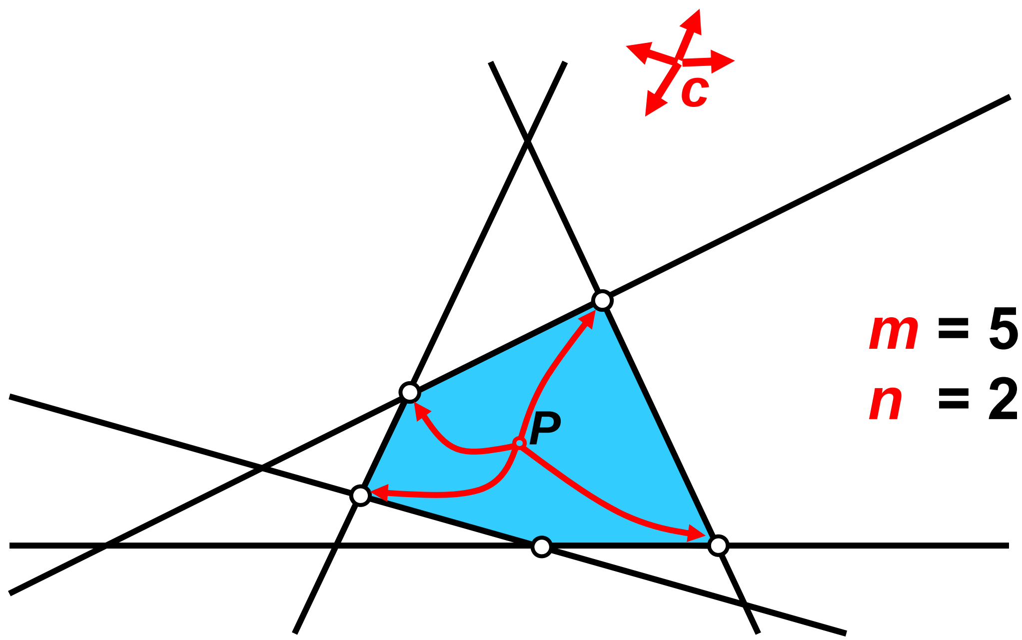
curvature at *t* : $\kappa(t) = \dfrac{\partial^2}{\partial t^2}\Psi(t)$

total curvature : $K = \int_a^b \|\kappa(t)\| dt$



Note: Curvature of Primal/Dual/Primal-dual path's (Sonnevend curvatue)

Polytopes & Arrangements : Curvature



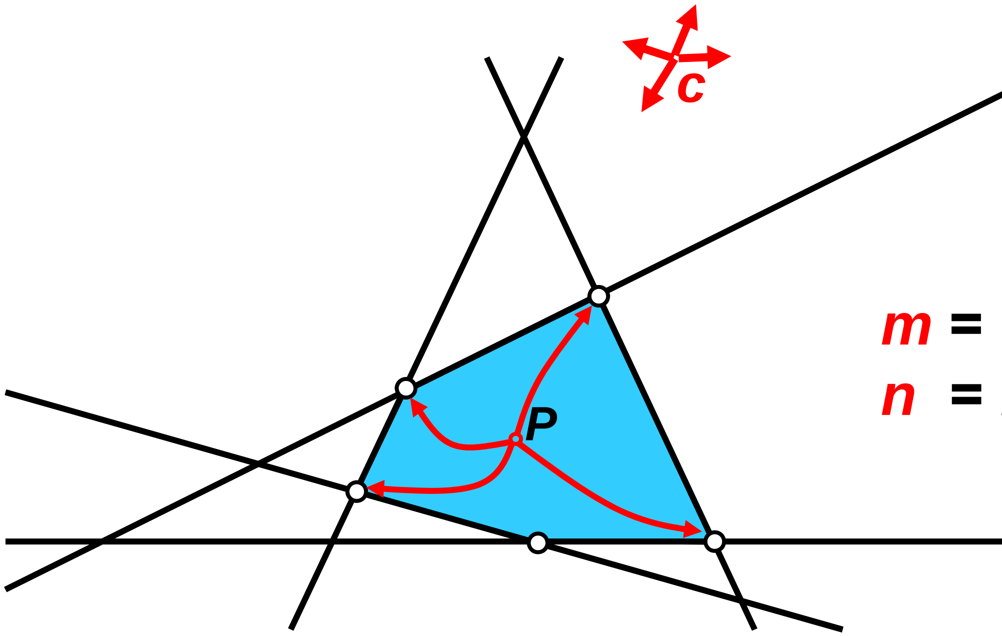
***m* = 5** : *inequalities*
***n* = 2** : *dimension*

