Counter-example(s) to the Hirsch Conjecture

Francisco Santos

Universidad de Cantabria, Spain
http://personales.unican.es/santosf/Hirsch

Efficiency of the simplex method — IPAM, January 18, 2011

The Hirsch conjecture

Let $\delta(P)$ denote the diameter of the graph of a polytope P.

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\delta(P) \leq n - d.$$

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There is an infinite family of non-Hirsch polytopes with diameter $\sim (1 + \epsilon)n$, even in fixed dimension. (Best so far: $\epsilon = 1/20$).

Why is $n - d$ a "reasonable" bound?

- It holds with equality in simplices ($n = d + 1$, $\delta = 1$) and cubes ($n = 2d$, $\delta = d$).
- If P and Q satisfy it, then so does $P \times Q$: $\delta(P \times Q) = \delta(P) + \delta(Q)$. In particular:

  For every $n \leq 2d$, there are polytopes in which the bound is tight (products of simplices).

- For every $n > d$, it is easy to construct unbounded polyhedra where the bound is tight.

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Hirsch $\Leftrightarrow$ d-step $\Leftrightarrow$ non-revisiting path.

Proof: Let $H(n, d) = \max\{\delta(P) : P$ is a d-polytope with n facets$\}$. Then:

$$\cdots \leq H(2d-1, d-1) \leq H(2d, d) \geq H(2d+1, d+1) \geq \cdots$$

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$$d_{P'}(u', v') \geq d_P(u, v).$$

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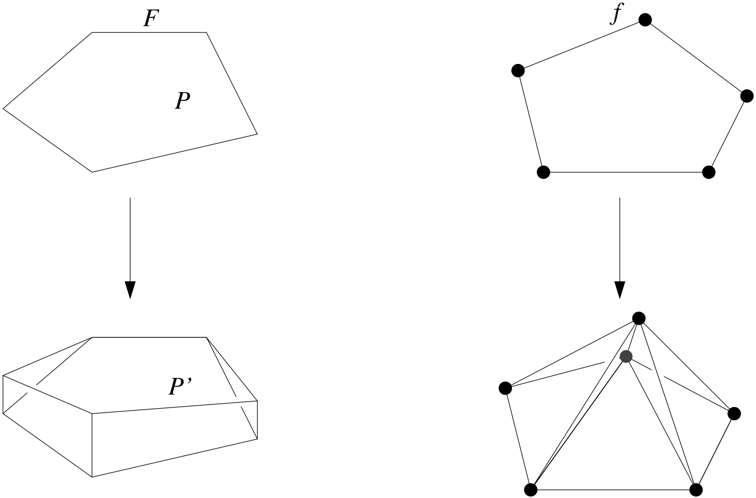

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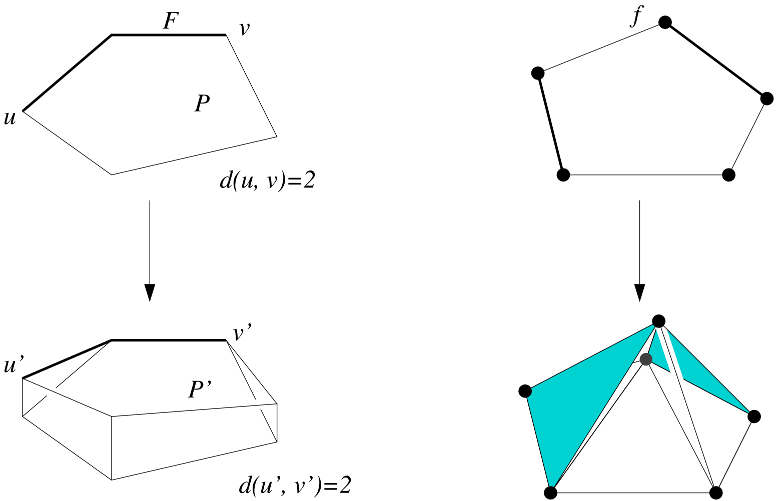

Two ingredients

The construction of the counter-example has two ingredients:

1. A strong d-step theorem for spindles/prismatoids.
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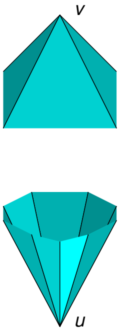
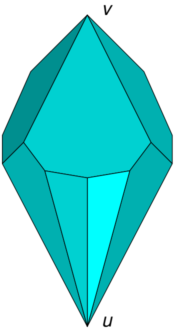
Definition

A *spindle* is a polytope P with two distinguished vertices u and v such that every facet contains either u or v (but not both).