$\lambda^{c}(P)$: total curvature of the primal central path of $\min\{c^T x : x \in P\}$

$\lambda(P)$: *largest* total **curvature** $\lambda^{c}(P)$ *over of all possible* ***c***

The Curvature of the Polytope

Polytopes & Arrangements: Diameter & Curvature



m = 5 : *inequalities*
n = 2 : *dimension*

$\lambda^{c}(\boldsymbol{P})$: total curvature of the primal central path

of $\min\{\boldsymbol{c}^{\mathrm{T}}x : x \in \boldsymbol{P}\}$

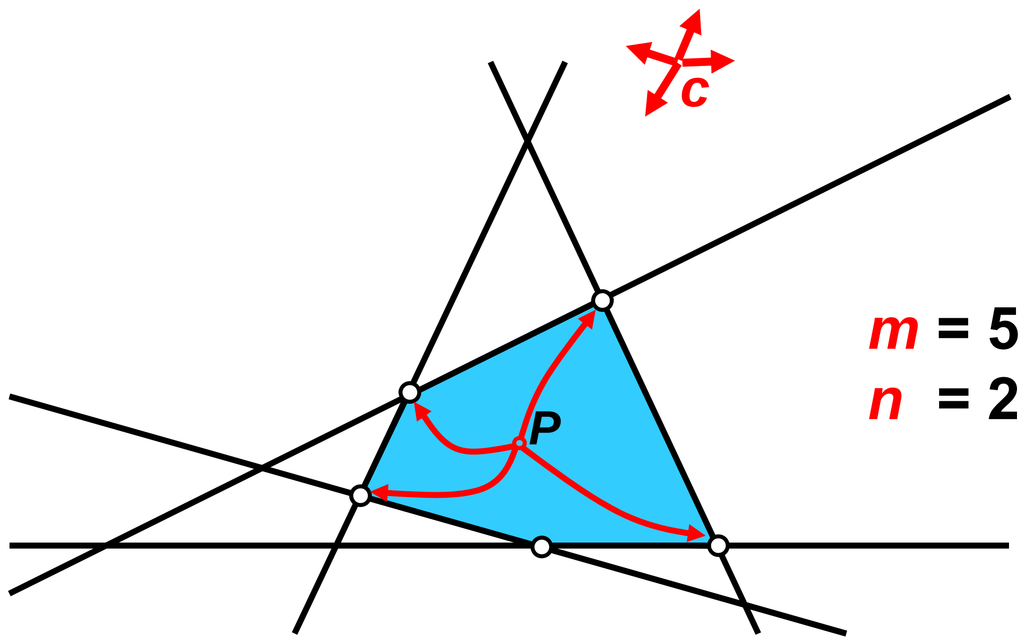
The Curvature of the Polytope

$\lambda(\boldsymbol{P})$: *largest* total curvature $\lambda^{c}(\boldsymbol{P})$ *over of all possible* $\boldsymbol{c}$

Continuous analogue of Hirsch Conjecture:

The curvature of the polytope $\lambda(\boldsymbol{P}) = O(\boldsymbol{m})$

Polytopes & Arrangements: Diameter & Curvature



$\boldsymbol{m}$ **= 5** : *inequalities*
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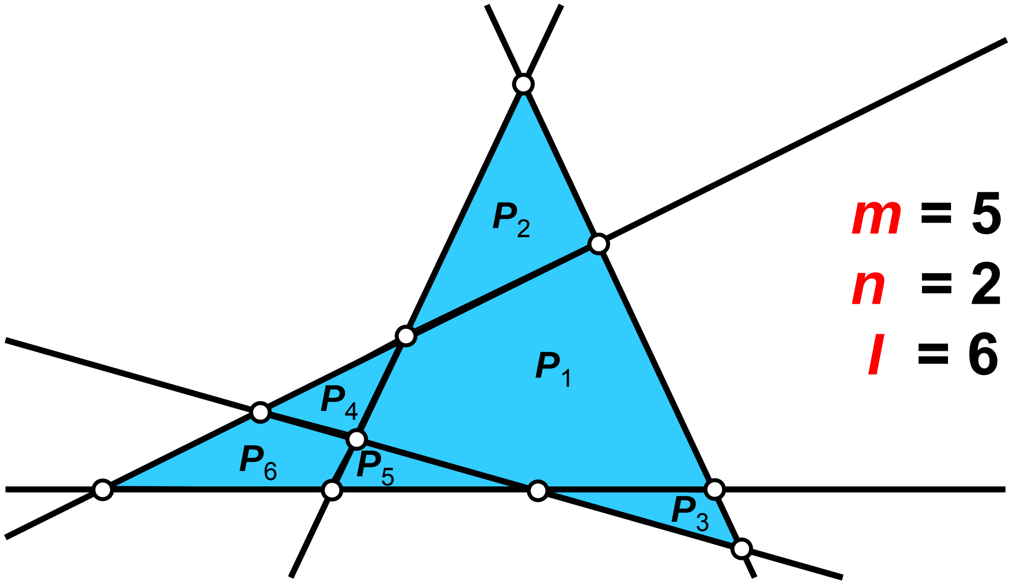
$\lambda^{c}(P)$: total curvature of the primal central path of $\min\{c^T x : x \in P\}$
$\lambda(P)$: *largest* total curvature $\lambda^{c}(P)$ *over of all possible* c

Continuous analogue of Hirsch Conjecture: $\lambda(P) = O(m)$

- **Dedieu-Shub hypothesis (2005):** $\lambda(P) = O(n)$
- **D.-T.-Z. (2006)** $\exists$ **polytope** P **such that:** $\lambda(P) \geq (1.5)^n$

Arrangements : *Diameter & Curvature*

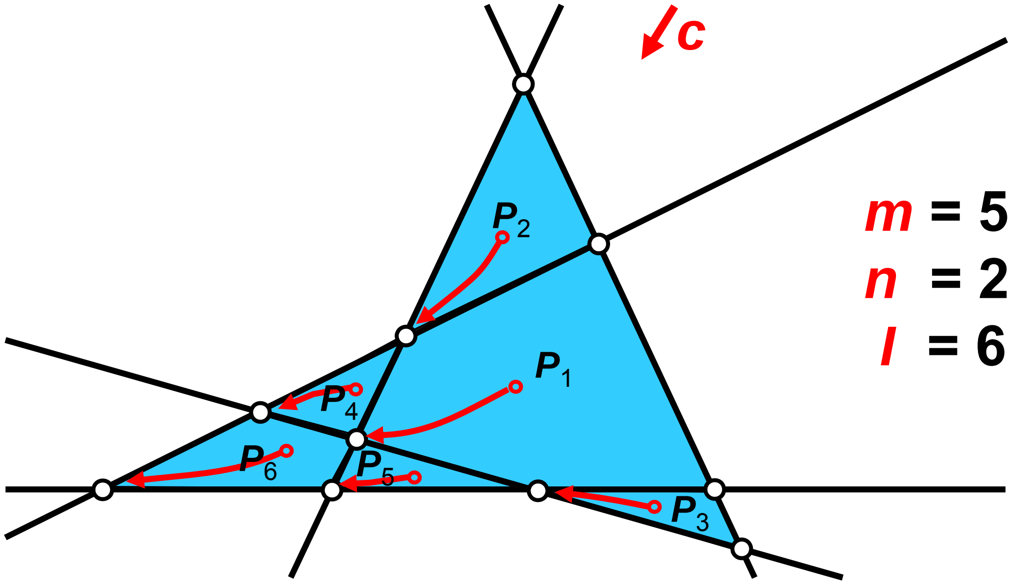


m **= 5** : *hyperplanes*
n **= 2** : *dimension*
l **= 6** : *bounded cells*

Simple arrangement:
$m > n$ & any n hyperplanes **intersect** at a **unique distinct point**