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The *length* of a spindle is the graph distance from u to v.

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3-spindles have length ≤ 3.

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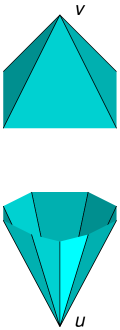

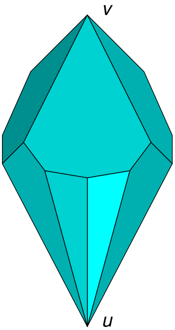

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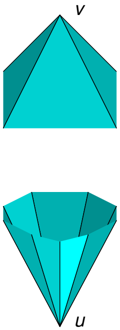

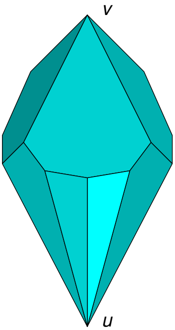

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Let P be a spindle of dimension d, with $n > 2d$ facets and length λ. Then there is another spindle P' of dimension $d+1$, with $n+1$ facets and length $\lambda+1$.

That is: we can increase the dimension, length and number of facets of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle P has length $> d$ then there is another spindle P' (of dimension $n-d$, with $2n-2d$ facets, and length $\geq \lambda + n - 2d > n - d$) that violates the Hirsch conjecture.

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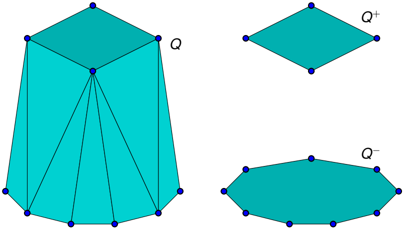

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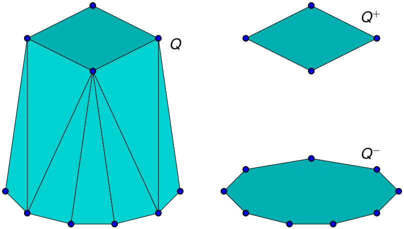

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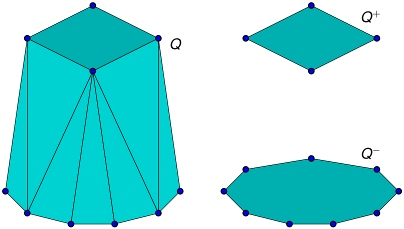

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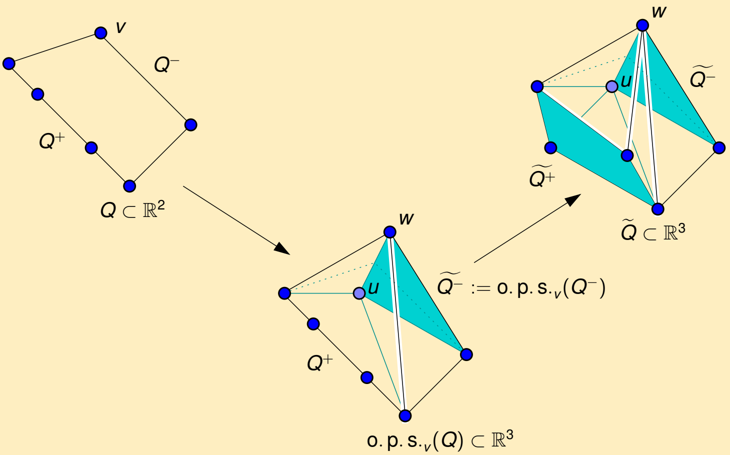
d-step theorem for prismatoids

Proof.



□

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So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant...*

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Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S., July 2010].
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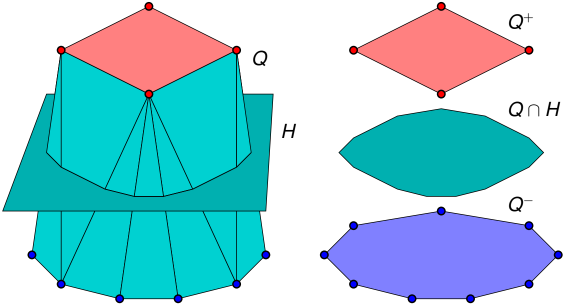
Combinatorics of prismatoids

Analyzing the combinatorics of a d-prismatoid Q can be done via an intermediate slice ...