For a simple arrangement,
the number of **bounded cells is** $l = \binom{m-1}{n}$

Polytopes& Arrangements: Diameter & Curvature



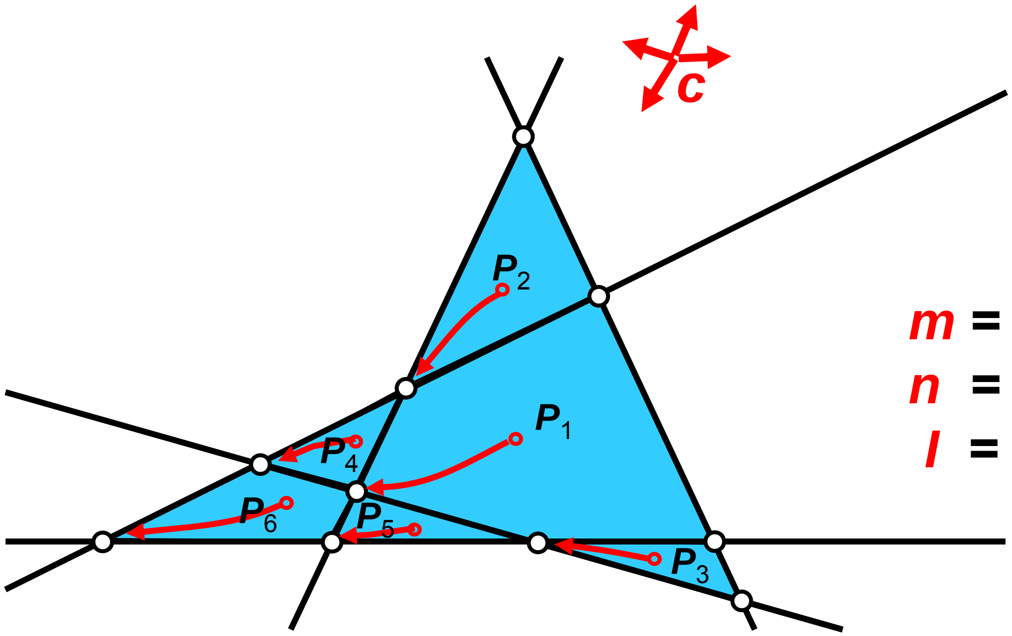
m = 5 : *hyperplanes*
n = 2 : *dimension*
I = 6 : *bounded cells*

$\lambda^c(A)$: average value of $\lambda^c(P_i)$ over the bounded cells P_i of A:

$$\lambda^c(A) = \frac{\sum_{i=1}^{i=I} \lambda^c(P_i)}{I} \qquad \text{with } I = \binom{m-1}{n}$$

❖ $\lambda^c(P_i)$: ***redundant*** **inequalities count**

Polytopes & Arrangements : Diameter & Curvature



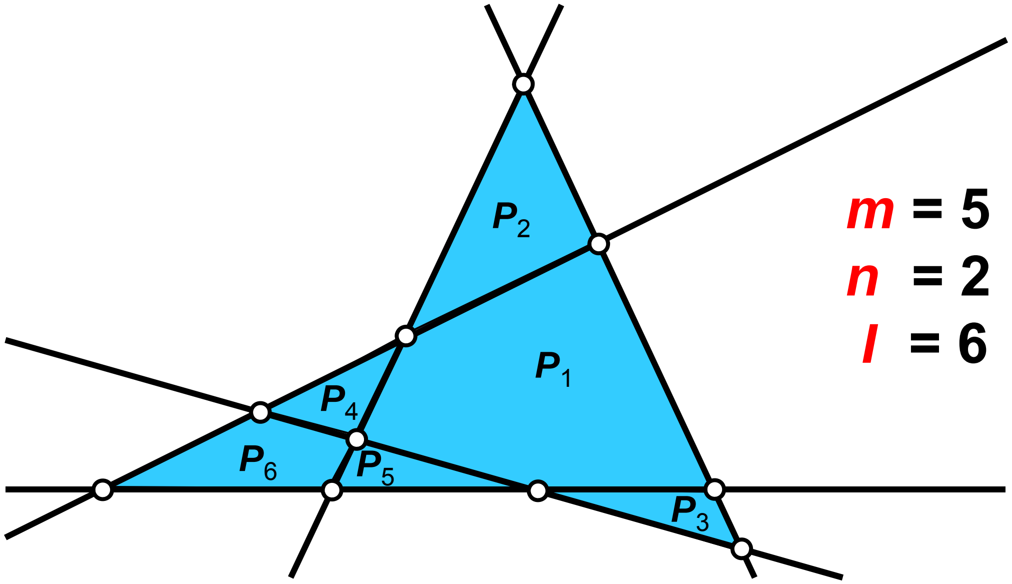
m = 5 : *hyperplanes*
n = 2 : *dimension*
I = 6 : *bounded cells*