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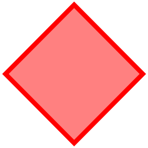
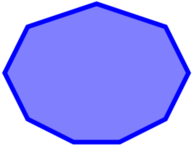
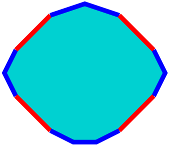
…which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-.

$\frac{1}{2}$  $+\ \frac{1}{2}$  $=$ 

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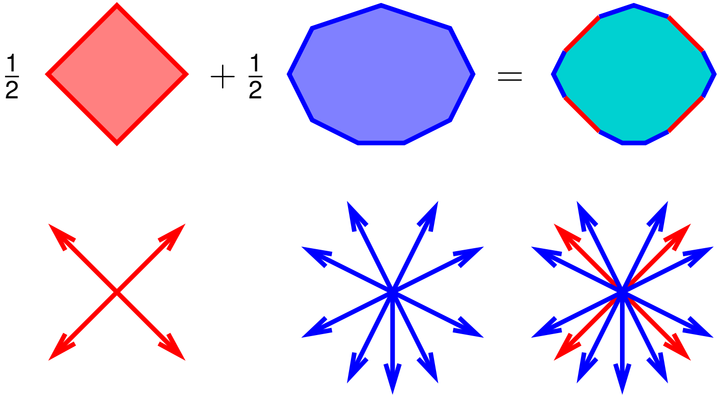
…which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-. The normal fan of $Q^+ + Q^-$ equals the “superposition” of those of Q^+ and Q^-.



Combinatorics of prismatoids

So: the combinatorics of Q follows from the superposition of the normal fans of Q^+ and Q^-.

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition ("map") of the $d-2$-sphere.

Conclusion

3-prismatoids ⇔ pairs of maps in the 1-sphere (a.k.a. circle).
4-prismatoids ⇔ pairs of maps in the 2-sphere.
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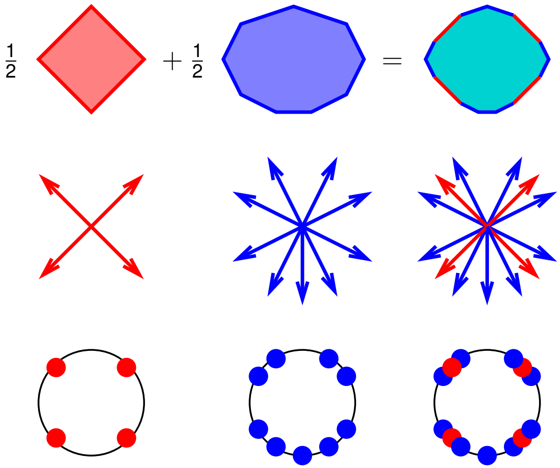
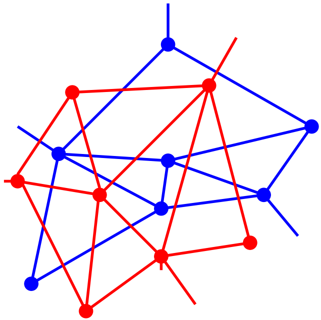
Example: a 3-prismatoid



Example: (part of) a 4-prismatoid



4-prismatoid of width > 4

$\Updownarrow$

pair of (geodesic, polytopal) maps in S^2 so that two steps do not let you go from a blue vertex to a red vertex.

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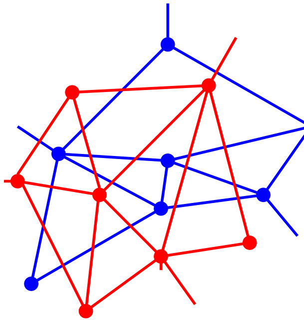

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Width in the pairs of maps picture.

Theorem

Let Q be a d-prismatoid with bases Q^+ and Q^- and let G^+ and G^- be the corresponding maps in the $(d-2)$-sphere (central projection of the normal fans of Q^+ and Q^-). Then, the width of Q equals 2 plus the minimum number of steps needed to go from a vertex of G^+ to a vertex of G^- in the (graph of) the superposition of the two maps.

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A sufficient condition for non-Hirschness.

Theorem

Let G^+ and G^- be the normal maps of the bases Q^+ and Q^- of a d-prismatoid Q (W.l.o.g., assume they intersect transversally). For Q to have width $\leq d$ it is **necessary** *that a facet F^+ of G^+ and a facet F^- of G^- exist such that F^+ contains a vertex of F^- and F^- one of F^+.*

Put differently: consider the directed bipartite graph with nodes being the facets of G^+ and a facet of G^-. For facets F^+ and F^- put an arrow $F^+ \to F^-$ if F^+ contains a vertex of F^-. then:

Corollary

For Q to have width $> d$ it is **sufficient** *that this graph does not have 2-cycles.*

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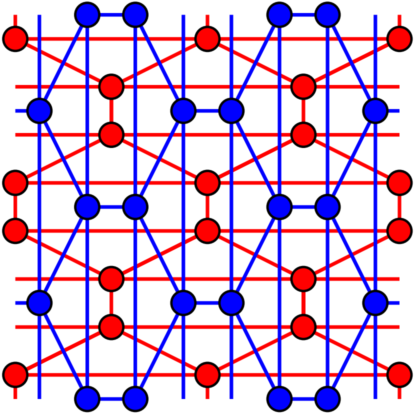
A 4-dimensional prismatoid of width > 4?

Replicating the following basic structure we can get a “non-Hirsch” periodic pair of maps in the plane:



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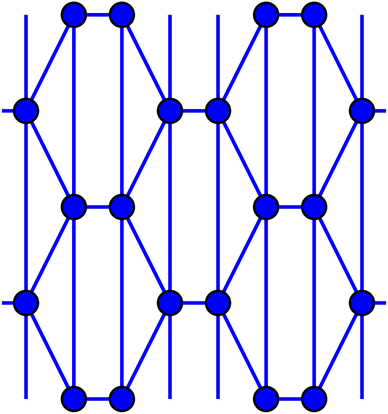

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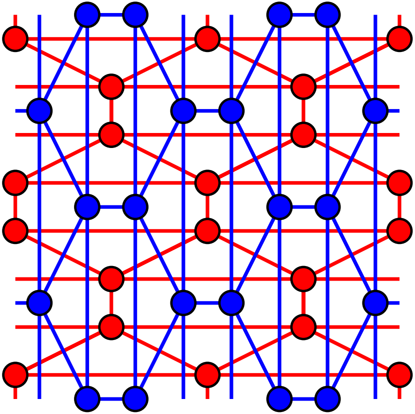

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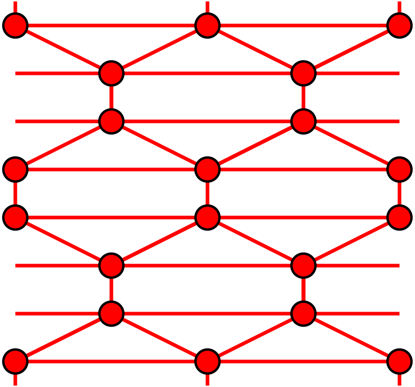

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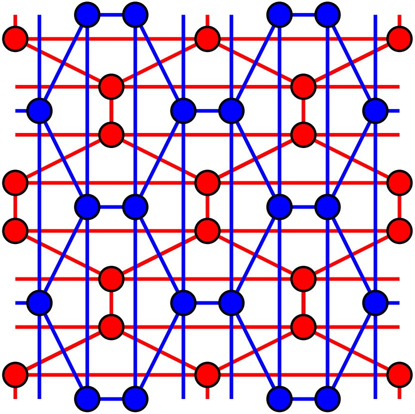

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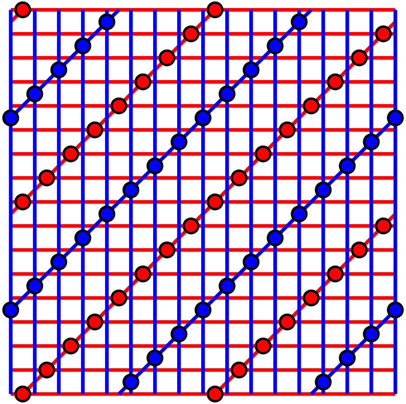

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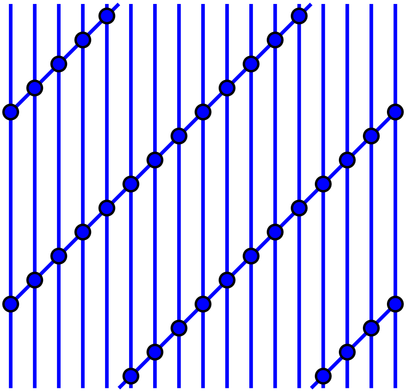