$\lambda^c(\boldsymbol{A})$: average value of $\lambda^c(\boldsymbol{P_i})$ over the bounded cells $\boldsymbol{P_i}$ of $\boldsymbol{A}$:

$\lambda(\boldsymbol{A})$: largest value of $\lambda^c(\boldsymbol{A})$ *over all possible* $\boldsymbol{c}$

Dedieu-Malajovich-Shub (2005): $\lambda(\boldsymbol{A}) \leq \boldsymbol{2n}\pi$

Polytopes & Arrangements : Diameter & Curvature



m **= 5** : *hyperplanes*
n **= 2** : *dimension*
I **= 6** : *bounded cells*

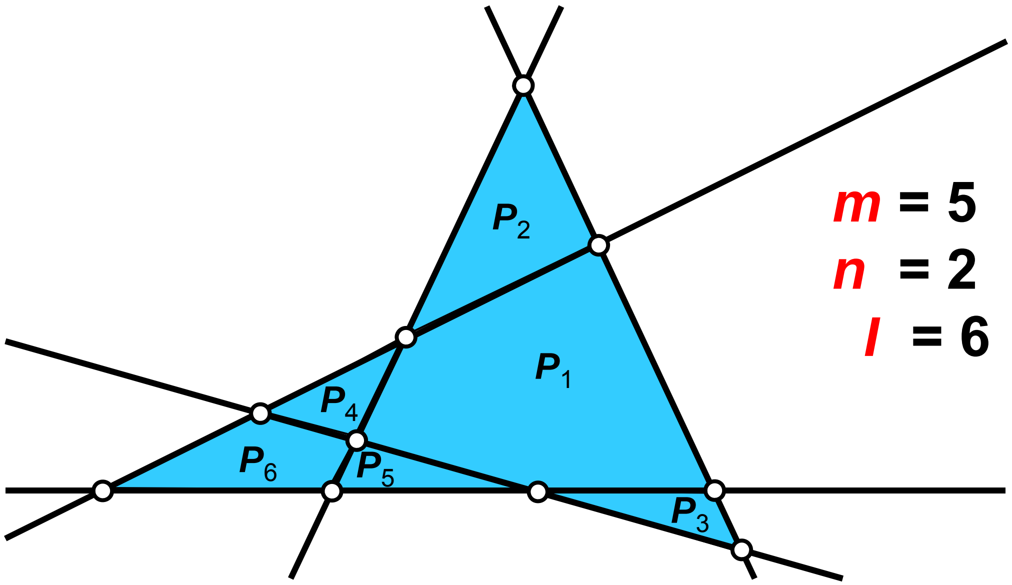
$\delta(\boldsymbol{A})$: average diameter over all bounded cells of ***A***:

$$\delta(\boldsymbol{A}) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I}$$

with $I = \binom{m-1}{n}$

❖ **δ(*A*): average diameter ≠ diameter of *A***
ex: δ(*A*)= 1.333…

Polytopes & Arrangements : Diameter & Curvature



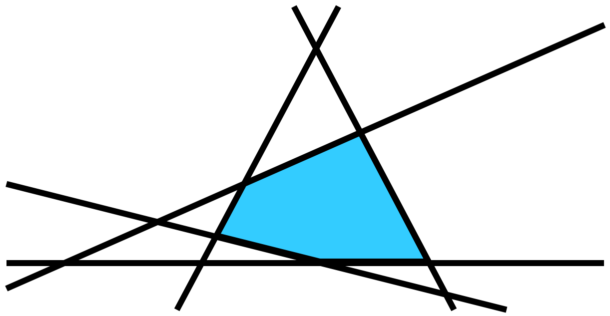
***m* = 5** : *hyperplanes*
***n* = 2** : *dimension*
***I* = 6** : *bounded cells*