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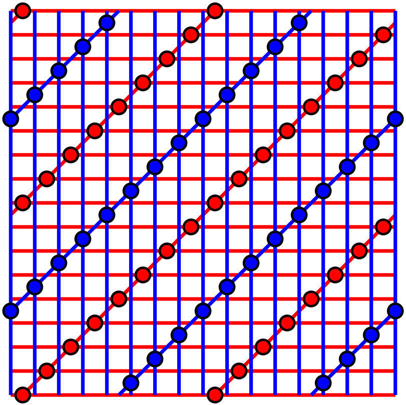

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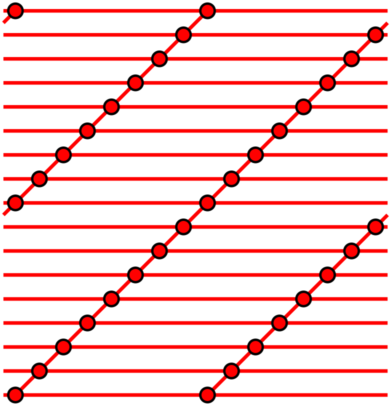

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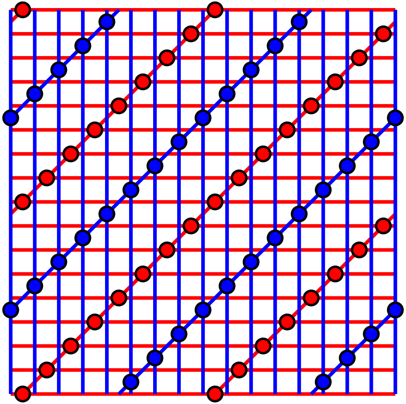

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So, “non-Hirsch pairs of maps” exist in the torus and other surfaces. Surprisingly enough:

Theorem (S., July 2010, Stephen-Thomas, Dec. 2010)

There is no non-Hirsch pair of maps in the 2-sphere.

Proof 1 (S.): Uses simply connectedness of the 2-sphere. Shows that a non-Hirsch pair of maps would imply an infinite sequence of loops each inside the previous one.

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5-dimensional prismatoids

A 5-prismatoid of width > 5

In dimension 5 (that is, with maps in the 3-sphere) we have room enough to construct “non-Hirsch pairs of maps”:

Theorem (S. May 2010)

The prismatoid Q of the next two slides, of dimension 5 and with 48 vertices, has width six.

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Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

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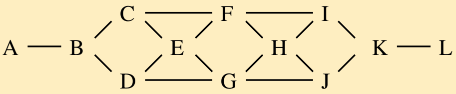
Proof 1.

It has been verified computationally that the dual graph of Q (modulo symmetry) has the following structure:



□

A 5-prismatoid of width > 5

$$Q := \text{conv} \left\{ \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 \\ & \pm 18 & 0 & 0 & 0 & 1 \\ & 0 & \pm 18 & 0 & 0 & 1 \\ & 0 & 0 & \pm 45 & 0 & 1 \\ & 0 & 0 & 0 & \pm 45 & 1 \\ & \pm 15 & \pm 15 & 0 & 0 & 1 \\ & 0 & 0 & \pm 30 & \pm 30 & 1 \\ & 0 & \pm 10 & \pm 40 & 0 & 1 \\ & \pm 10 & 0 & 0 & \pm 40 & 1 \end{matrix} \quad \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & \pm 18 & -1 \\ 0 & 0 & \pm 18 & 0 & -1 \\ \pm 45 & 0 & 0 & 0 & -1 \\ 0 & \pm 45 & 0 & 0 & -1 \\ 0 & 0 & \pm 15 & \pm 15 & -1 \\ \pm 30 & \pm 30 & 0 & 0 & -1 \\ \pm 40 & 0 & \pm 10 & 0 & -1 \\ 0 & \pm 40 & 0 & \pm 10 & -1 \end{matrix} \right\}$$

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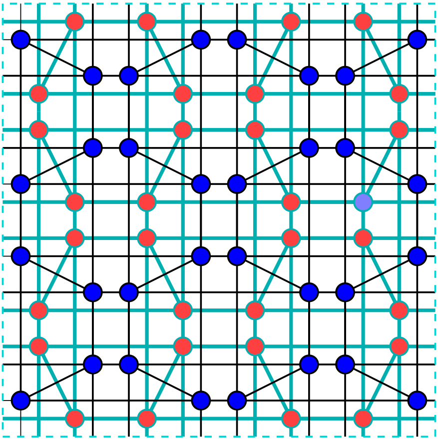