$\delta(\boldsymbol{A})$: average diameter of the bounded cells of ***A***:

Conjecture: $\delta(\boldsymbol{A}) \leq O(n)$

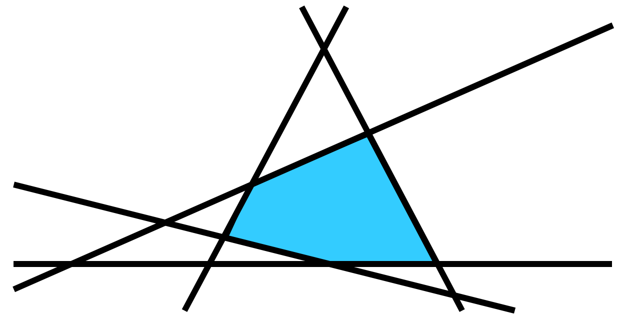
(*discrete analogue of Dedieu-Malajovich-Shub result*)

Polytopes & Arrangements : Diameter & Curvature

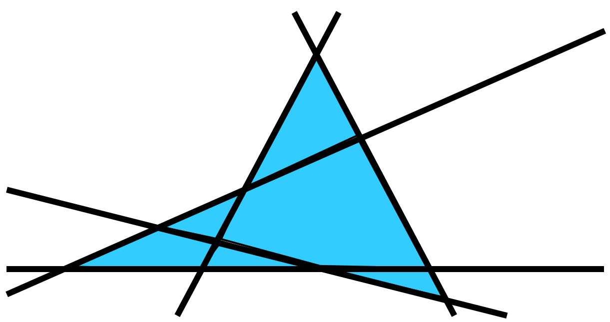


$\delta(\boldsymbol{P}) \leq \boldsymbol{m}\text{-}\boldsymbol{n}$ (Hirsch conjecture 1957, false)

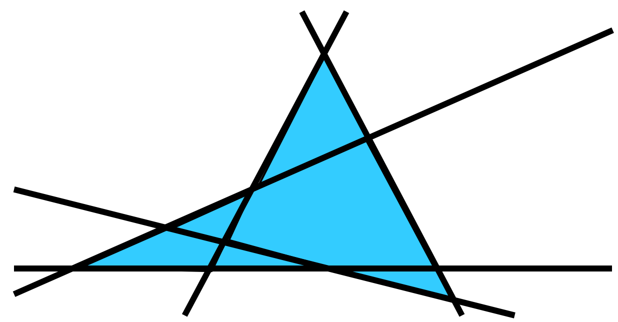
… maybe… $\delta(\boldsymbol{P}) \leq O(m)$



$\lambda(\boldsymbol{P}) \leq 2m\pi$ (conjecture D.-T.-Z. 2006)



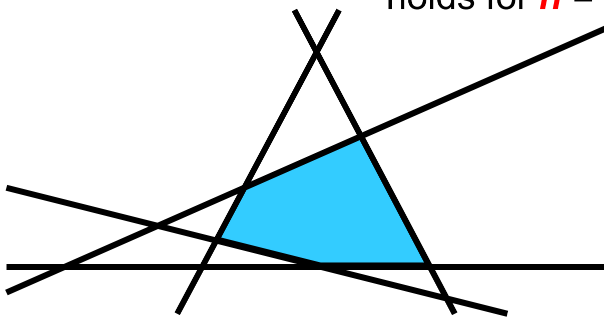
$\delta(\boldsymbol{A}) \leq O(n)$ (conjecture D.-T.-Z. 2006)



$\lambda(\boldsymbol{A}) \leq 2n\pi$ (Dedieu-Malajovich-Shub 2005)