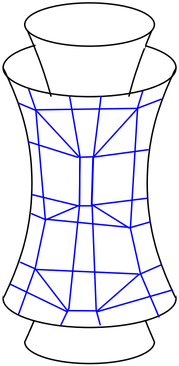

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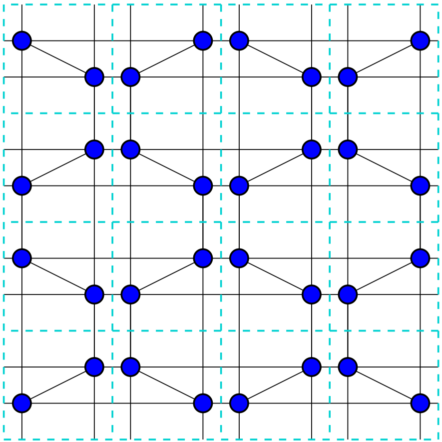

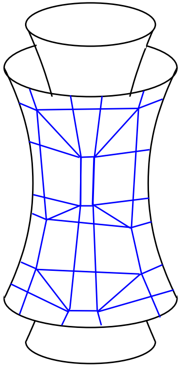

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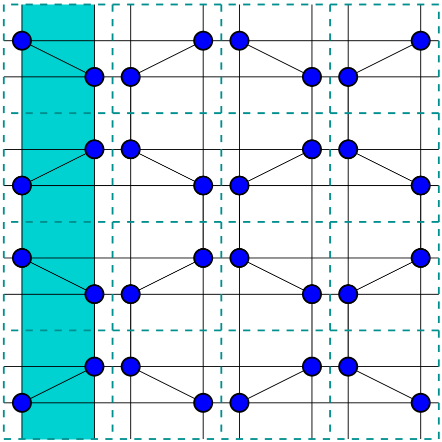

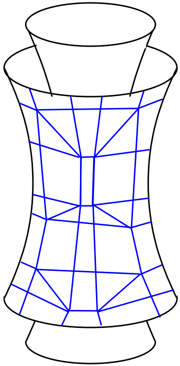

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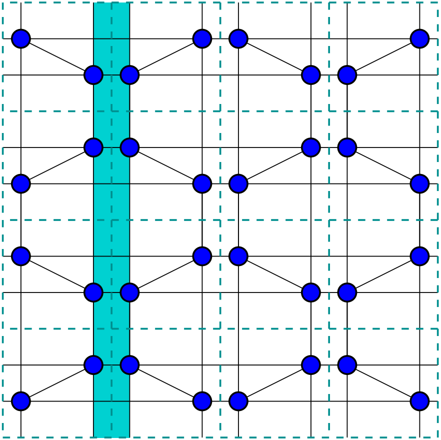

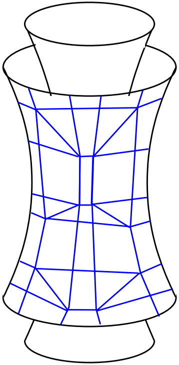

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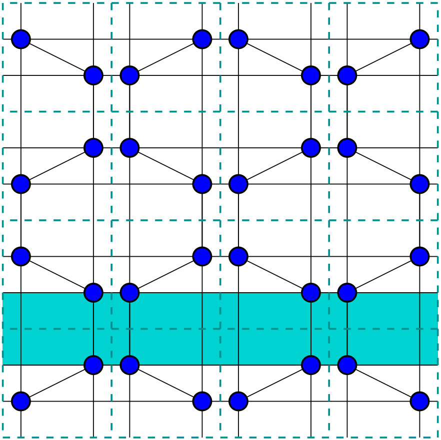

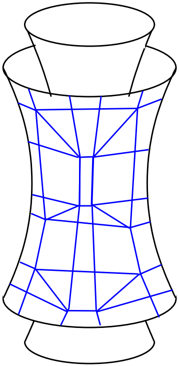

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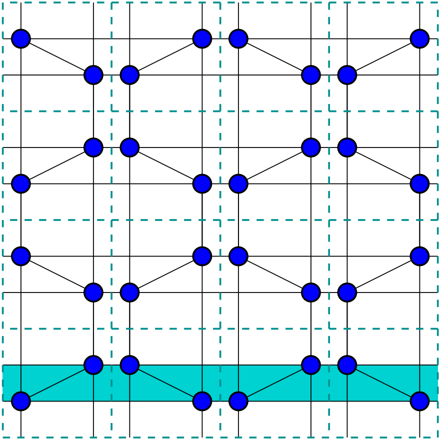

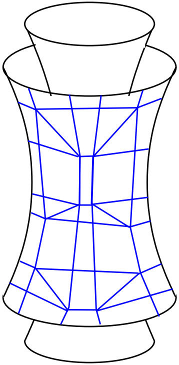

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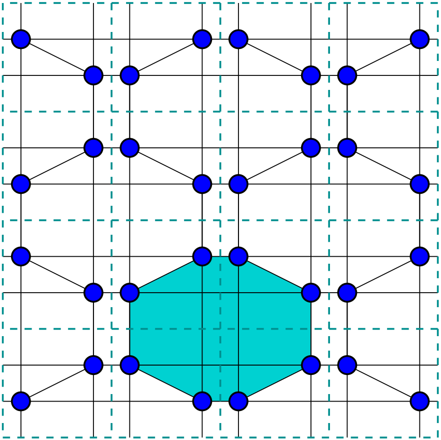
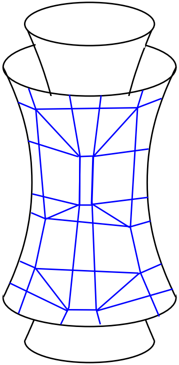


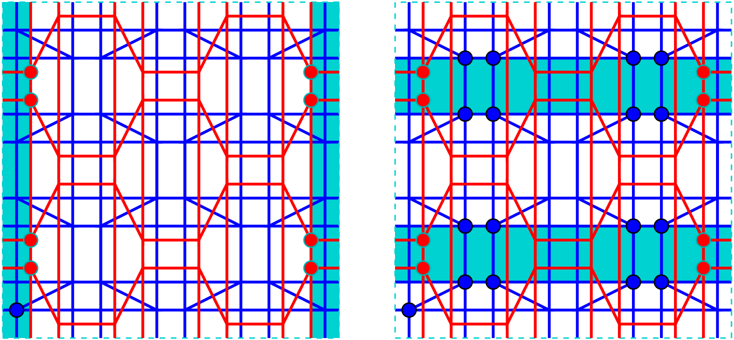


A smaller 5-prismatoid of width > 5

With the same ideas

Proof 2.

Show that there are no blue vertex a and red vertex b such that a is a vertex of the blue cell containing b and b is a vertex of the red cell containing a. □



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Theorem (S. 2010+)

The following prismatoid of dimension 5 with 28 vertices has width 6:

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$$Q := \text{conv} \left\{ \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} \pm 18 & 0 & 0 & 0 & 1 \\ 0 & 0 & \pm 30 & 0 & 1 \\ 0 & 0 & 0 & \pm 30 & 1 \\ 0 & \pm 5 & 0 & \pm 25 & 1 \\ 0 & 0 & \pm 18 & \pm 18 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 0 & 0 & \pm 18 & 0 & -1 \\ 0 & \pm 30 & 0 & 0 & -1 \\ \pm 30 & 0 & 0 & 0 & -1 \\ \pm 25 & 0 & 0 & \pm 5 & -1 \\ \pm 18 & \pm 18 & 0 & 0 & -1 \end{pmatrix} \end{array} \right\}$$

Corollary

There is a 23-polytope with 46 facets violating the Hirsch conjecture.

A smaller 5-prismatoid of width > 5

Theorem (Weibel 2011+)

There is a prismatoid of dimension 5 *with* 25 *vertices and width* 6.

A smaller 5-prismatoid of width > 5

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Corollary

There is a 20-polytope with 40 facets violating the Hirsch conjecture.

Asymptotic width in fixed dimension

If we fix the dimension d, the width of prismatoids is linear:

Theorem

The width of a d-dimensional prismatoid with n vertices cannot exceed $2^{d-3}n$.

Proof.

This is a general result for the (dual) diameter of a polytope [Barnette, Larman, ∼1970]. □

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Using the Strong d-step Theorem for 5-prismatoids it is impossible to violate the Hirsch conjecture by more than 50%.

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There are 5-dimensional prismatoids with n vertices and width $(\sqrt{n})$.