Polytopes & Arrangements : Diameter & Curvature

holds for $n = 2$ and $n = 3$

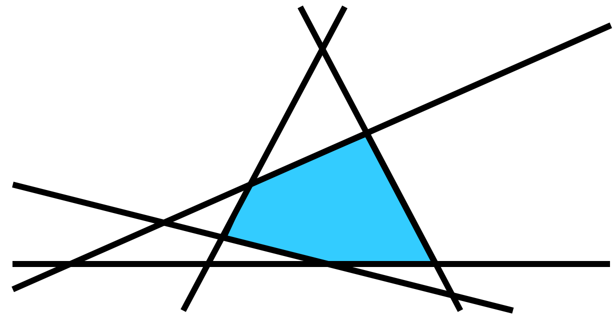


$m - n + 1 \leq \delta(P)$ Santos (2010)

$\delta(P) \leq O(m)$ *conjecture*

Hirsch holds for

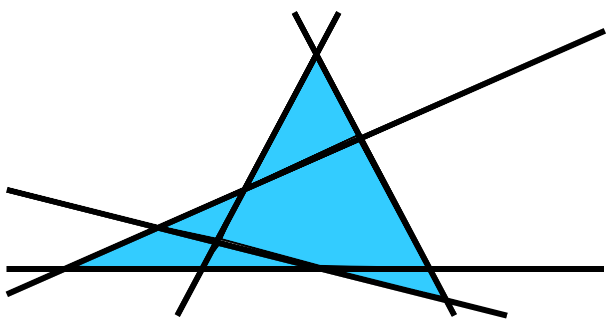
$n = 2$ and $n = 3$



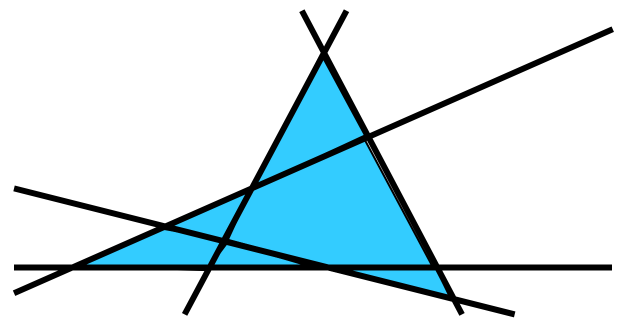
$\lambda(P) \leq 2m\pi$ *conjecture* D.-T.-Z. (2006)

$\lambda(P)=\Omega(m)$ D.-T.-Z (2006)

redundant inequalities count for $\lambda(P)$



$\delta(A) \leq n$ *conjecture* D.-T.-Z. (2006)



$\lambda(A) \leq 2n\,\pi$ Dedieu-Malajovich-Shub (2005)

$\delta(P) \leq m - n \Leftrightarrow \delta(P) \leq n$ for $m = 2n$ Klee-Walkup (1967)

$\lambda(P) = O(m) \Leftrightarrow \lambda(P) = O(n)$ for $m = 2n$ D.-T.-Z. (2006)

Open Problems, Conjectures:

Strong analogies between diameter and curvature

- **Is the linear/polynomial Hirsch (*d*-step) conjecture true?**
- **Is the average Hirsch conjecture true?**
- **What about "smoothed" Hirsch conjecture?**
- **Is there a strongly polynomial (maybe admissible) pivot algorithm to solve LO?**
- **From a given feasible basis is there a feasible pivot sequence with at most *m* (polynomial) pivots to an optimal basis?**

- **Is the $\lambda(P) \leq 2m\pi$ bound for the curvature of the central path true?**
- **What is the worst case total curvature of the Volumetric Path?**
- **What is the worst case total curvature of the Universal Barrier Path?**

Is there a strongly polynomial time algorithm to solve LO?

What about other 'old/new' algorithms, such as

Perceptron (rescaling) algorithms?

Von Neumann Algorithm?

Randomized/smooth(rescaling)/sequential projection?

Thinking parallel?!

How to utilize available/coming massively multi-core computers – how to design parallel algorithms?

Thank you! – Questions?