Sketch of proof

Start with the “simple, yet more drastic” pair of maps in the torus.

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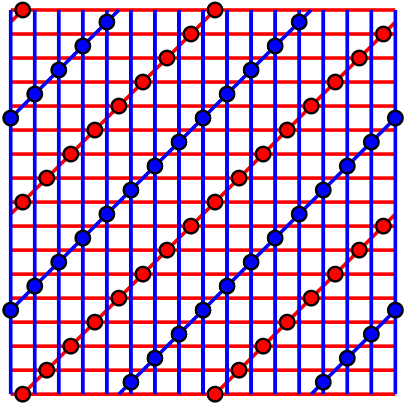

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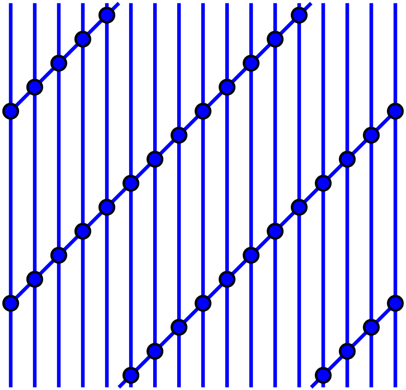

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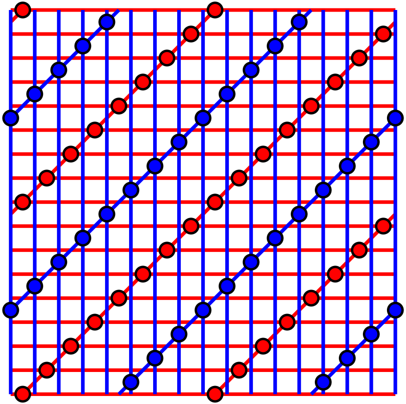

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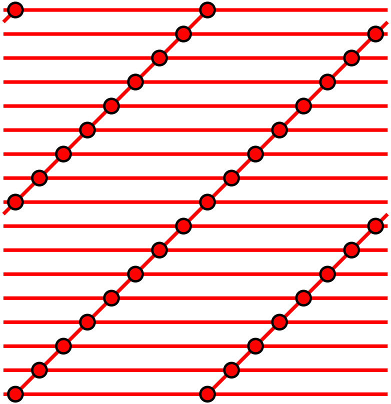

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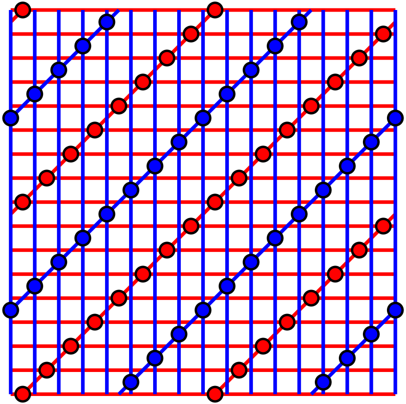
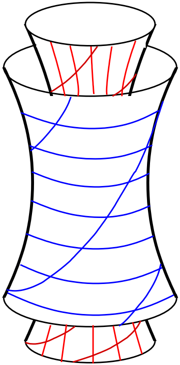


Asymptotic width in dimension five

Consider the red and blue maps as lying in two parallel tori in the 3-sphere.



Complete the tori maps to the whole 3-sphere (you need quadratically many cells for that).

Between the two tori you basically get the superposition of the two tori maps. □

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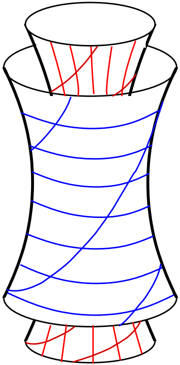

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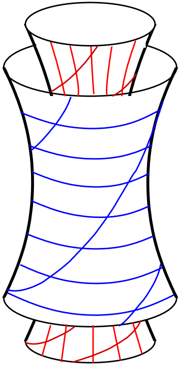

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Conclusion

- The counter-examples to the Hirsch conjecture break a "psychological barrier", but for applications they are so far irrelevant. They violate Hirsch by about 5%.
- The main open question(s) remains open: Is there a family of polytopes with superlinear diameter? Is the diameter of every polytope polynomially bounded?
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The diameter of a d-polytope with n-facets (more generally, of a connected layer family in the sense of Eisenbrand-Hähnle-Razborov-Rothvoss) cannot exceed

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The end

THANK YOU!